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A System for Evaluating the Admissibility of Rules for Intuitionistic Propositional Logic *

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Abstract

We give an evaluation system for the admissibility of rules in intuitionistic proportional logic. The system is based on recent work on the proof-theoretic semantics of IPL.

1 Introduction

Proof theory is an area of logic concerned with the study of constructions, called proofs, that certify the validity of formulae. In this paper, we restrict attention to the paradigm of *natural deduction* in the sense of Gentzen [43]. We assume general familiarity with this formalism, as covered in, for example, Troelstra and Schwichtenberg [44] and Negri [19].

A key concept within proof theory is the *admissibility* of rules. A rule is admissible for a logic when the logic is closed under that rule; that is, whenever the premisses are valid, the conclusion is also valid.

It is well-known that admissibility in intuitionistic propositional logic (IPL) is a subtle problem. Unlike in, for example, classical propositional logic (CPL), where all the admissible rules are derivable, there are rules that are admissible in IPL but not derivable. Perhaps the first such rule known is Harrop's Rule [13],

$$\frac{\neg\chi \to (\phi \vee \psi)}{(\neg\chi \to \phi) \vee (\neg\chi \to \psi)}$$

While several extensions of this rule have been studied — see, for example, Mints [18] and Citkin [2] — classifying all the admissible rules is subtle.

Friedman [6] asked if the problem is decidable; that is, is there a decision procedure such that given a rule it outputs 'yes' if the rule is admissible and 'no' otherwise. Rybakov [32] showed that the problem is indeed decidable, but also showed that there is no finite set of admissible rules that 'generates' all the admissible rules. Vissier [45] and de Jongh [3] provided

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a recursively enumerable set of rules that they conjectured would generate all the admissible rules for IPL. Following work by Ghilardi [10], Iemhoff [15] proved the conjecture correct.

As Schroeder-Heister [40] observe, the traditional approach to the correctness of an inference (i.e., an instance of a rule) concerns the material transmission of validity; that is, a rule is correct just in case the conclusion is valid whenever the premisses are valid. Hence, to address the question of an admissible rule in IPL, it suffices to have a characterization of validity for IPL that is amenable to evaluating such transmissions. In this paper, we consider the approach in which IPL is characterized by *valid* arguments.

Developed by a separate community in parallel to the question about the admissibility of rules, has been the problem of defining the *validity of proofs*. Beginning with remarks by Gentzen [43], Prawitz [28] initiated the study of such validity conditions by using his normalization results (see also Dummett [4]). This is now a central topic within the field of *proof-theoretic semantics* (P-tS) [41].

Since P-tS is broad subject, containing numerous subtle ideas, we restrict details in this paper to those pertinent to evaluating the admissibility of rules. In the Dummett-Prawitz tradition of P-tS, an essential theme is to justify the admissibility of the elimination rules through the introduction rules; for example, on the basis of

$$\frac{\begin{matrix}[\phi]\\ \psi\end{matrix}}{\phi \to \psi}\ \to_{\mathrm{I}}$$

as the introduction rule for implications ($\to$), one has the admissibility of the elimination rule for implication,

$$\frac{\phi \to \psi \quad \phi}{\psi}\ \to_{\mathrm{E}}$$

as follows: if there are proofs $\mathcal{D}_1$ and $\mathcal{D}_2$ witnessing $\phi \to \psi$ and ϕ, respectively, then (by $\to_{\mathrm{I}}$) one may compose them to yield a proof of ψ. Observe that this makes a central use of normalization as is thus intimately linked to the Curry-Howard Correspondence Theorem [14].

While P-tS concerns defining the validity of proofs, it also concerns defining the validity of formulae *in terms of* proofs. Recently, Sandqvist [35] has given such a P-tS for IPL — the details are given in Section 2 — it is called a *base-extension semantics* (B-eS) to distinguish it from the Dummett-Prawitz tradition discussed above. Recently, Gheorghiu and Pym [9] have shown that Sandqvist's B-eS [35] expresses the declarative content of the original notion of proof-theoretic validity by Prawitz [24] (see also Schroeder-Heister [38]).

Makinson [17] observes of P-tS/B-eS:

> Some readers may not be happy with the term 'semantics' for this construction, feeling that it — and perhaps any inferential semantics — is too syntactic in nature to deserve that title. They may prefer to use the neutral term 'evaluation system'; none of our formal results depend on the choice of terminology.

Indeed, the B-eS for IPL does provide an evaluation system for the logic, that is the interest in it for the present paper. The question of whether or not it is a 'semantics' is addressed elsewhere — see, for example, Dummett [4], Brandom [1], and Schroeder-Heister [41]. Note that admissibility of rules has been used by Piecha et al. [37, 21, 20, 23] to analyze the merits of various approaches to P-tS, particularly focusing on Harrop's Rule and Pierce's Law.

In this paper, we use the the completeness of valid arguments for IPL to characterize admissibility:

> a rule is admissible if whenever the premisses admit a valid argument, the conclusion admits a valid argument

Thus: using the B-eS for IPL as a characterization of the existence of a valid argument through the recent work by Gheorghiu and Pym [9], we immediately have an evaluation system. The resulting evaluation system is quite different from the aforementioned work by Iemhoff [15], and its relationship remains to be determined.

The paper begins in Section 2 with a summary of the required technical background on P-tS for this paper. The main problem (i.e., the admissibility of rules for IPL) is defined in Section 3, and we provide an evaluation system for admissibility (based on the P-tS for IPL). The paper concludes in Section 4 with a summary of contributions and discussion of future work.

We fix some notation conventions used throughout the paper. Throughout, fix a (denumerable) set of atomic propositions $\mathbb{ATOM}$ — the elements of which are called 'atoms,' 'propositional letters,' or 'propositional variables.' We take $\wedge$, $\vee$, $\to$, $\bot$ as the logical connectives of IPL. We use $A, B, C, \ldots$ to denote atoms and $\phi, \psi, \chi, \ldots$ to denote formulae; we may write $\neg\phi$ to abbreviate $\phi \to \bot$. We use $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \ldots$ to denote finite sets of atoms and $\Gamma, \Delta, \Sigma, \ldots$ to denote (possibly infinite) sets of (atomic and complex) formulae.

2 Background: Proof-theoretic Semantics for IPL

.

In *proof-theoretic semantics* (P-tS) [38, 5, 46], meaning and validity are characterized in terms of *proofs* — understood as objects denoting collections of acceptable inferences from accepted premises — and *provability*. To emphasize: it is not that one provides a proof system for the logic, but rather one explicates the meaning of the connectives *in terms of* proof systems. Indeed, as Schroeder-Heister [39] observes, since no formal system is fixed (only notions of inference), the relationship between semantics and provability remains the same as it has always been: soundness and completeness are desirable features of formal systems. Essentially, *proof* in P-tS plays the part of *truth* in traditional *model-theoretic semantics* (M-tS).

As a field, P-tS is wide and encompasses several distinct approaches. In this paper, we restrict attention to *base-extension semantics* (B-eS), which is a particular approach to P-tS. A B-eS is a characterization of a logic by judgment relation called *support* that is inductively defined according to the syntax of the logic. It is analogous to the *satisfaction* judgment in M-tS (cf. Kripke [16]). Crucially, the base case is given by 'derivability in a base' which is regarded as a *pre-logic* notion of proof; that is, bases are proof systems restricted to atoms, no logical constants permitted. Despite being structurally similar to M-tS, the subtle differences in the setup have significant consequences (discussed below).

In this paper, we follow the approach to B-eS by Sandqvist [33, 34, 35], Piecha et al. [21, 20, 23], and Gheorghiu et al. [7, 12]. While there are other versions than presented here (cf. Goldfarb [11] and Stafford and Nascimento [42]), they are not directly relevant for this work.

The B-eS for IPL begins by defining atomic rules. An *atomic rule* is a natural deduction rule of the following form, in which $p, p_1, \ldots, p_n$ are atoms and $\mathbb{P}_1,\ldots,\mathbb{P}_n$ are (possibly empty) sets of atoms:

$$\frac{}{p}\,\mathsf{a} \qquad \frac{\begin{matrix}[\mathbb{P}_1] \\ p_1\end{matrix} \quad \ldots \quad \begin{matrix}[\mathbb{P}_n] \\ p_n\end{matrix}}{p}\,\mathsf{r}$$

A *base* is a set of atomic rules. We write $\mathscr{B}, \mathscr{C}, \ldots$ to denote bases, and $\emptyset$ to denote the empty base (i.e., the base with no rules). We say $\mathscr{C}$ is an *extension* of $\mathscr{B}$ if $\mathscr{C}$ is a superset of $\mathscr{B}$, denoted $\mathscr{C} \supseteq \mathscr{B}$.

$\Vdash_{\mathscr{B}} p$	iff	$\vdash_{\mathscr{B}} p$	(At)
$\Vdash_{\mathscr{B}} \phi \to \psi$	iff	$\phi \Vdash_{\mathscr{B}} \psi$	($\to$)
$\Vdash_{\mathscr{B}} \phi \wedge \psi$	iff	$\Vdash_{\mathscr{B}} \phi$ and $\Vdash_{\mathscr{B}} \psi$	($\wedge$)
$\Vdash_{\mathscr{B}} \phi \vee \psi$	iff	$\forall \mathscr{C} \supseteq \mathscr{B} \, \forall p \in \mathbb{ATOM}$, if $\phi \Vdash_{\mathscr{C}} p$ and $\psi \Vdash_{\mathscr{C}} p$, then $\Vdash_{\mathscr{C}} p$	($\vee$)
$\Vdash_{\mathscr{B}} \bot$	iff	$\Vdash_{\mathscr{B}} p$ for any $p \in \mathbb{ATOM}$	($\bot$)
$\Gamma \Vdash_{\mathscr{B}} \phi$	iff	$\forall \mathscr{C} \supseteq \mathscr{B}$, if $\Vdash_{\mathscr{C}} \psi$ for any $\psi \in \Gamma$, then $\Vdash_{\mathscr{C}} \phi$	(Inf)
$\Gamma \Vdash \phi$	iff	$\Gamma \Vdash_{\mathscr{B}} \phi$ for all bases $\mathscr{B}$	

Figure 1: Base-extension Semantics for IPL

Bases are to be read as natural deduction systems except that atomic rules are taken *per se* and not closed under substitution when creating derivations:

Definition 1 (Derivability in a Base)**.** Derivability in a base $\mathscr{B}$ *is the smallest relation* $\vdash_{\mathscr{B}}$ *satisfying the following:*

- REF: $\mathbb{P} \vdash_{\mathscr{B}} p$
- APP$_1$: *If* $\mathsf{a} \in \mathscr{B}$*, then* $\mathbb{P} \vdash_{\mathscr{B}} c$.
- APP$_2$: *If* $\mathsf{r} \in \mathscr{B}$ *and* $\mathbb{P}, \mathbb{P}_1 \vdash_{\mathscr{B}} p_1, \ldots, \mathbb{P}, \mathbb{P}_n \vdash_{\mathscr{B}} p_n$*, then* $\mathbb{P} \vdash_{\mathscr{B}} c$.

Derivability in a base ($\vdash_{\mathscr{B}}$) is the base case of *support in a base* ($\Vdash_{\mathscr{B}}$) that gives the B-eS — that is, for any $p \in \mathbb{ATOM}$, $\Vdash_{\mathscr{B}} p$ iff $\vdash_{\mathscr{B}} p$. The inductive case is given by clauses, analogous to the definition of satisfaction in M-tS.

Definition 2 (Support for IPL)**.** *Support is the smallest relation* $\Vdash$ *satisfying the clauses of Figure 1.*

Lemma 3 (Sandqvist [36])**.** $\Gamma \vdash_{\mathrm{IPL}} \phi$ *iff* $\Gamma \Vdash \phi$.

Soundness — that is, $\Gamma \vdash_{\mathrm{IPL}} \phi$ implies $\Gamma \Vdash \phi$ — follows from showing that support admits all the rules of Gentzen's NJ [43]. Completeness — that is, $\Gamma \Vdash \phi$ implies $\Gamma \vdash_{\mathrm{IPL}} \phi$ — is more subtle. One constructs a special base $\mathscr{N}$ that 'simulates' Gentzen's NJ [43] and uses the clauses of the semantics to show that support in $\mathscr{N}$ is indeed equivalent to provability in NJ.

Remark 4. *The B-eS for IPL has the following clauses for disjunction ($\vee$):*

$$\Vdash_{\mathscr{B}} \phi \vee \psi \qquad \textit{iff} \qquad \forall \mathscr{C} \supseteq \mathscr{B} \, \forall p \in \mathbb{ATOM}, \textit{ if } \phi \Vdash_{\mathscr{C}} p \textit{ and } \psi \Vdash_{\mathscr{C}} p, \textit{ then } \Vdash_{\mathscr{C}} p$$

This is one significant consequence of the inferentialist setup: Piecha et al. [21, 20, 23] have shown that the standard meta-level 'or' clause — that is, $\Vdash_{\mathscr{B}} \phi \vee \psi$ *iff* $\Vdash_{\mathscr{B}} \phi$ *or* $\Vdash_{\mathscr{B}} \psi$ *— yields incompleteness. Observe the clause is closely related to the 'second-order' definition of disjunction (see Prawitz [28]) — that is,*

$$U + V = \forall X \, ((U \to X) \to (U \to X) \to X)$$

This adumbrates the categorical perspective on B-eS given by Pym et al. [31, 30] ◃

Remark 5. *The treatment of non-empty context is given by a* base-extension*,*

$$\Gamma \Vdash_{\mathscr{B}} \phi \qquad \textit{iff} \qquad \forall \mathscr{X} \supseteq \mathscr{B}, \textit{ if } \Vdash_{\mathscr{X}} \Gamma, \textit{ then } \Vdash_{\mathscr{X}} \phi$$

While prima facie *this use of base-extension appears to be related to pre-order in the standard possible-world semantics of IPL (cf. Kripke [16]), that remains to be determined.* ◃

Remark 6. *Piecha et al. [21, 20, 22, 23] has questioned the use of 'extension' in (Inf) and argue that the non-extension clause*

$$\Gamma \Vdash_{\mathscr{B}} \phi \qquad \text{iff} \qquad \text{if} \Vdash_{\mathscr{B}} \Gamma, \text{ then} \Vdash_{\mathscr{B}} \phi$$

perhaps better captures a 'definitional' view of atomic systems (as opposed to a 'knowledge' view). ◁

While the B-eS for IPL is intended as an inferential semantics (see, for example, Brandom [1] and Schroeder-Heister [41]), we reiterate the skepticism suggested by Makinson [17] (see Section 1): these 'inferential semantics' may simply be used as evaluation systems without any further philosophical discussion. This is the attitude we take in the present paper. We will use the B-eS of IPL to *evaluate* the admissibility of rules for IPL.

3 Evaluating the Admissibility of Rules

Having provided the technical background in Section 2, we now proceed to discuss how the B-eS for IPL by Sandqvist [35] may be used to evaluate the admissibility of rule for IPL. In Section 3.1, we setup the problem: What is a rule? What is admissibility? In Section 3.2 we provide an evaluation system for admissibility.

3.1 Rules, Substitution, and Admissibility

In this section, we make precise the notion of rule and admissibility — see also Iemhoff [15]. The discussion is terse as the ideas are doubtless familiar and we only require to fix a particular presentation of them.

Definition 7 (Rule)**.** *A rule is an expression of the form*

$$\frac{\phi_1 \quad \ldots \phi_n}{\phi}$$

We may write $\phi_1, \ldots \phi_n / \phi$ to denote the same rule.

Example 8 (Harrop's Rule)**.** *Harrop's Rule h may be presented as follows:*

$$\frac{\neg A \to (B \lor C)}{(\neg A \to B) \lor (\neg A \to B)}$$

It may be expressed inline as follows: $\neg A \to (B \lor C)/(\neg A \to B) \lor (\neg A \to B)$. The propositional letters A, B, and C may be replaced by any other pairwise different propositional variable A', B', C' without loss of generality. Any other replacement denotes a different rule. ◁

A rule is regarded as a general principle. An particular *instance* of a rule is given by uniformly replacing the propositional variables by formulas. To this end, we require the notion of substituion:

Definition 9 (Substitution)**.** *A substitution is a map $\sigma : \mathbb{ATOM} \to \mathbb{FORMULA}$. It extends to formulas inductively through the syntax:*

$$\sigma(\phi) := \begin{cases} \sigma(A) & \text{if } \phi = A \in \mathbb{ATOM} \\ \sigma(\phi_1) \circ \sigma(\phi_2) & \text{if } \phi = \phi_1 \circ \phi_2 \text{ for } \circ \in \{\land, \lor, \to\} \\ \bot & \text{if } \phi = \bot \end{cases}$$

Example 10 (Admissibility of Harrop's Law)**.** *Let σ be a substition such that $A \mapsto A'$, $B \mapsto B'$, and $C \mapsto C'$. The following is an instance of* h*:*

$$\frac{\neg A' \to (B' \vee C')}{(\neg A' \to B') \vee (\neg A' \to B')} \qquad \triangleleft$$

A rule is *admissible* for IPL iff any instance of it preserves validity for IPL:

Definition 11 (Admissible)**.** *A rule $\phi_1, \ldots \phi_n/\phi$ is admissible iff for all substitutions σ,*

$$\text{if} \vdash \sigma(\phi_i) \text{ for } i = 1, \ldots, n, \text{ then} \vdash \sigma(\phi)$$

Example 12 (Example 10 cont'd)**.** *Harrop [13] has shown that rule* h *is admissible.* ◁

Example 13. *The following rule* w *(i.e., weakening) is admissible (see, for example, Gentzen [43]):*

$$\frac{A \to B}{A \to (C \to B)} \qquad \triangleleft$$

This suffices for the background on admissibility. In the remainder of the paper, we characterize the admissibility through the proof-theoretic semantics of IPL.

3.2 Evaluation System

Since NJ characterizes IPL (see Gentzen [43]) we have the following:

> A rule $\phi_1, \ldots, \phi_n/\phi$ is admissible iff, for any substitution σ, if $\sigma(\phi_i)$ is NJ-provable for $i = 1 \ldots n$, then $\sigma(\phi)$ is NJ-provable.
>
> (AdmiProv)

On its own, this is neither a novel or insightful observation. However, coupled with P-tS, it becomes powerful: the point is that it only remains to characterize proofs for IPL and this is a central topic within P-tS — see, for example, Prawtiz [24, 25, 26, 27, 29] and Schroeder-Heister [38].

Given a base $\mathscr{B}$, Prawitz [24] provides a basic definition of a $\mathscr{B}$-valid argument. We elide the details as they do not matter. For present purposes, it suffices to regard a $\mathscr{B}$-valid argument as a $\mathsf{NJ} \cup \mathscr{B}$-derivation; in particular, a $\emptyset$-valid argument corresponds to a NJ-derivation.

Gheorghiu and Pym [9] have shown that the B-eS for IPL captures the basic notion of valid argument. Thus it characterizes provability for IPL.

Definition 14 (Entailment)**.** *Entailment is the smallest relation $\vDash$ satisfying the clauses of Figure 2.*

Remark 15. *Observe in Figure 2 that in the clause for $\to$, the definition shifts from $\vDash$ to $\Vdash$. This is an essential step for Lemma 16 below, as discussed by Gheorghiu and Pym [9] (cf. Piecha et al. [21, 20, 23]).* ◁

Lemma 16 (Gheorghiu and Pym [9])**.** *$\Gamma \vDash_{\mathscr{B}} \phi$ iff there is $\mathscr{B}$-valid argument from Γ to ϕ.*

Remark 17. *Observe that entailment (Figure 2) and support (Figure 1) differ in the clause governing disjunction ($\vee$): the former uses an 'introduction' clause and the latter uses an 'elimination' clause (in the sense of Gentzen's* NJ*). This is subtle and has been discussed by Gheorghiu et al. [9, 7, 8]. As observed in Remark 4, the elimination-form in support is essential for the completeness of support with respect to IPL, but the introduction-form in entailment delivers Lemma 16.* ◁

$\vDash_{\mathscr{B}} p$	iff	$\vdash_{\mathscr{B}} p$
$\vDash_{\mathscr{B}} \phi \wedge \psi$	iff	$\vDash_{\mathscr{B}} \phi$ and $\vDash_{\mathscr{B}} \psi$
$\vDash_{\mathscr{B}} \phi \vee \psi$	iff	$\vDash_{\mathscr{B}} \phi$ or $\vDash_{\mathscr{B}} \psi$
$\vDash_{\mathscr{B}} \phi \to \psi$	iff	$\phi \Vdash_{\mathscr{B}} \psi$
$\vDash_{\mathscr{B}} \bot$		never

Figure 2: Proof-theoretic Validity for IPL

It remains only to formally apply Lemma 16 to the problem of admissibility. We have the following for finite Γ,

$$\Gamma \mathrel{\vert\!\sim} \phi \qquad \text{iff} \qquad \text{if } \vDash_{\emptyset} \psi \text{ for } \psi \in \Gamma, \text{ then } \vDash_{\emptyset} \phi$$

The choice of $\mathrel{\vert\!\sim}$ for this symbol is a homage to Iemhoff [15] as it characterizes admissibility:

Theorem 18. *A rule $\phi_1, \ldots, \phi_n/\psi$ is admissible iff $\sigma(\phi_1), \ldots \sigma(\phi_n) \mathrel{\vert\!\sim} \sigma(\phi)$ for any sub. σ.*

Proof. Let $\phi_1, \ldots, \phi_n/\psi$ be an admissible rule. By Definition 11, for any substitution σ, if $\vdash \sigma(\phi_i)$ for $i = 1, \ldots n$, then $\vdash \sigma(\phi)$. By the soundness and completeness of Gentzen's NJ [43] with respect to IPL: for any formula χ,

$$\vdash \chi \qquad \text{iff} \qquad \text{there is an } \mathsf{NJ}\text{-proof of } \chi$$

We have thus establish AdmiProv. By Lemma 16,

$$\text{there is an } \mathsf{NJ}\text{-proof of } \chi \qquad \text{iff} \qquad \vDash_{\emptyset} \chi$$

Applying this to AdmiProv yields the desired result. ☐

The point of Theorem 18 is to connect admissibility with P-tS. However, the universal quantification over substitutions renders the theorem impractical for actually evaluating the admissibility of rule. Therefore, we use the following simplification that says it suffices to consider the form of the rule:

Corollary 19. *A rule $\phi_1, \ldots, \phi_n/\psi$ is admissible iff $\phi_1, \ldots, \phi_n \mathrel{\vert\!\sim} \psi$.*

Using Theorem 18 and Corollary 19, Figure 2 becomes an characterizes admissibility in IPL. To end this section, we now illustrates it by evaluating the admissibility of Harrop's Rule:

Example 20 (Admissibility of Harrop's Rule)**.** *By Corollary 19, we assume $\vDash_{\emptyset} \neg A \to (B \vee C)$ and require to show $\vDash_{\emptyset} (\neg A \to B) \vee (\neg A \to C)$. Using the clauses of Figure 2, it suffices to assume $\neg A \Vdash_{\emptyset} B \vee C$ and show $\neg A \Vdash_{\emptyset} B$ or $\neg A \Vdash_{\emptyset} C$. We will argue for the contra-positive:*

$$\text{if } \neg A \nVdash_{\emptyset} B \text{ and } \neg A \nVdash_{\emptyset} C, \text{ then } \neg A \nVdash_{\emptyset} B \vee C. \qquad (*)$$

To this end it suffices to find a base in which everything is derived if A is derived, but does not derive B, nor C, and does not support $B \vee C$.

Since (Inf) contains a universal quantifier over bases and $(*)$ is expressed in terms of the negation of the support judgement, we require to witness a base $\mathscr{X}$ such that $\Vdash_{\mathscr{X}} \neg A$, $\nVdash_{\mathscr{X}} B$, $\nVdash_{\mathscr{X}} C$, and $\nVdash_{\mathscr{X}} B \vee C$. Let $\mathscr{X}$ be the following rule for any $X \in \mathbb{ATOM}$:

$$\frac{A}{X} \qquad \frac{B}{D} \quad \frac{C}{D}$$

It is easy to check that $\Vdash_{\mathscr{X}} \neg A$, $\nVdash_{\mathscr{X}} B$ *and* $\nVdash_{\mathscr{X}} C$*; so, it remains only to check that* $\nVdash_{\mathscr{X}} B \lor C$*. By the* $\lor$*-clause in Figure 1, we require to show that there exists* $Y \in \mathbb{ATOM}$ *such that* $B \Vdash_{\mathscr{X}} Y$ *and* $B \Vdash_{\mathscr{X}} Y$ *but* $\nVdash_{\mathscr{X}} Y$*.*

Choose $Y = D$*. Clearly,* $\nVdash_{\mathscr{X}} D$*. Hence, it remains to verify* $B \Vdash_{\mathscr{X}} D$ *and* $B \Vdash_{\mathscr{X}} D$*. By (Inf) in Figure 1, for any* $\mathscr{Y}_1, \mathscr{Y}_2 \supseteq \mathscr{X}$*, if* $\Vdash_{\mathscr{Y}_1} B$ *and* $\Vdash_{\mathscr{Y}_2} C$*, then* $\Vdash_{\mathscr{Y}_1} D$ *and* $\Vdash_{\mathscr{Y}_2} D$*. This is immediate because of the rules in* $\mathscr{X}$*,*

$$\frac{B}{D} \quad \frac{C}{D}$$

That is, these rules are containing in both $\mathscr{Y}_1$ *and* $\mathscr{Y}_2$ *which also prove their premisses, respectively.* ◁

Remark 21. *Inspecting the evaluation in Example 20, we may wonder about the role of the negation in Harrop's Rule. For the argument to work, it is essential that the special base* $\mathscr{X}$ *does not, on its own, derive any atom. Hence, for example, the following rule is* not *admissible:*

$$\frac{A \to (B \lor C)}{(A \to B) \lor (A \to C)}$$

It is to satisfy the condition for $\bot$ *in the B-eS of IPL that we end up with the treatment of A in* $\mathscr{X}$ *— that is, as a premiss of a rule, rather than as an axiom. However, for the argument to go through, there's no need for A to be so prolific in* $\mathscr{X}$*. Accordingly,* mutatis mutandis *on Example 20, we see that the following rule is admissible:*

$$\frac{(A \to P) \to (B \lor C)}{((A \to P) \to B) \lor ((A \to P) \to C)}$$

This is Mint's Rule *[18]. It generalizes Harrop's Rule in the sense that instances of the latter are a subset of instances of the form in which* $P \mapsto \bot$*.* ◁

4 Conclusion

The admissibility of rules in intuitionistic logic is subtle. Unlike in classical logic, there's no finite system where all admissible rules are derivable, as shown by Rybakov [32]. However, Iemhoff [15] demonstrated that a recursively enumerable system (given by Vissier [45], de Jongh [3]) fully characterizes admissibility for IPL, building on the work of Ghilardi [10].

A rule is admissible for a logic if the conclusion is valid in the logic whenever (i.e., for any substitution instance) the premises are valid. One way to characterize validity for a logic is through proof theory; for example, Gentzen [43] gave a natural deduction system NJ that characterizes IPL. Hence, a proof-theoretic approach to the question of whether or not a given rule is admissible in IPL amounts to showing that there is a proof of the conclusion whenever there is a proof of the premises. Following work by Prawitz [24, 25, 26, 27] (see also Schroeder-Heister [38, 41]), the field of *proof-theoretic semantics* (P-tS) has emerged, in which one of the central problems is to characterize 'proof' for a logic.

Recent advances in proof-theoretic semantics, notably by Sandqvist [35] and Gheorghiu and Pym [9], offer an evaluation system for admissibility in IPL, building on the foundational work of Prawitz [24]. However, the precise relationship between this evaluation system and Iemhoff's calculus [15] remains an open question.

While the system is user-friendly, there's still a need to develop algorithmic methods for its application. The inherent non-determinism, particularly in the quantification of bases, prompts a call for systematic approaches to navigate these choices. This avenue remains a focus for future research, alongside investigations into the computational complexity of such procedures.

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