On Solé and Planat Criterion for the Riemann Hypothesis

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Abstract

There are several statements equivalent to the famous Riemann hypothesis. In 2011, Solé and Planat stated that the Riemann hypothesis is true if and only if the inequality $\zeta(2) \cdot \prod_{q \leq q_n} (1 + \frac{1}{q}) > e^{\gamma} \cdot \log \theta(q_n)$ holds for all prime numbers $q_n > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, $\zeta(x)$ is the Riemann zeta function and log is the natural logarithm. In this note, using Solé and Planat criterion, we prove that the Riemann hypothesis is true.

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1 Introduction

The Riemann hypothesis is the assertion that all non-trivial zeros have real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm. Leonhard Euler studied the following value of the Riemann zeta function (1734).

Proposition 1.1. *It is known that[1, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number (We also use the notation q_n to denote the nth prime number).

In mathematics, $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n).$$

Next, we have Solé and Planat Theorem:

Proposition 1.2. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers $q_n > 3$ if and only if the Riemann hypothesis is true [6, Theorem 4.2 pp. 5].*

A natural number N_k is called a primorial number of order k precisely when,

$$N_k = \prod_{i=1}^{k} q_i.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n) > \frac{e^{\gamma}}{\zeta(2)}$ is satisfied. There are several statements out from the Riemann hypothesis assumption:

Proposition 1.3. *We have [6, Proposition 3. pp. 3]:*

$$\lim_{k \to \infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 1.4. *Unconditionally on Riemann hypothesis, there are infinitely many primorial numbers N_k such that $R(N_k) > \frac{e^{\gamma}}{\zeta(2)}$ holds [6, Theorem 4.1 pp. 5].*

Proposition 1.5. *For $x \geq 2$, we have [2, Lemma 3.3. pp. 512]:*

$$\prod_{q \leq x} \left(1 - \frac{1}{q^2}\right) < \frac{1}{\zeta(2)} \cdot \left(1 + \frac{1}{x}\right).$$

Putting all together yields a proof for the Riemann hypothesis using the Chebyshev function.

2 What if the Riemann hypothesis were false?

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis might be false.

Lemma 2.1. *If the Riemann hypothesis is false, then there are infinitely many prime numbers q_n for which* $\mathsf{Dedekind}(q_n)$ *fails (i.e.* $\mathsf{Dedekind}(q_n)$ *does not hold).*

Proof. The Riemann hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound [6, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f was introduced in the Nicolas paper [5, Theorem 3 pp. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ [5, Theorem 3 (c) pp. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N} \ (y > y_0) \colon \ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N} \ (y > y_0) \colon \ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. Actually,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. $\square$

3 The Main Theorem

Theorem 3.1. *The Riemann hypothesis is true.*

Proof. We can show that for all sufficiently large prime numbers q_n if the inequality $R(N_n) > \frac{e^\gamma}{\zeta(2)}$ holds, then $R(N_{n+1}) > \frac{e^\gamma}{\zeta(2)}$ also holds. That result immediately implies that the inequality $\mathsf{Dedekind}(q_n)$ holds for all sufficiently large prime numbers q_n. The inequality $R(N_n) > \frac{e^\gamma}{\zeta(2)}$ is equal to

$$\frac{e^\gamma}{\zeta(2)} < \frac{\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)}{\log \theta(q_n)}.$$

That is the same as

$$e^\gamma \cdot \prod_{q \leq q_{n+1}} \left(1 - \frac{1}{q^2}\right) < \frac{\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)}{\log \theta(q_n)} \cdot \prod_{q > q_{n+1}} \left(\frac{q^2}{q^2 - 1}\right).$$

However, we know that

$$\prod_{q > q_{n+1}} \left(\frac{q^2}{q^2 - 1}\right) < \left(1 + \frac{1}{q_{n+1}}\right)$$

by Proposition 1.5. Certainly, that is equivalent to say that

$$e^\gamma \cdot \prod_{q \leq q_{n+1}} \left(1 - \frac{1}{q^2}\right) \cdot \frac{\log \theta(q_n)}{\log \theta(q_{n+1})} < \frac{\prod_{q \leq q_{n+1}} \left(1 + \frac{1}{q}\right)}{\log \theta(q_{n+1})}.$$

Hence, it is enough to show that

$$e^\gamma \cdot \prod_{q \leq q_{n+1}} \left(1 - \frac{1}{q^2}\right) \cdot \frac{\log \theta(q_n)}{\log \theta(q_{n+1})} \geq \frac{e^\gamma}{\zeta(2)}.$$

By Proposition 1.4, there must exist a large enough prime $q_{n'}$ such that $\mathsf{Dedekind}(q_{n'})$ holds and

$$\frac{\log \theta(q_{n'})}{\log \theta(q_{n+1})} \geq \alpha \cdot \prod_{q > q_{n+1}} \left(1 - \frac{1}{q^2}\right) \cdot \prod_{q_{n+1} \leq q \leq q_{n'}} \left(1 + \frac{1}{q}\right)$$

where $\alpha \gtrapprox 1$ tends to 1 as n grows by Proposition 1.3. Using the second Mertens' theorem, we obtain that

$$\begin{aligned}
&\log \log \theta(q_{n'}) - \log \log \theta(q_{n+1}) \\
&= B + \log \log \theta(q_{n'}) - B - \log \log \theta(q_{n+1}) \\
&\approx \sum_{q_{n+1} < q \leq q_{n'}} \frac{1}{q} \\
&\gtrapprox \log \left(\alpha \cdot \prod_{q > q_{n+1}} \left(1 - \frac{1}{q^2}\right)\right) + \sum_{q_{n+1} \leq q \leq q_{n'}} \log \left(1 + \frac{1}{q}\right)
\end{aligned}$$

where $B \approx 0.2614972128$ is the Meissel-Mertens constant [4, (17.) pp. 54] and the inequality $\frac{1}{q} > \log\left(1 + \frac{1}{q}\right)$ is satisfied for every prime q [3, pp. 1]. We would only need to prove that

$$e^{\gamma} \cdot \prod_{q \leq q_{n+1}} \left(1 - \frac{1}{q^2}\right) \cdot \frac{\log \theta(q_n)}{\log \theta(q_{n+1})} \geq \frac{\prod_{q \leq q_{n'}} \left(1 + \frac{1}{q}\right)}{\log \theta(q_{n'})}.$$

That would be

$$\frac{\log \theta(q_{n'})}{\log \theta(q_{n+1})} \geq \frac{q_{n+1}}{q_{n+1} - 1} \cdot \frac{\prod_{q \leq q_n} \left(\frac{q}{q-1}\right)}{e^{\gamma} \cdot \log \theta(q_n)} \cdot \prod_{q_{n+1} < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)$$

since

$$\left(1 + \frac{1}{q}\right) \cdot \left(\frac{q^2}{q^2 - 1}\right) = \left(\frac{q}{q-1}\right).$$

That is equivalent to

$$\frac{\log \theta(q_{n'})}{\log \theta(q_{n+1})} \geq \alpha \cdot \prod_{q > q_{n+1}} \left(1 - \frac{1}{q^2}\right) \cdot \prod_{q_{n+1} \leq q \leq q_{n'}} \left(1 + \frac{1}{q}\right)$$

which is trivially true under our assumptions. Consequently, if the inequality $\mathsf{Dedekind}(q_n)$ holds for all sufficiently large prime numbers q_n, then there won't exist infinitely many prime numbers q_n such that $\mathsf{Dedekind}(q_n)$ fails and so, the Riemann hypothesis must be true by Lemma 2.1. □

4 Conclusions

The Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, etc. In general, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics.

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