Representation of the Medial-Like Algebras

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Abstract

In this paper we characterize the regular medial algebras, the paramedial n-ary groupoids with a regular element, the paramedial algebras with a regular element and the regular paramedial algebras. Also, we characterize paramedial, co-medial and co-paramedial pairs of quasigroup operations and paramedial, co-medial and co-paramedial algebras with the quasigroup operations.

1 Introduction

An algebra, $A = (A, F)$, (without nullary operations) is called medial (entropic, abelian) if it satisfies the identity of mediality:

$$\begin{aligned} g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm})) = \\ f(g(x_{11}, \dots, x_{1m}), \dots, g(x_{n1}, \dots, x_{nm})), \end{aligned} \tag{1}$$

for every n-ary $f \in F$ and m-ary $g \in F$ [5]. The n-ary operation, f, is called idempotent if $f(x, \dots, x) = x$, for every $x \in A$. The algebra $A = (A, F)$ is called idempotent, if every operation $f \in F$ is idempotent. An idempotent medial algebra is a mode [10].

Let g and f be m-ary and n-ary operations on the set, A. We say that the pair of operations, (f, g), is medial (entropic), if the identity (1) holds in the algebra, $A = (A, f, g)$.

We say that the pair of operations, (f, g), is paramedial (or paraentropic), if the following identity holds in the algebra, $A = (A, f, g)$:

$$\begin{aligned} g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm})) = \\ f(g(x_{nm}, \dots, x_{n1}), \dots, g(x_{1m}, \dots, x_{11})). \end{aligned}$$

An algebra, $A = (A, F)$, (without nullary operations) is called paramedial if every pair of operations, $f, g \in F$, (not necessarily distinct) is paramedial.

Paramedial groupoids and paramedial quasigroups were studied in [1, 9, 11].

Let g and f be n-ary operations on the set, A. We say that the pair of n-ary operations, (f, g), is co-medial, if the following identity holds in the algebra, $A = (A, f, g)$:

$$\begin{aligned} g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1n}, \dots, x_{nn})) = \\ g(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})). \end{aligned}$$

The pair of n-ary operations, (f, g), is co-paramedial, if the following identity holds in the algebra, (A, f, g):

$$\begin{aligned} g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1n}, \dots, x_{nn})) = \\ g(f(x_{nn}, \dots, x_{n1}), \dots, f(x_{1n}, \dots, x_{11})). \end{aligned}$$

An algebra, $A = (A, F)$, is called a co-medial (co-paramedial) algebra, if every pair of the operations, $f, g \in F$, with the same arity is co-medial (co-paramedial).

In other words, the algebra, A, is medial (paramedial, co-medial, or co-paramedial) if it satisfies the hyperidentity of mediality (paramediality, co-mediality, or co-paramediality) [7, 8, 6].

2 Main Results

Let $A = (A, F)$ be an algebra and $f \in F$. We say that the element e, is the unit for the operation $f \in F$, if: $f(x, e, \ldots, e) \approx f(e, x, e, \ldots, e) \approx \ldots \approx f(e, \ldots, e, x) \approx x$, for every $x \in A$. The element e, is a unit for the algebra (A, F), if it is a unit for every operation, $f \in F$. The element e, is idempotent for the operation f, if: $f(e, \ldots, e) = e$. We say that the element e, is idempotent for the algebra (A, F), if it is an idempotent for every operation $f \in F$.

Definition 2.1. Let (f, g) be a pair of m-ary and n-ary operations of the algebra, (A, F). For any element e of A, let $\alpha_1, \ldots, \alpha_m$ be mappings of A into A defined by

$$\alpha_i : x \mapsto f(e, \ldots, e, x, e, \ldots, e),$$

with x at the i-th place. We call α_i the *i-th translation by e with respect to f*. An element e is called *i-regular with respect to f* if α_i is a bijection. An element e, is called i-regular for the pair operation, (f, g), if it is an i-regular with respect to the both operations f and g. The element e, is called i-regular for the algebra (A, F), if it is an i-regular element for every operation $f \in F$.

The element e is called regular with respect to the n-ary operation $f \in F$, if e is an i-regular element with respect to f for every $1 \leq i \leq n$. The element e is a regular element of the algebra (A, F), if e is a regular element with respect to the every operation $f \in F$.

Theorem 2.2. Let (A, F) be a medial algebra with the idempotent element e which is i- and j-regular element of (A, F) for fixed i and j $(i \neq j)$, then there exists a commutative semigroup $(A, +)$ with the unit element e, such that every operation $f \in F$ has the following linear representation:

$$f(x_1, \ldots, x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m,$$

where $\gamma_1, \ldots, \gamma_m$, are pairwise commuting endomorphisms of $(A, +)$, $m \geq 2$. Furthermore, γ_i, γ_j are automorphisms.

Definition 2.3. Let f be an m-ary operation and J be a non-empty subset of $\{1, 2, \ldots, m\}$, we will say that the element e is *J-regular* with respect to the operation f, if e is a j-regular element with respect to f, for all $j \in J$. The element e is J-regular element for the algebra (A, F), if e is a j-regular element with respect to every $f \in F$, for all $j \in J$, where $m = min\{|g| \mid g \in F\}$, and $m \geq 2$.

Definition 2.4. Let $f, g \in F$ be m-ary and n-ary operations $(m \leq n)$, $J \subseteq \{1, 2, \ldots, m\}$ (where J contains at leas two elements) and $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m, \ldots, a_n$ are J-regular elements of the algebra (A, f, g) . The pair operation (f, g) is *(i, J)-regular pair operation* (where $i \in J$), if for every $x \in A$ we have the following equation:

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m) = g(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

The pair operation (f, g) is a *J-regular pair operation* if (f, g) is (i, J)-regular for every $i \in J$. The pair operation (f, g) is a *regular pair operation* if (f, g) is J-regular pair operation for some $J \subseteq \{1, 2, \ldots, m\}$ (where J contains at leas two elements).

An algebra (A, F) is called a *regular algebra* if every pair operation of (A, F) be a regular pair operation.

Theorem 2.5. Let (A, f, g) be a regular medial algebra with m-ary operation f and n-ary operation g $(m \leq n)$, then there is a commutative semigroup with an unit element $(A, +)$, such that

$$f(x_1, \ldots, x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d_1,$$
$$g(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + d_2,$$

where, d_1, d_2 are fixed regular elements in $(A, +)$ and $\gamma_1, \ldots, \gamma_m$, $\lambda_1, \ldots, \lambda_n$, are commuting automorphisms of the semigroup $(A, +)$.

Corollary 2.6. Let (A, F) be a regular medial algebra, then there exists a commutative semigroup $(A, +)$, such that every operation $f \in F$ has the following representation:

$$f(x_1, \ldots, x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d,$$

where d is a fixed regular element in $(A, +)$ and $\gamma_1, \ldots, \gamma_m$ are commuting automorphisms of the semigroup $(A, +)$.

Corollary 2.7.[4] If (Q, f) is a medial n-ary quasigroup, then there exists an abelian group, $(Q, +)$, such that

$$f(x_1, \ldots, x_m) = \alpha_1 x_1 + \cdots + \alpha_m x_m + d,$$

where $\alpha_i \in Aut(Q, +)$, are pairwise commute, and $d \in Q$.

There exist various algebraic characterizations of different classes of n-ary operations (see, for instance, [2]).

Theorem 2.8. Let (A, f) be a paramedial n-ary groupoid such that A contains an n-ary regular subgroupoid, then there is a commutative semigroup with unit element $(A, +)$, such that

$$f(x_1, \ldots, x_n) = \gamma_1 x_1 + \cdots + \gamma_n x_n + d,$$

where, d is a fixed regular element in $(A, +)$ and $\gamma_1, \ldots, \gamma_n$, are automorphisms of the semigroup $(A, +)$, $n \geq 2$. Moreover: $\gamma_i \gamma_j = \gamma_{n-j+1} \gamma_{n-i+1}$, for $n > 2$, $i, j = 1, \ldots, n$, and for $n = 2$ we have: $\gamma_1^2 = \gamma_2^2$.

Corollary 2.9. Let (Q, f) be a paramedial n-ary quasigroup, then there exists an abelian group $(Q, +)$, such that

$$f(x_1, \ldots, x_n) = \gamma_1 x_1 + \cdots + \gamma_n x_n + d,$$

where, d is a fixed element in $(Q, +)$ and $\gamma_1, \ldots, \gamma_n$, are automorphisms of the abelian group $(Q, +)$, $n \geq 2$. Moreover: $\gamma_i \gamma_j = \gamma_{n-j+1} \gamma_{n-i+1}$, for $n > 2$, $i, j = 1, \ldots, n$, and for $n = 2$ we have: $\gamma_1^2 = \gamma_2^2$.

Theorem 2.10. Let (A, F) be a paramedial algebra with the regular idempotent element e, then there exists a commutative semigroup $(A, +)$ with the unit element e, such that every operation $f \in F$ has the following linear representation

$$f(x_1, \ldots, x_m) = \alpha_1 x_1 + \cdots + \alpha_m x_m,$$

where $\alpha_1, \ldots, \alpha_m$, are pairwise commuting automorphisms of $(A, +)$, $m \geq 2$.

Theorem 2.11. Let (A, f, g) be a regular paramedial algebra with m-ary operation f and n-ary operation g, then there is a commutative semigroup with unit element $(A, +)$, such that

$$f(x_1, \ldots, x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d_1,$$
$$g(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + d_2,$$

where, d_1, d_2 are fixed regular elements in $(A, +)$ and $\gamma_1, \ldots, \gamma_m$, $\lambda_1, \ldots, \lambda_n$, are commuting automorphisms of the semigroup $(A, +)$.

Theorem 2.12. Let (Q, F) is a binary paramedial algebra with quasigroup operations, then there exists an abelian group $(Q, +)$, such that every operation, $f_i \in F$, is represented by the following rule:

$$f_i(x, y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where $c_i \in Q$ and $\varphi_i, \psi_i \in Aut(Q,+)$, such that: $\varphi_i\varphi_j = \psi_j\psi_i,\ \varphi_i\psi_j = \varphi_j\psi_i,\ \psi_i\varphi_j = \psi_j\varphi_i$. The group, $(Q,+)$, is unique up to isomorphisms.

Theorem 2.13. Let (Q,F) is a binary co-medial algebra with quasigroup operations, then there exists an abelian group $(Q,+)$, such that every operation, $f_i \in F$, is represented by the following rule:

$$f_i(x,y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where $c_i \in Q$ and $\varphi_i, \psi_i \in Aut(Q,+)$, such that $\varphi_i\psi_j = \psi_i\varphi_j$. The group, $(Q,+)$, is unique up to isomorphisms.

Theorem 2.14. Let (Q,F) is a binary co-paramedial algebra with quasigroup operations, then there exists an abelian group $(Q,+)$, such that every operation, $f_i \in F$, is represented by the following rule:

$$f_i(x,y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where $c_i \in Q$ and $\varphi_i, \psi_i \in Aut(Q,+)$, such that $\varphi_i\varphi_j = \psi_i\psi_j$. The group, $(Q,+)$, is unique up to isomorphisms.

Further description of the contents of this section are available in [3].

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