Projectivity and unification in the varieties of locally finite monadic MV-algebras

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Abstract

A description of finitely generated free monadic MV-algebras and a characterization of projective monadic MV-algebras in locally finite varieties is given. It is shown that unification type of locally finite varieties is unitary.

1 Introduction

Monadic MV-algebras (monadic Chang algebras by Rutledge's terminology) were introduced and studied by Rutledge in [5] as an algebraic model for the predicate calculus qL of Lukasiewicz infinite-valued logic, in which only a single individual variable occurs. Rutledge followed P.R. Halmos' study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus the result of Rutledge in [5], showing the completeness of the monadic predicate calculus, has been a great interest. Adapting for the propositional case the axiomatization of monadic MV-algebras given by Rutledge in [5], we can define modal Lukasiewicz propositional calculus $MLPC$ as a logic which contains Lukasiewicz propositional calculus Luk, the formulas as the axioms schemes: $\alpha \to \exists\alpha$, $\exists(\alpha \vee \beta) \equiv \exists\alpha \vee \exists\beta)$, $\exists(\neg\exists\alpha) \equiv \neg\exists\alpha$, $\exists(\exists\alpha + \exists\beta) \equiv \exists\alpha + \exists\beta$, $\exists(\alpha + \alpha) = \exists\alpha + \exists\alpha$, $\exists(\alpha \cdot \alpha) = \exists\alpha \cdot \exists\alpha$ and closed under modus ponens and necessitation (if α, then $\forall\alpha$, where $\forall\alpha = \neg\exists\neg\alpha$).

Let L denote a first-order language based on $\cdot, +, \to, \neg, \exists$ and L_m denotes monadic propositional language based on $\cdot, +, \to, \neg, \exists$ and $Form(L)$ and $Form(L_m)$ - the set of all formulas of L and L_m, respectively. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation $\Psi : Form(L_m) \to Form(L)$ by putting: $\Psi(p) = p^*(x)$ if p is propositional variable, $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \to$, $\Psi(\exists\alpha) = \exists\Psi(\alpha)$.

Through this translation Ψ, we can identify the formulas of L_m with monadic formulas of L containing the variable x. Moreover, it is routine to check that $\Psi(MLPC) \subseteq QL$.

2 Monadic MV-algebras

The characterization of monadic MV-algebras as pair of MV-algebras, where one of them is a special kind of subalgebra, are given in [3, 2]. MV-algebras were introduced by Chang in [1] as an algebraic model for infinitely valued Lukasiewicz logic.

*The author is supported by the grant CNRS-SNRSF N 09/09 and the grant SNRSF N 31/08.

An MV-algebra is an algebra $A = (A, \oplus, \odot, ^*, 0, 1)$ where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold for all $x, y \in A$: $x \oplus 1 = 1, x^{**} = x, 0^* = 1, x \oplus x^* = 1, (x^* \oplus y)^* \oplus y = (x \oplus y^*) \oplus x, x \odot y = (x^* \oplus y^*)^*$.

An algebra $A = (A, \oplus, \odot, ^*, \exists, 0, 1)$ is said to be monadic MV-algebra (for short MMV-algebra) if $A = (A, \oplus, \odot, ^*, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities: $x \leq \exists x$, $\exists(x \vee y) = \exists x \vee \exists y$, $\exists(\exists x)^* = (\exists x)^*$, $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$, $\exists(x \odot y) = \exists x \odot \exists y$, $\exists(x \oplus y) = \exists x \oplus \exists y$.

We shall denote a monadic MV-algebra $A = (A, \oplus, \odot, ^*, \exists, 0, 1)$ by $(A, \exists)$, for brevity. Let $\exists A = \{x \in A : x = \exists x\}$.

A subalgebra A_0 of an MV-algebra A is said to be relatively complete if for every $a \in A$ the set $\{b \in A_0 : a \leq b\}$ has the least element.

A subalgebra A_0 of an MV-algebra A is said to be *m-relatively complete*, if A_0 is relatively complete and two additional conditions hold:

(#) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \odot a \Rightarrow v \geq a \& v \odot v \leq x)$,

(##) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \& v \oplus v \leq x)$.

Proposition 1. [3]. *Let $(A, \oplus, \odot, ^*, \exists, 0, 1)$ be a monadic MV-algebra. Then the MV-subalgebra $\exists A$ of MV-algebra $(A, \oplus, \odot, ^*, 0, 1)$ is m-relatively complete.*

Proposition 2. [3]. There exists a one-to-one correspondence between.
(1) monadic MV-algebras $(A, \exists)$;
(2) the pairs (A, A_0), where A_0 is m-relatively complete subalgebra of A.

3 Projective monadic MV-algebras

From the variety of monadic MV-algebras **MMV** select the subvariety $\mathbf{K_n}$ for $1 \leq n \neq \omega$, which is defined by the following equation [3]: (K_n) $x^n = x^{n+1}$, that is $\mathbf{K_n} = \mathbf{MMV} + (K_n)$.

Proposition 3.[3] *If $(A, \exists)$ is a totally ordered monadic MV-algebra, then $A = \exists A$.*

Proposition 4.[3] *If $(A, \exists)$ is a finite monadic MV-algebra with totally ordered $\exists A$, then MV-algebra A is isomorphic to a product of totally ordered MV-algebras A_i, $i \in I$, $A_i \cong \exists A$ and $\exists A$ is isomorphic to the diagonal subalgebra of the product.*

It is defined a unique monadic operator $\exists$ on S_n^k, where $S_n = (S_n; \oplus, \odot, ^*, 0, 1)$ and $S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, which corresponds to m-relatively complete linearly ordered MV-subalgebra, converting the algebra S_n^k into a simple monadic MV-algebra [3]. This subalgebra coincides with the greatest diagonal subalgebra, i.e. $d(S_n^k) = \{(x, ..., x) \in S_n^k : x \in S_n\}$. Denote this monadic MV-algebra by $(S_n^k, \exists_d)$. In this case the monadic operator $\exists_d$ is defined as follows: $\exists_d(x_1, ...x_k) = (x_j, ..., x_j)$, where $x_j = max(x_1, ..., x_k)$. The operator $\forall_d$ is defined dually: $\forall_d(x_1, ...x_k) = (x_i, ..., x_i)$, where $x_i = min(x_1, ..., x_k)$.

Notice that $\mathbf{K_n}$ is generated by $(S_p^k, \exists_d)$, $p = 1, ..., n$ and $k \in \omega$. Moreover, $\mathbf{K_n}$ is locally finite and there exists maximal $k \in \omega$, depending on n, such that $(S_n^k, \exists_d)$ is m-generated. The maximal k we denote by $t(n)$. There exists also a positive number $r(k, n)$ depending on k and n such that $(S_n^k, \exists_d)^{r(k,n)}$ is m-generated. So,

Theorem 5.

$$\prod_{p=1}^{n} \prod_{k=1}^{t(p)} (S_p^k, \exists_d)^{t(k,p)}$$

is a free m-generated algebra $F_{\mathbf{K_n}}(m)$ *in the variety* $\mathbf{K_n}$.

Let us notice, that exact description of one-generated free MMV-algebra in the variety $\mathbf{K_n}$ is given in [3].

Theorem 6. *The m-generated MMV-algebra A from* $\mathbf{K_n}$ *is projective iff A is isomorphic to* $(S_1^1, \exists_d) \times A'$.

Theorem 7. *Any subalgebra of the free m-generated algebra* $F_{\mathbf{K_n}}(m)$ *is projective.*

Let $\mathbf{V_n}$ be the variety generated by $\{S_1, ..., S_n\}$.
Let us observe that

$$\prod_{p=1}^{n} (S_p^1, \exists)^{t(1,p)}$$

is an algebra with trivial monadic operator $\exists$ (i. e. $\exists x = x$) which is isomorphic as an MV-algebra to the m-generated free MV-algebra $F_{\mathbf{V_n}}(m)$. Denote this algebra as $(F_{\mathbf{V_n}}(m), \exists)$. It holds

Theorem 8. *The MMV-algebra* $(F_{\mathbf{V_n}}(m), \exists)$ *is a retract of the algebra of the free m-generated algebra* $F_{\mathbf{K_n}}(m)$. *So,* $(F_{\mathbf{V_n}}(m), \exists)$ *is projective.*

4 Monadic operators on finite MV-algebras

Suppose that A is a finite MV-algebra. Then $A \cong S_{n_1} \times S_{n_2} \times ...S_{n_k}$ where the $n_i \geq 1$. Let $\Pi = \{K_1, K_2, ..., K_m\}$ be a partition of $\{1, 2, ..., k\}$. We shall say that Π is homogeneous if $i, j \in K_l$ implies $S_{n_i} = S_{n_j}$. Given such a Π, each K_i has associated a unique S_{n_j} which we shall denote by A_i. We clearly have $A \cong A_1^{K_1} \times ... \times A_m^{K_m}$. Since each K_i is finite, there is a monadic operator $\exists_i$ defined on $A_i^{K_i}$ such that $(A_i^{K_i}, \exists_i)$ is an MMV-algebra with $\exists_i(A_i^{K_i}) = A_i$. Setting $\exists = \exists_1 \times ... \times \exists_m$ and acting pointwise, we obtain a monadic operator $\exists$ on A, that is, $(A, \exists)$ is an MMV-algebra. If a $K_i \in \Pi$ has at least two members, then determined the monadic operator will not be trivial, that is will not be the identity operator.

Proposition 9.[2] *Suppose that A is a finite MV-algebra, say* $A = S_{n_1} \times S_{n_2} \times ...S_{n_k}$.

(i) *For each homogeneous partition* $\{K_1, K_2, ..., K_m\}$ *of* $\{1, 2, ..., k\}$, *there is a monadic operator defined on A. Conversely, each monadic operator defined on A is determined by some homogeneous partition of* $\{K_1, K_2, ..., K_m\}$.

(ii) *If* $A = S_n^k$, *then any partition on* $\{1, 2, ..., k\}$ *determines a monadic operator on A and conversely, each monadic operator on A comes from some partition of* $\{1, 2, ..., k\}$.

5 Unification problem

Let $\mathbf{V}$ be a variety of algebras and $F_{\mathbf{V}}(m)$ m-generated free algebra over the variety $\mathbf{V}$. Recall that an algebra A of $\mathbf{V}$ is finitely presented if it is a quotient of the form $A = F_{\mathbf{V}}(m)/\theta$, with θ a finitely generated congruence. Following [4], by an algebraic unification problem we mean a finitely presented algebra A of $\mathbf{V}$. An algebraic unifier for A is a homomorphism $u : A \to P$ with P a m-generated projective algebra in $\mathbf{V}$ and A is algebraically unifiable if such an algebraic unifier exists. Given another algebraic unifier $w : A \to Q$, we say that u is more general than w, written $w \preceq u$, if there is a homomorphism $g : P \to Q$ such that $w = gu$. The algebraic unification type of an algebraically unifiable finitely presented algebra A in the variety $\mathbf{V}$ is now defined exactly as in the symbolic case, using the partially order $\leq$ induced by the quasi-order $\preceq$. Let $U_{\mathbf{V}}(P)$ be the set of unifiers $\sigma : F_{\mathbf{V}}(m) \to F_{\mathbf{V}}(m)$ for the unification problem $P(x_1, ..., x_m)$; it is a quasi-ordered set. The problem $P(x_1, ..., x_m)$ is solvable iff $U_{\mathbf{V}}(P) \neq \emptyset$. Let $(\Sigma, \leq)$ be a poset, where $\leq$ is the ordering induced by the quasi-ordering identifying the equivalence classes with its elements. $Max\Sigma$ is said to be *basis* of unifiers for P.

We say that an equational theory E has:

1. Unification type 1 iff for every solvable unification problem P, $Card(Max\Sigma) = 1$.
2. Unification type ω iff for every solvable unification problem P, $Card(Max\Sigma) = n \neq 1$, $n \in \omega$.
3. Unification type ∞ iff for every solvable unification problem P, $Card(Max\Sigma)$ is infinite.
4. Unification type nullary, if none of the preceding cases applies.

We say that $\mathbf{V}$ has finitary unification type iff it has type 1 or ω.

Theorem 10. *The unification type of the equational class* $\mathbf{K_n}$ *is* 1*, i. e. unitary.*

References

[1] C. C. Chang, *Algebraic Analysis of Many-Valued Logics*, Trans. Amer. Math. Soc., 88(1958), 467-490.

[2] L.P. Belluce, R. Grigolia and A. Lettieri, *Representations of monadic MV - algebras*, Studia Logica, vol. 81, Issue October 15th, 2005, pp. 125-144.

[3] A. Di Nola, R. Grigolia, *On Monadic MV-algebras*, APAL, Vol. 128, Issues 1-3 (August 2004), pp. 125-139.

[4] S. Ghilardi, *Unification through projectivity*, J. Logic Comput., 7(6),733-752, 1997.

[5] J.D. Rutledge, *A preliminary investigation of the infinitely many-valued predicate calculus*, Ph.D. Thesis, Cornell University, 1959.