On Gottschalk's Surjunctivity Conjecture for Non-Uniform Cellular Automata Preprint Version Information

Xuan Kien Phung

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Xuan Kien Phung[1,2][0000−0002−4347−8931]

[1] Département de Mathématiques et de Statistique, Université de Montréal, Montréal, Québec, H3T 1J4, Canada
[2] Département d'informatique et de recherche opérationnelle, Université de Montréal, Montréal, Québec, H3T 1J4, Canada
phungxuankien1@gmail.com
xuan.kien.phung@umontreal.ca

Abstract. Gottschalk's surjunctivity conjecture for a group G states that it is impossible for cellular automata (CA) over the universe G with finite alphabet to produce strict embeddings of the full shift into itself. A group universe G satisfying Gottschalk's surjunctivity conjecture is called a surjunctive group. The surjunctivity theorem of Gromov and Weiss shows that every sofic group is surjunctive. In this paper, we study the surjunctivity of local perturbations of CA and more generally of non-uniform cellular automata (NUCA) with finite memory and uniformly bounded singularity over surjunctive group universes. In particular, we show that such a NUCA must be invertible whenever it is reversible. We also obtain similar results which extend to the class of NUCA a certain dual-surjunctivity theorem of Capobianco, Kari, and Taati for CA.

Keywords: Gottschalk's conjecture · surjunctivity · sofic group· cellular automata · non-uniform cellular automata · asynchronous cellular automata · reversibility

1 Introduction

In computational science and engineering, cellular automata (CA), especially reversible CA, arise as a fundamental and powerful model of simulation for various physical and biological systems [38] whose global evolution is described by spatially uniform local transition rules. In computer science, it is well-known that the CA called Game of Life of Conway [14] is Turing complete. Some notable recent mathematical theory of CA achieves a dynamical characterization of amenable groups by the Garden of Eden theorem for CA (pre-injectivity $\iff$ surjectivity, see Section 1.1 below) [22], [23], [10], [1] and establishes the equivalence between Kaplansky's stable finiteness conjecture [18] for group rings and Gottschalk's surjunctivity conjecture (injectivity $\implies$ surjectivity) [16] for linear CA [30], [34], [31], [9]. In this paper, we first explore Gottschalk's surjunctivity conjecture (see Section 1.2) over surjunctive group universes (see Definition 2), e.g. sofic groups such as amenable or residually finite groups, for non-uniform CA (NUCA) with finite memory. We then obtain an extension of a well-known dual version of Gottschalk's surjunctivity conjecture due to Capobianco, Kari, and Taati [6] (see Section 1.2) over the class of post-injunctive groups (see Definition 3), e.g. sofic groups, also for NUCA with finite memory. Our main results cover NUCA which are local perturbations of CA in which a finite number of cells can follow local transition rules different from the local transition rule of the underlying CA. Among applications, such NUCA have close connections with asynchronous CA. For example, if F is an asynchronous CA in which only a finite number of cells are allowed to update at each (discrete) time, e.g. fully asynchronous CA [13] and skew-asynchronous CA [36], then F can be identified with a sequence $(F_n)_{n \in \mathbb{N}}$ of local perturbations of the identity CA: the system F at time $n \in \mathbb{N}$ is exactly the NUCA $F_n \circ F_{n-1} \circ \cdots \circ F_0$.

We generalize our results to a certain class of global perturbations of CA with uniformly bounded singularity. Essentially by definition and basic properties of CA, every reversible CA with finite memory over a surjunctive group universe must be invertible (see Section 1.1 for the exact definitions). Gottschalk's surjunctivity conjecture amounts to say that every group is surjunctive. Hence, our results extend Gottschalk's surjunctivity conjecture by showing that over surjunctive group universes, reversible NUCA with finite memory and uniformly bounded singularity must be invertible (see Theorem A and Theorem C).

1.1 Basic definitions

We recall notions of symbolic dynamics. Given a discrete set A and a group G, a *configuration* $x \in A^G$ is a map $x \colon G \to A$. Two configurations $x, y \in A^G$ are *asymptotic* if $x|_{G \setminus E} = y|_{G \setminus E}$ for some finite subset $E \subset G$. The *Bernoulli shift* action $G \times A^G \to A^G$ is defined by $(g, x) \mapsto gx$, where $(gx)(h) = x(g^{-1}h)$ for $g, h \in G$, $x \in A^G$. The *full shift* A^G is equipped with the prodiscrete topology. For each $x \in A^G$, we define $\Sigma(x) = \overline{\{gx \colon g \in G\}} \subset A^G$ as the smallest closed subshift containing x.

Following von Neumann and Ulam [24], we define a CA over the group G (the *universe*) and the set A (the *alphabet*) as a G-equivariant and uniformly continuous self-map $A^G \circlearrowleft$ [7], [17]. One usually refers to each element $g \in G$ as a cell of the universe. Then every CA is uniform in the sense that all the cells follow the same local transition map. More generally, when different cells can evolve according to different local transition maps to break down the above uniformity, we obtain NUCA [11], [12], [29, Definition 1.1]:

Definition 1. *Given a group G and an alphabet A, let $M \subset G$ and let $S = A^{A^M}$ be the set of all maps $A^M \to A$. For $s \in S^G$, we define the NUCA $\sigma_s \colon A^G \to A^G$ by $\sigma_s(x)(g) = s(g)((g^{-1}x)|_M)$ for all $x \in A^G$ and $g \in G$.*

The set M is called a *memory* and $s \in S^G$ the *configuration of local transition maps or local defining maps* of σ_s. Every CA is thus a NUCA with finite memory and constant configuration of local defining maps. Conversely, we regard NUCA as *global perturbations* of CA. When $s, t \in S^G$ are asymptotic, σ_s and

σ_t are mutually *local perturbations* of each other. We also observe that NUCA with finite memory are precisely uniformly continuous selfmaps $A^G \circlearrowleft$. As for CA, such NUCA satisfy the closed image property [29, Theorem 4.4], several decidable and undecidable properties [11], [19], [35] and variants of the Garden of Eden theorem [25], [32].

We will analyze in this paper the relations between the following dynamic properties of σ_s. We say that σ_s is *pre-injective* if $\sigma_s(x) = \sigma_s(y)$ implies $x = y$ whenever $x, y \in A^G$ are asymptotic. Similarly, σ_s is *post-surjective* if for all $x, y \in A^G$ with y asymptotic to $\sigma_s(x)$, then $y = \sigma_s(z)$ for some $z \in A^G$ asymptotic to x. By the closed image property, it is known that post-surjectivity implies surjectivity for NUCA with finite memory. We say that σ_s is *stably injective*, resp. *stably post-surjective*, if σ_p is injective, resp. post-surjective, for every $p \in \Sigma(s)$.

The NUCA σ_s is said to be *reversible* or *left-invertible* if there exists a NUCA with finite memory $\tau\colon A^G \to A^G$ such that $\tau \circ \sigma_s = \mathrm{Id}_{A^G}$. A NUCA with finite memory is reversible if and only if it is stably injective (see [29, Theorem A]). We say that σ_s is *invertible* if it is bijective and the inverse map σ_s^{-1} is a NUCA with finite memory [29]. We define also *stable invertibility* which is in general stronger than invertibility, namely, σ_s is stably invertible if there exist $N \subset G$ finite and $t \in T^G$ where $T = A^{A^N}$ such that for every $p \in \Sigma(s)$, we have $\sigma_p \circ \sigma_q = \sigma_q \circ \sigma_p = \mathrm{Id}$ for some $q \in \Sigma(t)$. In fact, we will show (see Lemma 1) that every invertible NUCA with finite memory over a finite alphabet and a countable group universe is automatically stably invertible.

Remark 1. For CA, note that stable invertibility, resp. stable injectivity, resp. stable post-surjectivity, is equivalent to invertibility, resp. injectivity, resp. post-surjectivity since $\Sigma(s) = \{s\}$ if the configuration $s \in S^G$ is constant.

1.2 Main results

Gottschalk's surjunctivity conjecture [16] asserts that over any group universe, every CA with finite alphabet must be *surjunctive* (injectivity $\Longrightarrow$ surjectivity). In other words, it is impossible for CA to produce strict embeddings of the full shift into itself.

Definition 2. *A group G is said to be surjunctive if for every finite alphabet A, every injective CA $\tau\colon A^G \to A^G$ must be surjective.*

Every CA with finite memory is injective if and only if it is reversible. Therefore, a group G is surjunctive if and only if for every finite alphabet A, every reversible CA $\tau\colon A^G \to A^G$ must be invertible. Over the wide class of sofic group universes, the surjunctivity conjecture was famously shown by Gromov and Weiss in [15], [37].

Theorem 1 (Gromov-Weiss). *Every sofic group is surjunctive.*

The class of sofic groups was first introduced by Gromov [15] and includes all amenable groups and all residually finite groups. The question of whether there exists a non-sofic group is still open.

The situation for the surjunctivity of NUCA admits some complications. While injective and even reversible NUCA with finite memory may fail to be surjective (see [29, Example 14.3]), results in [29] show that reversible local perturbations of CA with finite memory over an amenable group or an residually finite group universe must be surjective. More generally, we obtain an extension (Theorem A) to cover all reversible, or equivalently stably injective, local perturbations of CA over surjunctive group universes. We also strengthen the conclusion by showing that such NUCA must be stably invertible (see Section 4).

Theorem A *Let M be a finite subset of a countable surjunctive group G. Let A be a finite alphabet and $S = A^{A^M}$. Suppose that σ_s is stably injective for some asymptotically constant $s \in S^G$. Then σ_s is stably invertible.*

Combining with Theorem 1 and the result in [29, Theorem B] where we can replace the stable injectivity by the weaker injectivity condition whenever G is an amenable group, we obtain the following general surjunctivity and invertibility result for NUCA which are local perturbations of CA.

Corollary 1. *Let M be a finite subset of a countable group G. Let A be a finite alphabet and $S = A^{A^M}$. Let $s \in S^G$ be an asymptotic constant configuration. Then σ_s is stably invertible in each of the following cases:*

(i) *G is an amenable group and σ_s is injective;*
(ii) *G is a sofic group and σ_s is reversible.*

Our next results concern a certain dual-surjunctivity version of Gottschalk's conjecture introduced by Capobianco, Kari, and Taati in [6] which states that every post-surjective CA over a group universe and a finite alphabet is also pre-injective. The authors settled in the same paper [6] the case of sofic group universes. See also [28] for some extensions.

Theorem 2 (Capobianco-Kari-Taati). *Let G be a sofic group and let A be a finite alphabet. Then every post-surjective CA $\tau \colon A^G \to A^G$ is pre-injective.*

The above result motivates the following notion of *post-injunctive* groups.

Definition 3. *A group G is post-injunctive if for every finite alphabet A, every post-surjective CA $\tau \colon A^G \to A^G$ must be pre-injective.*

By a similar technique as in the proof of Theorem A, we establish (see Section 5) the following extension of the above result of Capobianco, Kari, and Taati to cover the class of stably post-surjective local perturbations of CA over post-injunctive group universes.

Theorem B *Let G be a countable post-injunctive group and let A be a finite alphabet. Let $M \subset G$ be finite and $S = A^{A^M}$. Let $s \in S^G$ be asymptotically constant such that σ_s is stably post-surjective. Then σ_s is stably invertible.*

Our main results also hold for global perturbations of CA with *uniformly bounded singularity.* A NUCA $\sigma_s \colon A^G \to A^G$ has uniformly bounded singularity if for every finite subset $E \subset G$ with $1_G \in E = E^{-1}$, we can find a finite subset $F \subset G$ containing E such that the restriction $s|_{FE \setminus F}$ is constant. It is clear that σ_s has uniformly bounded singularity if s is asymptotically constant. When the universe G is a residually finite group, our notion of uniformly bounded singularity is closely related but not equivalent to the notion of (periodically) bounded singularity for NUCA defined in [29, Definition 10.1]. We obtain the following generalizations of Theorem A and Theorem B (see Section 6 and Section 7).

Theorem C *Let M be a finite subset of a countable surjunctive group G. Let A be a finite alphabet and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably injective for some $s \in S^G$ with uniformly bounded singularity. Then σ_s is stably invertible.*

Theorem D *Let G be a finitely generated post-injunctive group and let A be a finite alphabet. Let $M \subset G$ be finite and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably post-surjective for some $s \in S^G$ with uniformly bounded singularity. Then σ_s is stably invertible.*

1.3 Perspectives

Our main results motivate the following natural questions which have an affirmative answer for surjunctive group universes and post-injunctive group universes respectively.

Question 1. Let G be a group universe and let A be a finite alphabet. Let $M \subset G$ be finite and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably injective for some asymptotically constant $s \in S^G$. Is σ_s stably invertible?

Question 2. Let G be a group universe and let A be a finite alphabet. Let $M \subset G$ be finite and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably post-surjective for some asymptotically constant $s \in S^G$. Is σ_s stably invertible?

We can actually show (see Proposition 1 below) that Question 1 and Question 2 are equivalent when restricted to the class of linear NUCA with finite memory. Given a vector space V not necessarily finite, then V^G is a vector space with component-wise operations and a NUCA $\tau \colon V^G \to V^G$ is said to be *linear* if it is also a linear map of vector spaces or equivalently, if its local transition maps are linear.

Proposition 1. *Let G be a countable group and let A be a finite vector space alphabet. Question 1 has an affirmative answer for all linear* NUCA $\tau \colon A^G \to A^G$ *with finite memory and uniformly bounded singularity if and only if so does Question 2.*

Proof. Let $M \subset G$ be a finite subset and let $s \in S^G$ where $S = \mathcal{L}(A^M, A)$. By linearity, we define $s(g,m) \in \text{End}(A)$ for all $m \in M$ and $g \in G$ by $s(g)(v) = \sum_{m \in M} s(g,m)v(m)$ for all $v \in A^M$. By setting $s(g,m) = 0$ for $m \in G \backslash M$, we obtain for every $v \in A^G$ that $s(g)(v) = \sum_{h \in G} s(g,h)v(h)$. Let $T = \mathcal{L}(A^{*M^{-1}}, A^*)$ where A^* is the dual space of A. We use the right superscript $^\mathsf{T}$ to denote the transpose of linear maps. The dual configuration of local defining maps $s^* \in T^G$ is given by $s^*(g,m) := s(gm, m^{-1})^\mathsf{T}$ for all $g, m \in G$. We define $\sigma_s^* := \sigma_{s^*}$ to be the dual linear NUCA of σ_s. It is immediate from the definition that s has uniformly bounded singularity if and only if so does s^*. By [33, Lemma 5.2], note that $s^{**} = s$ and $\sigma_s^{**} = \sigma_s$ for all $s \in S^G$. We infer from the main results in [33] that

(i) σ_s is invertible $\Longleftrightarrow \sigma_{s^*}$ is invertible,
(ii) σ_s is stably injective $\Longleftrightarrow \sigma_{s^*}$ is stably post surjective,
(iii) σ_s is stably post-surjective $\Longleftrightarrow \sigma_{s^*}$ is stably injective.

Note that every invertible NUCA with finite memory over a finite alphabet and a countable group universe is automatically stably invertible (see Lemma 1). By identifying A with its dual vector space A^*, we can now conclude that Question 1 has a positive answer for all linear NUCA $A^G \to A^G$ of finite memory with uniformly bounded singularity if and only if so does Question 2. □

Linear NUCA with finite memory enjoy similar properties as linear CA such as the shadowing property [2], [4], [20], [3], [21], [5], [27], [26], [8], [33]. The recent result [33, Theorem D.(ii)] states that every stably post-surjective linear NUCA with finite vector space alphabet over a residually finite group universe is invertible whenever it is a local perturbation of a linear CA. When specialized to the class of linear NUCA, Theorem B thus generalizes [33, Theorem D.(ii)] because every residually finite group is sofic and thus post-injunctive by Theorem 2. To this end, we postulate another seemingly interesting question.

Question 3. Does an affirmative answer to Question 1 imply an affirmative answer to Question 2? Conversely, does an affirmative answer to Question 2 imply an affirmative answer to Question 1?

It is also natural to explore similar questions for the more general class of NUCA with uniformly bounded singularity.

1.4 Organization of the paper

In Section 2, we prove that the notions of invertibility and stable invertibility are equivalent for NUCA with finite memory (Lemma 1). As a consequence, we relate the surjectivity, the invertibility, and the stable invertibility of a stably injective NUCA in Corollary 2. As another application, we show that pre-injective stably post-surjective NUCA with finite memory must be stably invertible. We then fix in Section 3 the notations and recall the construction of induced local maps of NUCA in Section 3. The proofs of Theorem A, Theorem B, Theorem C, and Theorem D are given respectively in the subsequent Sections 4, 5, 6, and 7.

2 Invertibility vs stable invertibility

Lemma 1. *Let M be a finite subset of a countable group G. Let A be a finite alphabet and let $s \in S^G$ where $S = A^{A^M}$. Suppose that σ_s is invertible. Then σ_s is stably invertible.*

Proof. Since σ_s is invertible, there exist $N \subset G$ finite and $t \in T^G$ where $T = A^{A^N}$ such that $\sigma_s \circ \sigma_t = \sigma_t \circ \sigma_s = \mathrm{Id}_{A^G}$. For every $p \in \Sigma(s)$, we obtain from the relation $\sigma_s \circ \sigma_t = \mathrm{Id}_{A^G}$ and [29, Theorem 11.1] some $q \in \Sigma(t)$ such that $\sigma_p \circ \sigma_q = \mathrm{Id}_{A^G}$. Since $q \in \Sigma(t)$ and $\sigma_t \circ \sigma_s = \mathrm{Id}_{A^G}$, another application of [29, Theorem 11.1] shows that there exists $r \in \Sigma(s)$ such that $\sigma_q \circ \sigma_r = \mathrm{Id}_{A^G}$. It follows that σ_q is invertible and thus $\sigma_p \circ \sigma_q = \sigma_q \circ \sigma_p = \mathrm{Id}_{A^G}$. Hence, σ_s is stably invertible. □

Corollary 2. *Let G be a countable group and let A be a finite alphabet. Let $\tau \colon A^G \to A^G$ be a stably injective NUCA with finite memory. Then the following are equivalent:*

(i) *τ is surjective,*
(ii) *τ is invertible,*
(iii) *τ is stably invertible.*

Proof. It is trivial that (iii) $\Longrightarrow$ (i). The implication (i) $\Longrightarrow$ (ii) results from the definition of invertibility and [29, Theorem A]. Lemma 1 states that (ii) $\Longrightarrow$ (iii) and the proof is thus complete. □

Corollary 3. *Let G be a countable group and let A be a finite alphabet. Let $\tau \colon A^G \to A^G$ be a pre-injective stably post-surjective NUCA with finite memory. Then τ is stably invertible.*

Proof. The fact that τ is invertible follows from [29, Theorem 13.4] (see also [6]). We can thus conclude from Lemma 1 that τ is stably invertible. □

3 Induced local maps of NUCA

Let G be a group and let A be an alphabet. For every subset $E \subset G$ and $x \in A^E$ we define $gx \in A^{gE}$ by $gx(gh) = x(h)$ for all $h \in E$. In particular, $gA^E = A^{gE}$. Let $M \subset G$ and let $S = A^{A^M}$ be the collection of all maps $A^M \to A$. For every finite subset $E \subset G$ and $w \in S^E$, we define a map $f_{E,w}^{+M} \colon A^{EM} \to A^E$ as follows. For every $x \in A^{EM}$ and $g \in E$, we set:

$$f_{E,w}^{+M}(x)(g) = w(g)((g^{-1}x)|_M). \tag{3.1}$$

In the above formula, note that $g^{-1}x \in A^{g^{-1}EM}$ and $M \subset g^{-1}EM$ since $1_G \in g^{-1}E$ for $g \in E$. Therefore, the map $f_{E,w}^{+M} \colon A^{EM} \to A^E$ is well defined. Consequently, for every $s \in S^G$, we have a well-defined induced local map $f_{E,s|_E}^{+M} \colon A^{EM} \to A^E$ for every finite subset $E \subset G$ which satisfies:

$$\sigma_s(x)(g) = f_{E,s|_E}^{+M}(x|_{EM})(g) \tag{3.2}$$

for every $x \in A^G$ and $g \in E$. Equivalently, we have for all $x \in A^G$ that:

$$\sigma_s(x)|_E = f^{+M}_{E,s|_E}(x|_{EM}). \tag{3.3}$$

For every $g \in G$, we have a canonical bijection $\gamma_g \colon G \mapsto G$ induced by the translation $a \mapsto g^{-1}a$. For each subset $K \subset G$, we denote by $\gamma_{g,K} \colon gK \to K$ the restriction to gK of γ_g. Now let $N \subset G$ and $T = A^{A^N}$. Let $t \in T^G$. With the above notations, we have the following auxiliary lemma.

Lemma 2. *Suppose that* $1_G \in M \cap N$. *Then for every* $g \in G$, *the condition* $\sigma_t(\sigma_s(x))(g) = x(g)$ *for all* $x \in A^G$ *is equivalent to the condition*

$$t(g) \circ \gamma_{g,N} \circ f^{+M}_{gN,s|_{gN}} \circ \gamma^{-1}_{g,NM} = \pi,$$

where $\pi \colon A^{NM} \to A$ *is the projection* $z \mapsto z(1_G)$.

Proof. For every $g \in G$ and $x \in A^G$, we deduce from Definition 1 and the relation (3.3) that

$$\begin{aligned}
\sigma_t(\sigma_s(x))(g) &= t(g)\left((g^{-1}\sigma_s(x))|_N\right)\\
&= t(g)\left(\gamma_{g,N}\left((\sigma_s(x))|_{gN}\right)\right)\\
&= t(g)\left(\gamma_{g,N} \circ f^{+M}_{gN,s|_{gN}}(x|_{gNM})\right)\\
&= t(g) \circ \gamma_{g,N} \circ f^{+M}_{gN,s|_{gN}}(x|_{gNM})\\
&= t(g) \circ \gamma_{g,N} \circ f^{+M}_{gN,s|_{gN}} \circ \gamma^{-1}_{g,NM}\left((g^{-1}x)|_{NM}\right)
\end{aligned}$$

from which the conclusion follows as $x(g) = (g^{-1}x)(1_G)$. □

4 Proof of Theorem A

Proof. The theorem is trivial if G is finite since every injective selfmap of a finite set is also surjective. Thus, without loss of generality we can suppose that G is infinite. Up to enlarging M if necessary, we can also assume that $1_G \in M$. As $s \in S^G$ is asymptotically constant, we can find a constant configuration $c \in S^G$ and a finite subset $F \subset G$ such that $s|_{G \setminus F} = c|_{G \setminus F}$. Note that as G is infinite, we have $c \in \Sigma(s)$. Since σ_s is stably injective, we deduce that $\sigma_c \colon A^G \to A^G$ is an injective CA. We infer from the surjunctivity of the group G that σ_c is also surjective. By Corollary 2 and Remark 1, it follows that σ_c is invertible. Therefore, there exist a nonempty finite subset $N \subset G$ and a constant configuration $d \in T^G$, where $T = A^{A^N}$, such that $1_G \in N$ and

$$\sigma_c \circ \sigma_d = \sigma_d \circ \sigma_c = \mathrm{Id}_{A^G}\,. \tag{4.1}$$

The set of NUCA with finite memory over the universe G and the alphabet A forms a monoid with respect to the composition operation and we obtain from [29, Theorem 6.2] a configuration $q \in Q^G$, where $Q = A^{A^{MN}}$, such that

$\sigma_s \circ \sigma_d = \sigma_q$. Note that MN is a memory set of σ_q. As both σ_s and σ_d are injective, $\sigma_q = \sigma_s \circ \sigma_d$ is also injective. Since $\sigma_c \circ \sigma_d = \mathrm{Id}_{A^G}$ and $\sigma_s \circ \sigma_d = \sigma_q$ and s is asymptotic to c, we deduce that the configuration $q \in Q^G$ is asymptotic to the constant configuration π^G where $\pi \colon A^{MN} \to A^{\{1_G\}}$ is the canonical projection $x \mapsto x(1_G)$. More precisely, $q|_{G\backslash F} = \pi^{G\backslash F}$ since $s|_{G\backslash F} = c|_{G\backslash F}$ so that for every $g \in G\backslash F$ (see (3) and the proof of [29, Theorem 6.2]):

$$q(g) = s(g) \circ f^{+N}_{M, g^{-1}d|_M} = c(g) \circ f^{+N}_{M, g^{-1}d|_M} = \pi$$

where the last equality results from (4.1).

Let $E = FMN \subset G$ then E is finite and $F \subset E$ as $1 \in M \cap N$. It follows that $q|_{G\backslash E} = \pi^{G\backslash E}$. Consider the map $\Phi \colon A^E \to A^E$ induced by the restriction of σ_p to A^E. More specifically, for every $x \in A^E$, we define

$$\Phi(x) = \sigma_q(y)|_E$$

for any configuration $y \in A^G$ extending x, that is, $y|_E = x$. To check that Φ is well-defined, let $z \in A^G$ be another configuration such that $z|_E = x$. Let $g \in E\backslash F$ then we have $q(g) = \pi$ and thus

$$\begin{aligned}\sigma_q(z)(g) &= q(g)((g^{-1}z)|_{MN}) = \pi((g^{-1}z)|_{MN}) \\ &= z(g) = x(g) = y(g) \\ &= \pi((g^{-1}y)|_{MN}) = q(g)((g^{-1}y)|_{MN}) \\ &= \sigma_q(y)(g).\end{aligned}$$

Now let $g \in F$. Then $(g^{-1}z)|_{MN} = (g^{-1}y)|_{MN}$ since $(g^{-1}z)(h) = z(gh) = x(gh) = y(gh) = (g^{-1}y)(h)$ for all $h \in MN$. Therefore,

$$\sigma_q(z)(g) = q(g)((g^{-1}z)|_{MN}) = q(g)((g^{-1}z)|_{MN}) = q(g)((g^{-1}u)|_{MN}) = \sigma_q(z)(g).$$

We conclude that $\sigma_q(z)|_E = \sigma_q(y)|_E$ and thus Φ is well-defined. Observe that

$$\sigma_q = \Phi \times \mathrm{Id}_{A^{G\backslash E}}.$$

Since σ_q and $\mathrm{Id}_{A^{G\backslash E}}$ are injective, we deduce that Φ is also injective. Consequently, Φ must be surjective as A^E is finite. It follows that $\sigma_q = \Phi \times \mathrm{Id}_{A^{G\backslash E}}$ is surjective. Combining with (4.1) and the surjectivity of σ_c, we find that $\sigma_s = \sigma_q \circ (\sigma_d)^{-1} = \sigma_q \circ \sigma_c$ is also surjective. Since σ_s is stably injective by hypothesis, we can thus conclude from Corollary 2 that σ_s is stably invertible and the proof is complete. □

5 Proof of Theorem B

Proof. The proof follows the same lines, *mutatis mutandis*, as in the proof of Theorem A. We can suppose that G is infinite since every surjective selfmap of a finite set is also injective. Up to enlarging M, we can also assume that

$1_G \in M$. By hypothesis, there exist a constant configuration $c \in S^G$ and a finite subset $F \subset G$ with $s|_{G\setminus F} = c|_{G\setminus F}$. Note again that $c \in \Sigma(s)$ as G is infinite. We infer from the stable post-surjectivity of σ_s that the CA $\sigma_c \colon A^G \to A^G$ is post-surjective. Since G is post-injunctive, σ_c is pre-injective and thus invertible by Corollary 3 and Remark 1. Hence, we can find finite subset $N \subset G$ and a constant configuration $d \in T^G$, where $T = A^{A^N}$, such that $1_G \in N$ and

$$\sigma_c \circ \sigma_d = \sigma_d \circ \sigma_c = \mathrm{Id}_{A^G} \, . \tag{5.1}$$

By [29, Theorem 6.2], there exists a configuration $q \in Q^G$, where $Q = A^{A^{MN}}$, such that $\sigma_s \circ \sigma_d = \sigma_q$. As σ_s and σ_d are surjective, so is $\sigma_q = \sigma_s \circ \sigma_d$. Let $\pi \colon A^{MN} \to A^{\{1_G\}}$ be the canonical projection $x \mapsto x(1_G)$. As $s|_{G\setminus F} = c|_{G\setminus F}$, we have $q|_{G\setminus F} = \pi^{G\setminus F}$ since for every $g \in G\setminus F$ (see the proof of [29, Theorem 6.2]):

$$q(g) = s(g) \circ f^{+N}_{M, g^{-1}d|_M} = c(g) \circ f^{+N}_{M, g^{-1}d|_M} = \pi$$

where the last equality follows from (5.1). Let $E = FMN \subset G$ then $F \subset E$ as $1 \in M \cap N$. It follows that $q|_{G\setminus E} = \pi^{G\setminus E}$. As in the proof of Theorem A, we can write $\sigma_q = \Phi \times \mathrm{Id}_{A^{G\setminus E}}$ where $\Phi \colon A^E \to A^E$ is the map given by $\Phi(x) = \sigma_q(y)|_E$ for every $x \in A^E$ and any $y \in A^G$ such that $y|_E = x$. Then Φ is surjective because σ_q and $\mathrm{Id}_{A^{G\setminus E}}$ are both surjective. It follows that Φ must be injective as A^E is finite. Consequently, $\sigma_q = \Phi \times \mathrm{Id}_{A^{G\setminus E}}$ is injective. Thus by (5.1), we have $\sigma_s = \sigma_q \circ (\sigma_d)^{-1} = \sigma_q \circ \sigma_c$ is injective and thus pre-injective. By the stable post-surjectivity of σ_s and Corollary 3, we conclude that σ_s is stably invertible and the proof is thus complete. □

6 Proof of Theorem C

We first prove the following technical lemma which will enable a reduction of Theorem C to Theorem A.

Lemma 3. *Let A be an alphabet and let G be a group. Let $M \subset G$ be a finite subset and $S = A^{A^M}$. Suppose that $\sigma_t \circ \sigma_s = \mathrm{Id}_{A^G}$ for some $s, t \in S^G$ and s has uniformly bounded singularity. Then for each $E \subset G$ finite, there exist asymptotically constant configurations $p, q \in S^G$ such that $p|_E = s|_E$, $q|_E = t|_E$, and $\sigma_q \circ \sigma_p = \mathrm{Id}_{A^G}$.*

Proof. Up to enlarging M, we can assume that $1_G \in M = M^{-1}$. Let $E \subset G$ be a finite subset. Up to enlarging E, we can suppose without loss of generality that $M \subset E$ and $E = E^{-1}$. As s has uniformly bounded singularity, there exist a constant configuration $c \in A^{A^M}$ and a finite subset $F \subset G$ containing E^3 such that $s|_{FE^3\setminus F} = c|_{FE^3\setminus F}$. We define an asymptotically constant configuration $p \in S^G$ by setting $p|_{FE} = s|_{FE}$ and $p|_{G\setminus FE} = c|_{G\setminus FE}$. We fix $g_0 \in FE^2\setminus FE$ and define $q \in S^G$ by $q(g) = t(g_0)$ if $g \in G\setminus FE$ and $q(g) = t(g)$ if $g \in FE$. Then q is asymptotic to the constant configuration $d \in S^G$ defined by $d(g) = t(g_0)$ for all $g \in G$. Since $1_G \in M$, we have a projection $\pi \colon A^{M^2} \to A$ given by $x \mapsto x(1_G)$.

Since $E \subset E^3 \subset F \subset FE$, we deduce from our construction that $p|_E = s|_E$ and $q|_E = t|_E$. To conclude, we only need to check that $\sigma_q \circ \sigma_p = \mathrm{Id}_{A^G}$.

Let $g \in FE^2 \backslash FE$. Then $gE \subset FE^3 \backslash F$ since $E = E^{-1}$. Consequently, $s|_{gE} = c|_{gE} = p|_{gE}$ and thus $s|_{gM} = c|_{gM} = p|_{gM}$ since $M \subset E$. Hence, the condition $\sigma_t(\sigma_s(x))(g) = x(g)$ for all $x \in A^G$ is equivalent to $t(g) \circ f^{+M}_{M,c|_M} = \pi$ by Lemma 2. Similarly, the condition $\sigma_q(\sigma_p(x))(g) = x(g)$ for all $x \in A^G$ amounts to $q(g) \circ f^{+M}_{M,c|_M} = \pi$. Since $q(g) = t(g_0)$ and $\sigma_t \circ \sigma_s = \mathrm{Id}_{A^G}$, we conclude from the above discussion that $\sigma_q(\sigma_p(x))(g) = x(g)$ for all $x \in A^G$.

Let $g \in FE$. Since $s|_{FE^3 \backslash F} = c|_{FE^3 \backslash F}$, $p|_{FE} = s|_{FE}$, and $p|_{G \backslash FE} = c|_{G \backslash FE}$ by construction, we have $p|_{FE^3} = s|_{FE^3}$. In particular, $p|_{gM} = s|_{gM}$ since $gM \subset (FE)E = FE^2 \subset FE^3$. Therefore, we can infer from the relations $\sigma_t \circ \sigma_s = \mathrm{Id}_{A^G}$ and $q(g) = t(g)$ that $\sigma_q(\sigma_p(x))(g) = x(g)$ for all $x \in A^G$.

Finally, let $g \in G \backslash FE^2$. Since $p|_{G \backslash FE} = c|_{G \backslash FE}$, $q|_{G \backslash FE} = d|_{G \backslash FE}$, and since c, d are constant, we deduce that $q(g) = d(g_0) = t(g_0)$ and $p|_{gM} = c|_{gM}$. The condition $\sigma_q(\sigma_p(x))(g) = x(g)$ for all $x \in A^G$ is thus equivalent to $t(g_0) \circ f^{+M}_{M,c|_M} = \pi$ by Lemma 2. But since $\sigma_t \circ \sigma_s = \mathrm{Id}_{A^G}$ and $s|_{g_0 M} = c|_{g_0 M}$, another application of Lemma 2 shows that $t(g_0) \circ f^{+M}_{M,c|_M} = \pi$. Thus, $\sigma_t(\sigma_s(x))(g) = x(g)$ for all $x \in A^G$. Therefore, we conclude that $\sigma_q \circ \sigma_p = \mathrm{Id}_{A^G}$. The proof is complete. □

We are now in position to prove Theorem C.

Proof of Theorem C. Since σ_s is stably injective by hypothesis, we deduce from [29, Theorem A] that there exist a finite subset $N \subset G$ and $t \in T^G$, where $T = A^{A^N}$, such that $\sigma_t \circ \sigma_s = \mathrm{Id}_{A^G}$. Up to enlarging M and N, we can assume that $1_G \in M = N$ and thus $S = T$. By Corollary 2, it suffices to show that σ_s is surjective to conlude that σ_s is stably invertible. We suppose on the contrary that σ_s is not surjective. Since $\Gamma = \sigma_s(A^G)$ is closed in A^G with respect to the prodiscrete topology by [29, Theorem 4.4], there must exist a nonempty finite subset $E \subset G$ such that $\Gamma_E \subsetneq A^E$. Since s has uniformly bounded singularity, we infer from Lemma 3 that there exist asymptotically constant configurations $p, q \in S^G$ such that $p|_E = s|_E$, $q|_E = t|_E$, and $\sigma_q \circ \sigma_p = \mathrm{Id}_{A^G}$. Let $\Lambda = \sigma_p(A^G)$ then it follows from $p|_E = s|_E$ that $\Lambda_E = \Gamma_E \subsetneq A^E$. We deduce that σ_p is not surjective. In particular, σ_p is not invertible. On the other hand, the condition $\sigma_q \circ \sigma_p = \mathrm{Id}_{A^G}$ shows that σ_p is stably injective by [29, Theorem A]. We can thus apply Theorem A to deduce that σ_p is invertible and thus surjective. Therefore, we obtain a contradiction and the proof is complete. □

7 Proof of Theorem D

The proof of Theorem D generalizes the proof of Theorem A and Theorem B.

Proof. We will show that σ_s is injective. Suppose on the contrary that there exist an element $h \in G$ and configurations $u, v \in A^G$ such that $\sigma_s(u) = \sigma_s(v)$ but $u(h) \neq v(h)$. Up to enlarging M, we can suppose without loss of generality that $M = M^{-1}$ (that is, M is symmetric) and $1_G, h \in M$. When G is finite, σ_s

is trivially invertible as a surjective selfmap of a finite set. Hence, we assume in what follows that the group G is infinite.

Let $c_1, ..., c_n \in S^G$ be all the constant configurations in S^G where $n = |S|$. Let $\Delta \subset G$ be a finite generating set of G such that $1_G \in \Delta = \Delta^{-1}$ (that is, Δ is symmetric) and $M \subset \Delta$. Up to enlarging M, we can suppose that $M = \Delta$. For each $k \geqslant 1$, let $E_k = \Delta^k$ then E_k is a finite symmetric subset of G containing 1_G and we have an exhaustion $G = \bigcup_{k=1}^{\infty} E_k$. Since s has uniformly bounded singularity, we can find a finite subset $F_k \subset G$ such that $E_k^5 \subset F_k$ and $s|_{F_k E_k^5 \setminus F_k} \in S^{F_k E_k^5 \setminus F_k}$ is constant for every $k \geqslant 1$. Since S is finite, there exist $m \in \{1, 2, ..., n\}$ and an infinite sequence $1 \leqslant k_1 < k_2 < k_3 < \dots$ of integers such that for $c = c_m$, we have $E_{k_i}^5 \subset F_{k_i}$ for every $i \geqslant 1$ and

$$s|_{F_{k_i} E_{k_i}^5 \setminus F_{k_i}} = c|_{F_{k_i} E_{k_i}^5 \setminus F_{k_i}}. \tag{7.1}$$

For every $i \geqslant 1$, we claim that $F_{k_i} E_{k_i}^2 \setminus F_{k_i} E_{k_i} \neq \varnothing$. Indeed, we would have otherwise $F_{k_i} E_{k_i}^2 \subset F_{k_i} E_{k_i}$. Hence, $F_{k_i} E_{k_i}^r \subset F_{k_i} E_{k_i}$ by induction on $r \geqslant 2$ and

$$G = F_{k_i} G = F_{k_i} \bigcup_{r \geqslant 2} E_{k_i}^r = \bigcup_{r \geqslant 2} F_{k_i} E_{k_i}^r \subset F_{k_i} E_{k_i}$$

which is a contradiction since G is infinite. Therefore, for every $i \geqslant 1$, we can fix some $g_i \in F_{k_i} E_{k_i}^2 \setminus F_{k_i} E_{k_i}$. As $1_G \in E_{k_i}$ and E_{k_i} is symmetric, it follows that

$$g_i E_{k_i} \subset F_{k_i} E_{k_i}^3 \setminus F_{k_i}. \tag{7.2}$$

It follows from (7.1) and (7.2) that $c|_{g_i E_{k_i}} = s|_{g_i E_{k_i}}$. Since $G = \bigcup_{i=1}^{\infty} E_{k_i}$ and $\Sigma(s) = \overline{\{gs \colon g \in G\}} \subset S^G$, we deduce that $c \in \Sigma(s)$. We thus infer from the stable post-surjectivity of σ_s that $\sigma_c \colon A^G \to A^G$ is a post-surjective CA. Hence, σ_c is pre-injective by the post-injunctivity of the group universe G. Theorem C then implies that σ_s is an invertible CA. Therefore, there exists a CA $\tau \colon A^G \to A^G$ such that

$$\tau \circ \sigma_c = \sigma_c \circ \tau = \mathrm{Id}_{A^G}. \tag{7.3}$$

Without loss of generality, we can assume that M is also a memory set of τ up to enlarging M. Thus $\tau = \sigma_d$ for some constant configuration $d \in S^G$.

From [29, Theorem 6.2] we obtain a configuration $q \in Q^G$, where $Q = A^{A^{M^2}}$, such that $\sigma_d \circ \sigma_s = \sigma_q$. Let $\pi \colon A^{M^2} \to A^{\{1_G\}}$ denote the canonical projection $z \mapsto z(1_G)$ and fix $j \geqslant 1$ large enough such that $M^2 \subset E_{k_j}$. We claim that

$$q|_{F_{k_j} E_{k_j}^4 \setminus F_{k_j} E_{k_j}} = \pi^{F_{k_j} E_{k_j}^4 \setminus F_{k_j} E_{k_j}}. \tag{7.4}$$

Indeed, let $g \in F_{k_j} E_{k_j}^4 \setminus F_{k_j} E_{k_j}$. Then $g E_{k_j} \subset F_{k_j} E_{k_j}^5 \setminus F_{k_j}$ as E_{k_j} is symmetric. Hence, $gM \subset g E_{k_j} \subset F_{k_j} E_{k_j}^5 \setminus F_{k_j}$. It follows that $s|_{gM} = c|_{gM}$ by (7.1) and thus $g^{-1}s|_M = g^{-1}c|_M$. Since $\sigma_d \circ \sigma_c = \mathrm{Id}_{A^G}$ by (7.3), the claim (7.4) follows since (see the proof of [29, Theorem 6.2]):

$$q(g) = d(g) \circ f^{+M}_{M, g^{-1}s|_M} = d(g) \circ f^{+M}_{M, g^{-1}c|_M} = \pi.$$

Therefore, as in the proof of Theorem A, we deduce from (7.4) that

$$\sigma_q = \Phi \times \left(\mathrm{Id}_{A^{F_{k_j} E^3_{k_j} \setminus F_{k_j} E^2_{k_j}}} \right) \times \Psi \tag{7.5}$$

where $\Phi\colon A^{F_{k_j} E^2_{k_j}} \to A^{F_{k_j} E^2_{k_j}}$ and $\Psi\colon A^{G \setminus F_{k_j} E^3_{k_j}} \to A^{G \setminus F_{k_j} E^3_{k_j}}$ are well-defined maps given by the following formula:

(i) $\Phi(x) = \sigma_q(y)|_{F_{k_j} E^2_{k_j}}$ for $x \in A^{F_{k_j} E^2_{k_j}}$ and $y \in A^G$ with $y|_{F_{k_j} E^2_{k_j}} = x$,

(ii) $\Psi(x) = \sigma_q(y)|_{G \setminus F_{k_j} E^3_{k_j}}$ for $x \in A^{G \setminus F_{k_j} E^3_{k_j}}$ and $y \in A^G$ with $y|_{G \setminus F_{k_j} E^3_{k_j}} = x$.

Recall the choice of the configurations $u, v \in A^G$ in the first paragraph of the proof with $u(h) \neq v(h)$ and $\sigma_s(u) = \sigma_s(v)$ for some $h \in M \subset F_{k_j} E^2_{k_j}$. It follows that $u|_{F_{k_j} E^2_{k_j}} \neq v|_{F_{k_j} E^2_{k_j}}$ and

$$\begin{aligned} \Phi\left(u|_{F_{k_j} E^2_{k_j}}\right) = \sigma_q(u)|_{F_{k_j} E^2_{k_j}} &= \sigma_d\left(\sigma_s(u)\right)|_{F_{k_j} E^2_{k_j}} \\ &= \sigma_d\left(\sigma_s(v)\right)|_{F_{k_j} E^2_{k_j}} = \sigma_q(v)|_{F_{k_j} E^2_{k_j}} = \Phi\left(v|_{F_{k_j} E^2_{k_j}}\right). \end{aligned}$$

We deduce that Φ is not injective and thus it is not surjective as a selfmap of the finite set $A^{F_{k_j} E^2_{k_j}}$. Consequently, (7.5) implies that σ_q is not surjective. We arrive at a contradiction since $\sigma_q = \sigma_d \circ \sigma_s$ is surjective as a composition of surjective maps. Hence, σ_s must be injective. By Corollary 3, we conclude that σ_s is stably invertible and the proof is complete. ◻

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