Morgan-Stone Lattices

Alexej Pynko

August 3, 2023

MORGAN-STONE LATTICES

ALEXEJ P. PYNKO

Abstract. *Morgan-Stone (MS) lattices* are axiomatized by the constant-free identities of those axiomatizing *Morgan-Stone (MS) algebras*. Applying the technique of characteristic functions of prime filters as homomorphisms from lattices onto the two-element chain one and their products, we prove that the variety of MS lattices is the abstract hereditary multiplicative class generated by a six-element one with an equational disjunctive system expanding the direct product of the three- and two-element chain distributive lattices, in which case subdirectly-irreducible MS lattices are exactly isomorphic copies of nine non-one-element subalgebras of the six-element generating MS lattice, and so we get a 29-element non-chain distributive lattice of varieties of MS lattices subsuming the four-/three-element chain one of "De Morgan"/Stone lattices/algebras (viz., constant-free versions of De Morgan algebras)/(more precisely, their term-wise definitionally equivalent constant-free versions, called *Stone lattices*). Among other things, we provide an REDPC scheme for MS lattices. Laying a special emphasis onto the equational unbounded approximation of MS algebras (viz., the greatest variety of MS lattices without members with bounds but expandable to no MS algebra), we find a 29-element non-chain distributive lattice of its sub-quasi-varieties, subsuming the fifteen-element one of the [quasi-]equational join (viz., the [quasi-]variety generated by the union) of De Morgan and Stone lattices, in its turn, subsuming the eight-element one of those of the variety of De Morgan lattices found earlier, each of the rest being the quasi-equational join of its intersection with the variety of De Morgan lattices and the variety of Stone lattices. In this connection, we also prove that any relatively simple sub-quasi-variety of the equational unbounded approximation of MS-algebras is a variety of De Morgan lattices.

1. Introduction

The notion of *De Morgan lattice*, being originally due to [15], has been independently explored in [10] under the term *distributive i-lattice* w.r.t. their subdirectly-irreducibles and the lattice of varieties. They satisfy so-called *De Morgan identities*. On the other hand, these are equally satisfied in *Stone algebras* (cf., e.g., [7]). This has inevitably raised the issue of unifying such varieties. Perhaps, a first way of doing it within the framework of De Morgan algebras (viz., bounded De Morgan lattices; cf., e.g., [1]) has been due to [2] (cf. [23]) under the term *Morgan-Stone (MS) algebra* providing a description of their subdirectly-irreducibles, among which there are those being neither De Morgan nor Stone algebras. Here, we study unbounded MS algebras naturally called *Morgan-Stone (MS) lattices*. Demonstrating the usefulness of the technique of the characteristic functions of prime filters and functional products of former ones as well as disjunctive systems, we briefly discuss the issues of subdirectly-irreducible Morgan-Stone lattices and their varieties. Likewise, summarizing construction of REDPC schemes (cf. [6]) for distributive lattice[expansion]s originally being due to [8] [and [12, 21]], we provide that for Morgan-Stone lattices and an enhanced one for the {quasi-}equational join of De Morgan and Stone lattices. Nevertheless, the culminating issue of this study is to

2020 *Mathematics Subject Classification.* 06D15, 06D30, 08A30, 08B05, 08B26, 08C15.
Key words and phrases. De Morgan lattice, Stone algebra, quasi-variety, REDPC..

find the lattice of sub-quasi-varieties of the equational unbounded approximation of MS algebras upon the basis of that of the variety of De Morgan lattices found in [17]. In this connection, we also prove that any relatively simple sub-quasi-variety of the equational unbounded approximation of MS algebras is a variety of De Morgan lattices. In general, we seek to expand our results to bounded MS lattices subsuming MS algebras, whenever it is at all possible. This equally concerns the very last point mentioned above but not the previous much more extended issue because of the well-known infiniteness of the lattice of quasi-varieties of De Morgan algebras.

The rest of the work is as follows. Section 2 is a concise summary of basic set-theoretical and algebraic issues underlying the work. Then, in Section 3 we briefly summarize general issues concerning REDPC in the sense of [6] as well as equational implicative/disjunctive systems in the sense of [20]/[19] in connection with simplicity/"subdirect irreducibility". Next, Section 4 is devoted to preliminary study of Morgan-Stone lattices. Further, Section 5 is a thorough collection of culminating results on sub-quasi-varieties of the equational unbounded approximation of Morgan-Stone algebras. Finally, Section 6 is a concise collection of open issues and necessary statements.

2. General background

2.1. **Set-theoretical background.** Non-negative integers are identified with the sets/ordinals of lesser ones, "their set/ordinal"|"the ordinal‖set class" being denoted by $\omega|(\infty\|\Upsilon)$. Unless any confusion is possible, one-element sets are identified with their elements.

For any sets A, B and D as well as $\theta \subseteq A^2$, $h : A \to B$ and $g : A^2 \to A$, let $\wp_{[K]}((B,)A)$ be the set of all subsets of A (including B) [of cardinality in $K \subseteq \infty$, $D \subseteq_K A$ standing for $D \in \wp_K(A)$], $((\Delta_A|\nu_\theta)\|(A/\theta)\|\chi^B_A) \triangleq (\{\langle a, a|\theta[\{a\}]\rangle \mid a \in A\}\|\nu_\theta[A]\|(((A \cap B) \times \{1\}) \cup ((A \setminus B) \times \{0\})))$, $A^{*|+} \triangleq (\bigcup_{m\in(\omega\setminus(0|1))} A^m)$, $h_* : A^* \to B^* : a \mapsto (a \circ h)$, $g_+ : A^+ \to A, \langle[\langle a, b\rangle,]c\rangle \mapsto [g]([g_+(\langle a, b\rangle),]c)$ and $\varepsilon_B : (\Upsilon^B)^2 \to \wp(B), \langle d, e\rangle \mapsto \{b \in B \mid \pi_b(d) = \pi_b(e)\}$, A-tuples {viz., functions with domain A} being written in the sequence form $\bar{t}$ with t_a, where $a \in A$, standing for $\pi_a(\bar{t})$. Then, for any $(\bar{a}|C) \in (A^*|\wp(A))$, by induction on the length (viz., domain) of any $\bar{b} = \langle[\bar{c}, d]\rangle \in A^*$, put $((\bar{a} * \bar{b})|(\bar{b}(\cap/\setminus)C)) \triangleq ((([\langle]\bar{a}[*\bar{c}, d\rangle])|(\langle[\bar{c}(\cap/\setminus)C(, d)]\rangle))$ $|[(\text{provided } d \in / \notin C)]$. Likewise, given any $\bar{S} \in \wp(D)^B$ and $\bar{f} \in \prod_{b\in B} S_b^A$, let $(\prod \bar{f}) : A \to (\prod_{b\in B} S_b), a \mapsto \langle f_b(a)\rangle_{b\in B}$, in which case

$$\ker(\prod \bar{f}) = (A^2 \cap (\bigcap_{b\in B} (\ker f_b))), \tag{2.1}$$

$$\forall b \in B : f_b = ((\prod \bar{f}) \circ \pi_b), \tag{2.2}$$

$f_0 \times f_1$ standing for $(\prod \bar{f})$, whenever $B = 2$.

A *lower/upper cone of* a poset $\mathcal{P} = \langle P, \leqq\rangle$ is any $C \subseteq P$ such that, for all $a \in C$ and $b \in P$, $(a \geqq / \leqq b) \Rightarrow (b \in C)$. Then, an $a \in S \subseteq P$ is said to be *minimal/maximal in* S, if $\{a\}$ is a lower/upper cone of S, their set being denoted by $(\min/\max)_{\mathcal{P}|\leqq}(S)$, in case of the equality of which to S, this being called an *anti-chain of* $\mathcal{P}$.

An $X \in Y \subseteq \wp(A)$ is said to be *[K-]meet-irreducible in* Y, [where $K \subseteq \infty$], if $\forall Z \in \wp_{[K]}(Y) : ((A\cap(\bigcap Z)) = X) \Rightarrow (X \in Z)$, their set being denoted by $\mathrm{MI}^{[K]}(Y)$, "finitely-" standing for "ω-" within any related context. Next, a $\mathcal{U} \subseteq \wp(A)$ is said to be *upward-directed*, if $\forall \mathcal{S} \in \wp_\omega(\mathcal{U}) : \exists T \in (\mathcal{U} \cap \wp(\bigcup \mathcal{S}, A))$, subsets of $\wp(A)$ closed under unions of upward directed subsets being called *inductive*. Further, a *[finitary] closure operator over* A is any unary operation on $\wp(A)$ such that $\forall X \in \wp(A), \forall Y \in$

$\wp(X) : (X \cup C(C(X)) \cup C(Y)) \subseteq C(X)[= (\bigcup C[\wp_\omega(X)])]$. Finally, a *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ containing A and closed under intersections of subsets containing A, any $\mathcal{B} \subseteq \mathcal{C}$ with $\mathcal{C} = \{A \cap (\bigcap \mathcal{S}) \mid \mathcal{S} \subseteq \mathcal{B}\}$ being called a *(closure) basis of* $\mathcal{C}$ and determining the closure operator $C_\mathcal{B} \triangleq \{\langle Z, A \cap (\bigcap(\mathcal{X} \cap \wp(Z, A)))\rangle \mid Z \in \wp(A)\}$ over A with $(\operatorname{img} C_\mathcal{B}) = \mathcal{C}$. Conversely, $\operatorname{img} C$ is a closure system over A with $C_{\operatorname{img} C} = C$, being inductive iff C is finitary, and forming a complete lattice under the partial ordering by inclusion with meet/join $(\Delta_{\wp(A)}/C)(A \cap ((\bigcap/\bigcup)\mathcal{S}))$ of any $\mathcal{S} \subseteq (\operatorname{img} C)$, C and $\operatorname{img} C$ being called *dual to* one another. Then, $C(X) \in (\operatorname{img} C)$ is said to be *generated by* an $X \subseteq A$, elements of $C[\wp_{\omega/\{n\}}(A)]$ /"with $n \in (\omega|\{1\})$" being said to be *finitely/n-generated|principal*.

Remark 2.1. Due to Zorn Lemma, according to which any non-empty inductive set has a maximal element, $MI^{[K]}(\mathcal{C})$ is a basis of any inductive closure system $\mathcal{C}$. $\square$

A *filter/ideal on* A is any $\mathcal{F} \subseteq \wp(A)$ such that, for all $\mathcal{S} \in \wp_\omega(\wp(A))$, $(\mathcal{S} \subseteq \mathcal{F}) \Leftrightarrow ((A \cap ((\bigcap/\bigcup)\mathcal{S})) \in \mathcal{F})$ "the set $\operatorname{Fi}(A)$ of them being an inductive closure system over $\wp(A)$ with dual finitary closure operator (of filter generation) Fg_A such that

$$\operatorname{Fg}_A(\mathcal{T}) = \wp(A \cap (\bigcap \mathcal{T}), A), \tag{2.3}$$

for all $\mathcal{T} \in \wp_\omega(\wp(A))$"/. Then, an *ultra-filter on* A is any filter $\mathcal{U}$ on A such that $\wp(A) \setminus \mathcal{U}$ is an ideal on A.

2.2. **Algebraic background.** Unless otherwise specified, we deal with a fixed but arbitrary finitary functional signature Σ, Σ-algebras/"their carriers" being denoted by same capital Fraktur/Italic letters (with same indices, if any) "with denoting the class of all [one-element] ones by $\mathsf{A}_\Sigma^{[=1]}$"/. Given any $\alpha \in (\infty \setminus 1)$, let $\operatorname{Tm}_\Sigma^\alpha$ be the carrier of the absolutely-free Σ-algebra $\mathfrak{Tm}_\Sigma^\alpha$, freely-generated by the set $V_\alpha \triangleq \{x_\beta\}_{\beta \in \alpha}$ of *(first α) variables*, and $\operatorname{Eq}_\Sigma^\alpha \triangleq (\operatorname{Tm}_\Sigma^\alpha)^2$, $\phi \approx /(\lessapprox \mid \gtrapprox)\psi$, where $\phi, \psi \in \operatorname{Tm}_\Sigma^\alpha$ /"and $\wedge \in \Sigma$", meaning $\langle \phi/(\phi \wedge \psi), \psi/(\phi|\psi)\rangle$ "and being called a Σ-*equation of rank* α"/. /"Likewise, for any Σ-algebra $\mathfrak{A}$ and $a, b \in A$, $(a(\leqslant \mid \geqslant)^\mathfrak{A} b)\|[a,b]_\mathfrak{A}$ stands for $((a|b) = (a \wedge^\mathfrak{A} b))\|\{c \in A \mid a \leqslant^\mathfrak{A} c \leqslant^\mathfrak{A} b\}$." Then, any $\langle \Gamma, \Phi\rangle \in (\wp_{\infty/(1[\cup\omega])}(\operatorname{Eq}_\Sigma^\alpha) \times \operatorname{Eq}_\Sigma^\alpha)$ /"with $\alpha \in \omega$" is called a Σ-*implication/-[quasi-]identity of rank* α, written as $\Gamma \to \Phi$ and identified with Φ, if $\Gamma = \varnothing$, as well as treated as the universal infinitary/first-order strict Horn sentence $\forall_{\beta\in\alpha} x_\beta((\bigwedge \Gamma) \to \Phi)$, the class/set of those of any /finite rank true in a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ being called the *implicational/[quasi-]equational theory of* K and denoted by $(\mathcal{I}/[\mathcal{Q}]\mathcal{E})(\mathsf{K})$.

Subclasses of A_Σ "closed under $\mathbf{I}|\mathbf{S}_{(>1)}|\mathbf{P}^{[\mathrm{SD}\|\mathrm{U}]}$"/"containing each Σ-algebra with finitely-generated subalgebras in them"/"containing no infinite finitely-generated member" are referred to as *"abstract|(non-trivially-)hereditary|[ultra-\|sub-]multiplicative"/local/locally-finite* (cf. [14]). Then, a *skeleton* {*of* a(n abstract) $\mathsf{K} \subseteq \mathsf{A}_\Sigma$} is any $\mathsf{S} \subseteq \mathsf{A}_\Sigma$ without pair-wise distinct isomorphic members {such that $\mathsf{S} \subseteq \mathsf{K} \subseteq \mathbf{IS}$ (i.e., $\mathsf{K} = \mathbf{IS}$)}. Given a $\mathsf{K} \subseteq \mathsf{A}_\Sigma \ni \mathfrak{A}$, set $\hom^{[\mathrm{S}]}(\mathfrak{A}, \mathsf{K}) \triangleq \{h \in \hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}[, (\operatorname{img} h) = B]\}$ and $\operatorname{Co}_\mathsf{K}(\mathfrak{A}) \triangleq \{\theta \in \operatorname{Co}(\mathfrak{A}) \mid (\mathfrak{A}/\theta) \in \mathsf{K}\}$, whose elements are called K-*(relative)congruences of* $\mathfrak{A}$, $\mathfrak{A} \preceq \mathsf{K}$ standing for $\mathfrak{A} \in \mathbf{IS}\mathsf{K}$ and thus providing a quasi-ordering on A_Σ, in which case, by the Homomorphism Theorem, we have

$$(\ker[\hom^{\mathrm{S}|}(\mathfrak{A}, \mathsf{K})]\{\setminus(\varnothing|\{A^2\})\}) = \operatorname{Co}_{(\mathbf{I}|(\mathbf{IS}_{\{>1\}}))\mathsf{K}}(\mathfrak{A}), \tag{2.4}$$

and so "by the Homomorphism Theorem"|, for all $\mathfrak{B} \in \mathsf{A}_\Sigma$ and $h \in \hom^{\mathrm{S}|(\mathrm{S}\|)}(\mathfrak{B}|\mathfrak{A}, \mathfrak{A}|\mathfrak{B})$:

$$\begin{aligned}(2.5)\quad & \forall \theta \in (\operatorname{Co}_{[(\mathbf{I}|(\mathbf{I}\|(\mathbf{IS})))\mathsf{K}]}(\mathfrak{B}) \cap \wp((\ker h)|\Delta_B, B^2)) : \\ & h_*^{|(-1)}[\theta] \in (\operatorname{Co}_{[(\mathbf{I}|(\mathbf{I}\|(\mathbf{IS})))\mathsf{K}]}(\mathfrak{A}) \cap \wp(\Delta_A|(\ker h), A^2)),\end{aligned}$$

$$h_*^{(-1)|}[h_*^{|(-1)}[\theta]] = (\theta \cap (B|h[A])^2)$$

"yielding an isomorphism between the posets $\mathrm{Co}_{[\mathbf{I}\mathsf{K}]}(\mathfrak{B}) \cap \wp(\ker h, B^2)$ and $\mathrm{Co}_{[\mathbf{I}\mathsf{K}]}(\mathfrak{A})$ ordered by inclusion as well as"$\|$ $|$"implying:

$$h_*^{-1}[\mathrm{Cg}^{\mathfrak{B}}_{[(\mathbf{I}\|(\mathbf{IS}))\mathsf{K}]}(h_*[X]) = \| \supseteq \mathrm{Cg}^{\mathfrak{A}}_{[(\mathbf{I}\|(\mathbf{IS}))\mathsf{K}]}(X \cup (\ker h)), \tag{2.6}$$

for all $X \subseteq A^2$", while, as, for any set I, $\overline{\mathfrak{B}} \in \mathsf{A}_\Sigma^I$ and $\bar{f} \in (\prod_{i \in I} \hom(\mathfrak{A}, \mathfrak{B}_i))$:

$$(\prod \bar{f}) \in \hom(\mathfrak{A}, \prod_{i \in I} \mathfrak{B}_i), \tag{2.7}$$

by (2.1) and (2.2) with [finite] $I \triangleq \mathrm{Co}_{(\mathbf{I}\|(\mathbf{IS}))\mathsf{K}}(\mathfrak{A})$ [if either A is finite or, by (2.4), both $\mathfrak{A}$ is finitely-generated and K as well as all its members are finite] for B, $\overline{\mathfrak{B}} \triangleq \langle \mathfrak{B}/i \rangle_{i \in I}$, $D \triangleq (\bigcup_{i \in I} B_i)$ and $\bar{f} \triangleq \langle \nu_i \rangle_{i \in I}$, we get:

$$(\mathfrak{A} \in \mathbf{IP}^{\mathrm{SD}}_{[\omega]}(\{\mathbf{I}\}\|(\{\mathbf{I}\}\mathbf{S}))\mathsf{K})) \Leftrightarrow ((A^2 \cap (\bigcap \ker[\hom^{\mathrm{S}\|}(\mathfrak{A}, \mathsf{K})])) = \Delta_A), \tag{2.8}$$

whereas, since, for any $I \triangleq \Theta \subseteq \mathrm{Co}_{\langle \mathsf{K} \rangle}(\mathfrak{A})$, $\theta \triangleq (A^2 \cap (\bigcap \Theta)) \in \mathrm{Co}(\mathfrak{A})$, $\overline{\mathfrak{B}} \triangleq \langle \mathfrak{A}/i \rangle_{i \in I} \in (\mathsf{A}_\Sigma \langle \cap \mathsf{K} \rangle)^I$ as well as, by the Homomorphism Theorem, $\bar{f} \triangleq \langle \nu_\theta^{-1} \circ \nu_i \rangle_{i \in I} \in (\prod_{i \in I} \hom(\mathfrak{A}/\theta, \mathfrak{B}_i))$, taking (2.1), (2.2) and (2.7) into account, we see that $e \triangleq (\prod \bar{f})$ is an embedding of $\mathfrak{A}/\theta$ into $\mathfrak{C} \triangleq (\prod_{i \in I} \mathfrak{B}_i)$ such that $\mathfrak{C} \restriction (\mathrm{img}\, e)$, being isomorphic to $\mathfrak{A}/\theta$, is a subdirect product of $\overline{\mathfrak{B}}$ $\langle$ in which case $(\mathfrak{A}/\theta) \in \mathbf{IP}^{\mathrm{SD}}\mathsf{K}$, and so, providing K is both abstract and sub-multiplicative, $\theta \in \mathrm{Co}_{\mathsf{K}}(\mathfrak{A})\rangle$. In particular, [providing K is both abstract and sub-multiplicative], $\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A})$ is a closure system over A^2, the dual closure operator being denoted by $\mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{K}]}$.

Remark 2.2. By (2.4), the $|$-right alternative of (2.5) with $h = \nu_\vartheta$, where $\vartheta \in \mathrm{Co}_{\mathbf{IP}^{\mathrm{SD}}([\mathbf{I}]\|([\mathbf{I}]\mathbf{S}))\mathsf{K}}(\mathfrak{A})$, $\mathfrak{B} = (\mathfrak{A}/\vartheta)$ and $\theta = \Delta_B$ as well as (2.8), since $\vartheta = h_*^{-1}[\theta]$, while h_*^{-1} preserves intersections, $\mathrm{Co}_{(\mathbf{I}\|(\mathbf{IS}))\mathsf{K}}(\mathfrak{A})$ is a basis of the closure system $\mathrm{Co}_{\mathbf{IP}^{\mathrm{SD}}([\mathbf{I}]\|([\mathbf{I}]\mathbf{S}))\mathsf{K}}(\mathfrak{A})$ over A^2. □

Given any Σ-algebra $\mathfrak{A}$ and any function f with $(\mathrm{dom}\, f) = A$ and $(\ker f) \in (\mathrm{Co}(\mathfrak{A})/\{\Delta_A\})$, we have its *homomorphic/isomorphic image/copy* $f[\mathfrak{A}]$ *by* f with carrier $f[A]$ and operations $\varsigma^{f[\mathfrak{A}]} \triangleq f_*[\varsigma^{\mathfrak{A}}]$, for each $\varsigma \in \Sigma$, in which case $f \in \hom^{\mathrm{S}}(\mathfrak{A}, f[\mathfrak{A}])$, and so $f[\mathfrak{A}] \in (\mathbf{H}|\mathbf{I})\mathfrak{A}$, such exhausting all members of $(\mathbf{H}|\mathbf{I})\mathfrak{A}$.

According to [22], *pre-varieties* are abstract hereditary multiplicative subclasses of A_Σ (these are exactly model classes of theories constituted by Σ-implications of unlimited rank, and so are also called *implicative/implicational*; cf., e.g., [3]/[17]), $\mathbf{PV}(\mathsf{K}) \triangleq \mathbf{ISP}\mathsf{K} = \mathbf{IP}^{\mathrm{SD}}(\mathbf{I})\mathbf{S}_{[>1]}\mathsf{K} = \mathrm{Mod}(\mathfrak{I}(\mathsf{K}))$ being the least one including and so called *generated by* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$. Likewise, *[quasi-]varieties* are [ultra-multiplicative] pre-varieties closed under $\mathbf{H}^{[\mathrm{I}]}[\triangleq \mathbf{I}]$ (these are exactly model classes of sets of Σ-[quasi-]identities of unlimited finite rank, and so are local and also called *[quasi-]equational*; cf., e.g., [14]), $[\mathbf{Q}]\mathbf{V}(\mathsf{K}) \triangleq \mathbf{H}^{[\mathrm{I}]}\mathbf{SP}[\mathbf{P}^{\mathrm{U}}]\mathsf{K} = \mathrm{Mod}([\mathfrak{Q}]\mathfrak{I}(\mathsf{K}))$ being the least one including and so called *generated by* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$. Then, ((pre-/quasi-)varieties generated by finite classes of finite Σ-algebras are called *finitely-generated*, in which case, by [(2.8)] (and [5, Corollary 2.3]), they are locally-finite (and quasi-equational)/. Further, intersections of a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ with [pre-/quasi-]varieties are called its *relative sub-[pre-/quasi-]varieties*, in which case, for any $\mathcal{E} \subseteq \mathrm{Eq}^\omega_\Sigma$,

$$(\mathbf{IP}^{\mathrm{SD}}(\mathsf{K}) \cap \mathrm{Mod}(\mathcal{E})) = \mathbf{IP}^{\mathrm{SD}}(\mathsf{K} \cap \mathrm{Mod}(\mathcal{E})), \tag{2.9}$$

and so $\mathsf{S} \mapsto (\mathsf{S} \cap \mathsf{K})$ and $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ are inverse to one another isomorphisms between the lattices of relative sub-varieties of $\mathbf{IP}^{\mathrm{SD}}\mathsf{K}$ and those of K.

Then, a [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is said to be *[(relatively)] congruence-distributive*, if, for each $\mathfrak{A} \in \mathsf{P}$, $\mathrm{Co}_{[(\mathsf{P})]}(\mathfrak{A})$ is distributive.

Remark 2.3. Given a [quasi-equational] pre-variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ and $\alpha \in (\infty \setminus 1)$, by the $|$-right alternative of (2.4) with $\mathsf{K} = \mathsf{P}$ and $\mathfrak{A} = \mathrm{Tm}^\alpha_\Sigma$, any Σ-implication $\Gamma \to \Phi$ of rank α is true in P iff $\Phi \in \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\Gamma)$ [in which case, by the Compactness Theorem for ultra-multiplicative classes of algebras (cf., e.g., [14]), $\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}$ is finitary, and so is $\mathrm{Cg}^{\mathfrak{B}}_{\mathsf{P}}$, for any $\mathfrak{B} \in \mathsf{A}_\Sigma$, in view of the left $\|$-alternative of (2.6), when taking $\alpha = |B|$ and h to extend any bijection from V_α onto B]. □

Furthermore, [given an abstract $\mathsf{K} \subseteq \mathsf{A}_\Sigma$] an $\mathfrak{A} \in (\mathsf{A}_\Sigma[\cap \mathsf{K}])$ is said to be */*K*-{relatively }]simple/(K-)subdirectly-irreducible /*(where $K \subseteq \infty$), if $\Delta_A \in (\max_\subseteq / \mathrm{MI}^{(K)})(\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A}) \setminus (\{A^2\}/\varnothing))$, in which case $|A| \neq 1$, the class of ⟨those of⟩ them ⟨which are in a $\mathsf{K}' \subseteq (\mathsf{A}_\Sigma[\cap \mathsf{K}])$⟩ being denoted by $(\mathrm{Si} \,/\, \mathrm{SI}^{(K)})_{[\mathsf{K}]}\langle(\mathsf{K}')\rangle$,[1] and so, by (2.4) and (2.8),

$$(\mathrm{Si} \,|\, \mathrm{SI})_{[\mathbf{IP}^{\mathrm{SD}}(\mathbf{S})\mathsf{K}'']}(\mathbf{IP}^{\mathrm{SD}}(\mathbf{S})\mathsf{K}'') \subseteq \mathbf{I}(\mathbf{S}_{>1})\mathsf{K}'', \tag{2.10}$$

for any $\mathsf{K}'' \subseteq \mathsf{A}_\Sigma$. Then, a [pre-]variety P is said to be *[{relatively}] (finitely) semi-simple/subdirectly-representable*, if

$$(\mathrm{SI}^{(\omega)}_{[\{\mathsf{P}\}]}(\mathsf{P})/\mathsf{P}) \subseteq | = (\mathrm{Si}_{[\{\mathsf{P}\}]}(\mathsf{P})/\mathbf{IP}^{\mathrm{SD}}(\mathrm{Si} \,/\, \mathrm{SI}^{(\omega)})_{[\{\mathsf{P}\}]}(\mathsf{P})),$$

any variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ being well-known, due to Birkgoff's Theorem, to be subdirectly-representable. More generally, we have:

Remark 2.4. Given any [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ and $\mathfrak{A} \in (\{\mathsf{Q}\cap\}\mathsf{A}_\Sigma)$, by Remarks 2.1, 2.2, 2.3 and the right $\|$-alternative of (2.5), $\mathrm{MI}^{(\omega)}(\mathrm{Co}_{\mathsf{Q}}(\mathfrak{A})) = \mathrm{Co}_{\mathrm{SI}^{(\omega)}_{\mathsf{Q}}(\mathsf{Q})}(\mathfrak{A})$ is a basis of both $\mathrm{Co}_{\mathsf{Q}}(\mathfrak{A})$ and $\mathrm{Co}_{\mathbf{IP}^{\mathrm{SD}}\,\mathrm{SI}^{(\omega)}_{\mathsf{Q}}(\mathsf{Q})}(\mathfrak{A})$, in which case these are equal {and so, since $\nu_{\Delta_A} \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{A}/\Delta_A)$ is injective, $\mathfrak{A} \in \mathbf{IP}^{\mathrm{SD}}\,\mathrm{SI}^{(\omega)}_{\mathsf{Q}}(\mathsf{Q})$}. In particular, Q is [relatively] (finitely) subdirectly-representable. □

Recall that, according to [13], a[n implicational] $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ is *congruence-permutable*, i.e., for each $\mathfrak{A} \in \mathsf{K}$ and all $\theta, \vartheta \in \mathrm{Co}(\mathfrak{A})$, $(\theta \circ \vartheta) \subseteq (\vartheta \circ \theta)$, if[f] it has a *congruence-permutation term*, viz., a $\pi \in \mathrm{Tm}^3_\Sigma$ such that K satisfies the Σ-identities in $\{x_1 \approx (\sigma_i(\pi)) \mid i \in \{0, 2\}\}$, where, for every $j \in 3$, $\sigma_j \triangleq [x_j/x_1; x_k/x_0]_{k \in (3 \setminus \{j\})}$. Likewise, a *minority|majority term for* K |{with $\Sigma_+ \triangleq \{\wedge, \vee\} \subseteq \Sigma$ and the Σ_+-reducts of members of K being lattices} is any $\mu \in \mathrm{Tm}^3_\Sigma$ such that K satisfies the Σ-identities in $\{x_{(1-\min(2-i,i))|0} \approx (\sigma_i(\mu)) \mid i \in 3\}$ |{$\mu_+ \triangleq (\wedge_+ \langle x_i \vee (x_{\max(1-i,0)} \wedge x_{2+\min(i,1-i)})\rangle_{i \in 3})$ being so}, in which case it is so "as well as a congruence-permutation term"| for the variety generated by K, and so this is congruence-distributive [16], while, for any congruence-permutation term π for K, $\pi[x_1/\mu]$ is a majority|minority term for K "and so $\mu[x_1/\mu]$ is a majority term for K"|. Finally, a *(ternary) |dual discriminator (term) for* K is any $\delta \in \mathrm{Tm}^3_\Sigma$ such that, for each $\mathfrak{A} \in \mathsf{K}$, $\delta^{\mathfrak{A}} = ((\pi_{2|0} \restriction (\Delta_A \times A)) \cup (\pi_{0|2} \restriction ((A^2 \setminus \Delta_A) \times A)))$, in which case $\mathfrak{A}$ is simple, because, for every $\theta \in (\mathrm{Co}(\mathfrak{A}) \setminus \{\Delta_A\})$, any $\langle a, b\rangle \in (\theta \setminus \Delta_A) \neq \varnothing$ and all $c \in A$, we have $(a|c) = \delta^{\mathfrak{A}}(a, b, c)\ \theta\ \delta^{\mathfrak{A}}(a, a, c) = (c|a)$, so getting $\theta = A^2$, while δ is a |dual discriminator for $\mathbf{ISP}^{\mathrm{U}}\mathsf{K}$ as well as a minority|majority term for K, whereas, for any congruence-permutation term π for K, $\pi[x_1/\delta]$ is a dual| discriminator for K "and so $\delta[x_1/\delta]$ is a dual discriminator for K"|, {⟨quasi-/pre-⟩varieties generated by classes of} Σ-algebras with [dual] discriminator δ being called *[dual] δ-discriminator*, with denoting the class of [dual] δ-discriminator members of a $\mathsf{C} \subseteq \mathsf{A}_\Sigma$ by $\mathsf{C}^{[\partial]}_\delta$. Then, [dual] δ-discriminator quasi-varieties are exactly quasi-equational [dual] δ-discriminator pre-varieties.

[1] This is abstract ⟨whenever K' is so⟩, in view of (2.5).

2.2.1. *Filtral congruences.* Let I be a set, $\mathcal{F}$ a{n ultra-}filter on I [$\mathsf{P} \subseteq \mathsf{A}_\Sigma$ a (quasi-equational) pre-variety], $\overline{\mathfrak{A}} \in (\mathsf{A}_\Sigma[\cap \mathsf{P}])^I$ and $\mathfrak{B}$ a subalgebra of its direct product. Then, by (2.5), for each $i \in I$, $(B^2 \cap (\ker \pi_i)) = ((\pi_i \upharpoonright B)_2^{-1}[\Delta_{A_i}] \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$, as $(\pi_i \upharpoonright B) \in \hom(\mathfrak{B}, \mathfrak{A}_i)$ and $\Delta_{A_i} \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}_i)$, in which case, for all $K \subseteq J \subseteq I$, the closure system $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$ on B^2 contains $\theta^B_J \triangleq (B^2 \cap \varepsilon_I^{-1}[\wp(J, I)]) = (B^2 \cap (\bigcap_{j \in J} \ker \pi_j)) \subseteq \theta^B_K$, $\Theta^B_{\mathcal{F}} \triangleq \{\theta^B_L \mid L \in \mathcal{F}\}$ being thus upward-directed (and so $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$, being inductive, in view of Remark 2.3, contains $\theta^B_{\mathcal{F}} \triangleq (\bigcup \Theta^B_{\mathcal{F}}) = (B^2 \cap \varepsilon_I^{-1}[\mathcal{F}])$, called $\langle \mathcal{F}\text{-}\rangle\{$*ultra-*$\}$*filtral*). Clearly, for any $\mathcal{X} \subseteq \mathrm{Fi}(I)$ |"with $(\bigcup \mathcal{X}) \in \mathrm{Fi}(I)$",

$$\theta^B_{\wp(I) \cap ((\bigcap | \bigcup)\mathcal{X})} = (B^2 \cap ((\bigcap | \bigcup)\{\theta^B_{\mathcal{F}} \mid \mathcal{F} \in \mathcal{X}\})). \tag{2.11}$$

A [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is said to be *[relatively] (subdirectly) ⟨finitely/principally⟩ filtral*, if every ⟨finitely-generated/principal⟩ [P-]congruence of each member of $\mathbf{SP}$ $\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})(\cap \mathbf{P}^{\mathrm{SD}}\, \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P}))$ is filtral (cf. [6] for the equational case).

2.2.1.1. Filtrality versus semi-simplicity.

Lemma 2.5. *Any [relatively] subdirectly principally filtral [pre-]variety* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ *is [relatively] semi-simple.*

Proof. Consider any $\mathfrak{A} \in \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})$, in which case $|A| > 1$, and any $\theta \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$ as well as any $\bar{a} \in (\theta \setminus \Delta_A) \neq \varnothing$, in which case $\mathfrak{B} \triangleq \mathfrak{A}^1 \in \mathbf{P}^{\mathrm{SD}}\, \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})$, while $h \triangleq (\pi_0 \upharpoonright B) \in \hom^{\mathrm{S}}(\mathfrak{B}, \mathfrak{A})$ is injective, whereas $B^2 \ni \bar{b} \triangleq (\bar{a} \circ h^{-1}) \in \vartheta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(\bar{b}) = \theta^B_{\mathcal{F}}$, for some $\mathcal{F} \in \mathrm{Fi}(1)$, and so, by (2.5), $\eta \triangleq h_*^{-1}[\theta] \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B}) \cap \wp(\vartheta, B^2))$, while $\theta = h_*[\eta]$, whereas $\varnothing = \varepsilon_1(\bar{b}) \in \mathcal{F}$. Then, $\mathcal{F} = \wp(1)$, in which case $\eta \supseteq \vartheta = B^2$, and so $\theta \supseteq h_*[B^2] = A^2$. Thus, $\mathfrak{A} \in \mathrm{Si}_{[\mathsf{P}]}(\mathsf{P})$, as required. □

2.2.1.2. Filtrality versus congruence-distributivity.

Lemma 2.6 (cf. [9] for the []()-non-optional case). *Let* $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ *be a [quasi-]variety, I a set,* $\overline{\mathfrak{A}} \in \mathsf{Q}^I$, $\mathfrak{B} \in \mathbf{S}(\prod \overline{\mathfrak{A}})$ *and* $\theta \in \mathrm{MI}^{(\omega)}(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}))$. *Suppose* $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ *is distributive. Then, there is an ultra-filter* $\mathcal{U}$ *on* I *such that* $\theta^B_{\mathcal{U}} \subseteq \theta$.

Proof. By (2.11), $S \triangleq \{\mathcal{F} \in \mathrm{Fi}(I) \mid \theta^B_{\mathcal{F}} \subseteq \theta\} \ni \{I\}$ is inductive, for $\mathrm{Fi}(I)$ is so, in which case, by Zorn Lemma, it, being non-empty, has a maximal element $\mathcal{U}$, and so, for any $\mathcal{X} \in \wp_\omega(\wp(I))$ such that $Y \triangleq (\bigcup \mathcal{X}) \in \mathcal{U}$, $(\mathcal{X} \cap \mathcal{U}) \neq \varnothing$, as, for each $Z \in \mathcal{X}$, $\theta^B_{\mathcal{F}_Z} \in \mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ with $\mathcal{U} \subseteq \mathcal{F}_Z \triangleq \mathrm{Fg}_I(\mathcal{U} \cup \{Z\}) \in \mathrm{Fi}(I)$, while $\mathcal{U} = \mathrm{Fg}_I(\mathcal{U}) = \mathrm{Fg}_I(\mathcal{U} \cup \{Y\}) = (\wp(I) \cap (\bigcap\{\mathcal{F}_Z \mid Z \in \mathcal{X}\}))$, in view of (2.3), since Fg_I is finitary, whereas, by (2.11), $\theta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{U}}) = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup (B^2 \cap (\bigcap\{\theta^B_{\mathcal{F}_Z} \mid Z \in \mathcal{X}\}))) = (B^2 \cap (\bigcap\{\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{F}_Z}) \mid Z \in \mathcal{X}\}))$, that is, for some $Z \in \mathcal{X}$, $\theta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{F}_Z}) \supseteq \theta^B_{\mathcal{F}_Z}$, i.e., $\mathcal{U} \subseteq \mathcal{F}_Z \in S$, viz., $Z \in \mathcal{F}_Z = \mathcal{U}$, as required. □

This, by (2.5), Birkgoff's and the Homomorphism Theorems [as well as [5, Corollary 2.3]/[20, Lemma 2.1]], immediately yields:

Corollary 2.7. *Let* K *be a [finite/] class of [finite/] Σ-algebras (with {dual} discriminator δ) and* $\mathsf{P} \triangleq \mathbf{H}^{\langle \mathrm{I} \rangle}\mathbf{SP}\mathsf{K}$. *Suppose* P *is a ⌈relatively⌉ congruence-distributive [/locally-finite] ⌈quasi-⌉variety. Then,*

$$(\mathsf{P}^{\{\partial\}}_\delta \subseteq \mathrm{Si}_{\lfloor\lceil \mathsf{P} \rceil\rfloor}(\mathsf{P}) \subseteq) \mathrm{SI}^{\omega|\infty}_{\lceil \mathsf{P} \rceil}(\mathsf{P}) \subseteq \mathbf{H}^{(\|\langle \mathrm{I} \rangle\|)}\mathbf{SP}^{\mathrm{U}}\mathsf{K}[\subseteq \mathbf{H}^{(\|\langle \mathrm{I} \rangle\|)}\mathbf{S}\mathsf{K}](\subseteq \mathsf{P}^{\{\partial\}}_\delta)$$

[in which case its members are finite, and so $\mathrm{SI}^{\omega}_{\lceil \mathsf{P} \rceil}(\mathsf{P}) = \mathrm{SI}_{\lceil \mathsf{P} \rceil}(\mathsf{P})$*]/. In particular, {dual} (δ-)discriminator quasi-varieties are exactly [semi-simple] {dual} (δ-)discriminator varieties.*

Corollary 2.8. *Let* $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ *be a ([relatively] semi-simple) [quasi-]variety,* $I \in \Upsilon$, $\overline{\mathfrak{A}} \in \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})^I$, $\mathfrak{D} \triangleq (\prod \overline{\mathfrak{A}})$, $\mathfrak{B} \in \mathbf{S}\{\mathfrak{D}\}$ *and* $\theta \in (\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\})$. *Suppose*

$\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})^I$ *is both ultra-multiplicative and non-trivially-hereditary* {*while* $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ *is distributive*}. *Then,* θ *is maximal in* $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}$ *if*{*f*} *it is ultra-filtral.* {*(In particular, all elements of* $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ *are filtral.)*}

Proof. First, assume $\theta = \theta^B_{\mathcal{U}}$, for some ultra-filter $\mathcal{U}$ on I, in which case $\mathfrak{C} \triangleq (\mathfrak{D}/\theta^D_{\mathcal{U}}) \in \mathbf{P}^{\mathrm{U}}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$, while $h \triangleq (\Delta_B \circ \nu_{\theta^D_{\mathcal{U}}}) \in \hom(\mathfrak{B},\mathfrak{C})$, whereas $(\ker h) = (\Delta_B)^{-1}_*[\theta^D_{\mathcal{U}}] = \theta$, and so by (2.4) and Footnote 1, as $\theta \neq B^2$, $(\mathfrak{B}/\theta) \in \mathbf{IS}_{>1}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$. Then, by (2.5), $\theta \in \max(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\})$. {Conversely, assume $\theta \in \max(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}) \subseteq \mathrm{MI}(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}))$, in which case, by Lemma 2.6, there is some ultra-filter $\mathcal{U}$ on I such that, as $\theta \neq B^2$, $(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}) \ni \theta^B_{\mathcal{U}} \subseteq \theta$, and so, by the "if" part, $\theta = \theta^B_{\mathcal{U}}$. (Then, Remarks 2.1, 2.3, 2.4, (2.5) and (2.11) complete the argument.)} □

2.2.2. *Subdirect products versus subalgebras.*

Lemma 2.9 (cf. [11]). *Let* $\mathfrak{A} \in \mathsf{A}_\Sigma$ *and* $\mathfrak{B}$ *a subalgebra of* $\mathfrak{A}$. *Then,* $h^B_A \triangleq \{\langle \bar{a}, b\rangle \in (A^\omega \times B) \mid |\omega \setminus \varepsilon_\omega(\bar{a}, \omega \times \{b\})| \in \omega\} \supseteq (\bigcup\{\{\langle \omega \times \{b\}, b\rangle\} \cup \{\langle ((\omega \setminus \{i\}) \times \{b\}) \cup \{\langle i, a\rangle\}, b\rangle \mid i \in \omega, a \in A\} \mid b \in B\})$ *is a function forming a subalgebra of* $\mathfrak{A}^\omega \times \mathfrak{B}$, *in which case it is a surjective homomorphism from* $\mathfrak{C}^B_A \triangleq (\mathfrak{A}^\omega \restriction (\mathrm{dom}\, h^B_A))$ *onto* $\mathfrak{B}$, *and so* $\mathfrak{C}^B_A$ *is a subdirect product of* $\omega \times \{\mathfrak{A}\}$.

2.2.2.1. Filtrality versus non-trivial hereditarity of simplicity.

Corollary 2.10. *Let* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ *be a [relatively] subdirectly principally filtral [pre-]variety. Then,* $(\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma)(\setminus \mathsf{A}^{=1}_\Sigma)$ *is (non-trivially-)hereditary.*

Proof. Let $\mathfrak{A} \in (\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma)$ and $\mathfrak{B}$ a non-one-element subalgebra of $\mathfrak{A}$, in which case $|A| \neq 1$, and so, by Lemma 2.9, $h \triangleq h^B_A$ is a surjective homomorphism from the subdirect product $\mathfrak{C} \triangleq \mathfrak{C}^B_A$ of $(\omega \times \{\mathfrak{A}\}) \in \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})^\omega$ onto $\mathfrak{B}$. Consider any $\theta \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B}) \setminus \{\Delta_B\})$ and take any $\langle a, b\rangle \in (\theta \setminus \Delta_B) \neq \varnothing$, in which case, by (2.5), $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{C}) \ni \vartheta \triangleq h^{-1}_*[\theta] \ni \langle \bar{c}, \bar{d}\rangle \triangleq \langle \omega \times \{a\}, \omega \times \{b\}\rangle$, while $h_*[\vartheta] = \theta$, and so $\vartheta \supseteq \eta \triangleq \mathrm{Cg}^{\mathfrak{C}}_{[\mathsf{P}]}(\langle \bar{c}, \bar{d}\rangle) = \theta^C_{\mathcal{F}}$, for some $\mathcal{F} \in \mathrm{Fi}(\omega)$. Then, $\varnothing = \varepsilon_\omega(\bar{c}, \bar{d}) \in \mathcal{F}$, in which case $\mathcal{F} = \wp(\omega)$, and so $\vartheta \supseteq \eta = C^2$. Thus, $\theta \supseteq h_*[C^2] = B^2$, in which case $\theta = B^2$, and so $\mathfrak{B} \in \mathrm{Si}_{[\mathsf{P}]}(\mathsf{P})$, as required. □

2.2.3. *Locality versus local finiteness.* As an immediate consequence of [20, Lemma 2.1], in its turn, being that of [5, Corollary 2.3], we, first, have the following useful universal observation:

Corollary 2.11. *Any abstract hereditary local subclass of a locally-finite quasi-variety is ultra-multiplicative.*

Aside from quasi-varieties as such, certain representative subclasses of them are local as well.

2.2.3.1. Local subclasses of local pre-varieties.

Lemma 2.12. *Let* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ *be a [local (more specifically, quasi-equational) pre-]variety. Then,* $(\mathrm{SI}^\omega \mid \mathrm{Si})_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma)$ *is local.*

Proof. Consider any $\mathfrak{B} \in (\mathsf{P} \setminus ((\mathrm{SI}^\omega \mid \mathrm{Si})_{[\mathsf{P}]})(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma))$, in which case there are some $\bar{a} \in (B^2 \setminus \Delta_B) \neq \varnothing$, $n \in (\omega|\{1\})$ and $\bar{\theta} \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B}) \setminus (\mathrm{img}\,\bar{\vartheta}^B))^n$, where, for any $C \subseteq B$, $\bar{\vartheta}^C \triangleq (\langle \Delta_C\rangle|\langle \Delta_C, C^2\rangle)$, "such that $(B^2 \cap (\bigcap(\mathrm{img}\,\bar{\theta}))) = \Delta_B$"|, and so some $\langle \bar{b}^{i,j}\rangle^{j\in(1|2)}_{i\in n} \in (\prod^{j\in(1|2)}_{i\in n}((\theta_i \setminus \vartheta^B_j) \cup (\vartheta^B_j \setminus \theta_i))) \neq \varnothing$. Let $\mathfrak{A}$ be the finitely-generated subalgebra of $\mathfrak{B}$ generated by $\{a_0, a_1\} \cup \{b^{i,j}_k \mid i \in n, j \in (1|2), k \in 2\}$, in which case, by (2.5) with $h = \Delta_A$, $\bar{\eta} \triangleq \langle \theta_i \cap A^2\rangle_{i\in n} \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus (\mathrm{img}\,\bar{\vartheta}^A))^n$, as $\langle \bar{b}^{i,j}\rangle^{j\in(1|2)}_{i\in n} \in (\prod^{j\in(1|2)}_{i\in n}((\eta_i \setminus \vartheta^A_j) \cup (\vartheta^A_j \setminus \eta_i)))$, so $\mathfrak{A} \in (\mathsf{P} \setminus ((\mathrm{SI}^\omega\,\mathrm{Si})_{[\mathsf{P}]})(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma))$, for $\bar{a} \in (A^2 \setminus \Delta_A)$ "and $(A^2 \cap (\bigcap(\mathrm{img}\,\bar{\eta}))) = (A^2 \cap (\bigcap(\mathrm{img}\,\bar{\theta}))) = (A^2 \cap \Delta_B) = \Delta_A$"|. □

2.2.3.1.1. Finite semi-simplicity versus semi-simplicity and local finiteness. Lemma 2.12 immediately yields:

Corollary 2.13. *Any locally-finite [relatively] semi-simple [local (more specifically, quasi-equational) pre-]variety* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ *with hereditary* $\mathrm{SI}^{\omega}_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma$ *is [relatively] finitely semi-simple.*

3. Preliminaries: quaternary equational schemes

A *quaternary* Σ*-(equational)scheme* is any $\mho \subseteq \mathrm{Eq}^4_\Sigma$. This is called an *implication scheme for* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, if this satisfies the Σ-implication:

$$(\{x_0 \approx x_1\} \cup \mho) \to (x_2 \approx x_3). \tag{3.1}$$

Likewise, it is called an *identity|reflexive|symmetric|transitive* one, if K satisfies the Σ-implications of the form $(\varnothing|\varnothing|\mho|(\mho \cup (\mho[x_{2+i}/x_{3+i}]_{i\in 2}))) \to \Psi$, where $\Psi \in (\mho([x_3/x_2]|[x_{2+i}/x_i]_{i\in 2}|[x_3/x_2, x_2/x_3]|[x_3/x_4]))$, reflexive symmetric transitive ones being also called *equivalence* ones. Then, $\mho$ is called a *congruence* one, if it is an equivalence one, while, for each $\varsigma \in \Sigma$ of arity $n \in (\omega \setminus 1)$, K satisfies the Σ-implications of the form $(\bigcup_{j\in n}(\mho[x_{2+i}/x_{2+i+(2\cdot j)}]_{i\in 2})) \to \Psi$, where $\Psi \in (\mho[x_{2+i}/\varsigma(\langle x_{2+i+(2\cdot j)}\rangle_{j\in n})]_{i\in 2})$.] Finally, $\mho$ [being finite] is called a *"restricted equationally definable principal {relative} congruence (REDP{R}C)"/"(equational) implicative|disjunctive scheme/system for* a "{pre-}variety"/ $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, if, for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^4$, $(\forall \theta \in (\mathrm{Co}_{\{\mathsf{K}\}}(\mathfrak{A})/\{\Delta_A\}) : (\langle a_0, a_1\rangle \in | \notin \theta) \Rightarrow (\langle a_3, a_3\rangle \in \theta)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i]_{i\in I}$ [cf. [6]/[20]|[19]] /"and so for $\mathbf{IS}[\mathbf{P}^{\mathrm{U}}]\mathsf{K}$, ⟨pre-varieties generated by classes of⟩ Σ-algebras with [finite] implicative|disjunctive system $\mho$ being called ⟨*[finitely]*⟩ $\mho$*-implicative|-disjunctive* with the class of $\mho$-implicative|-disjunctive members of any $\mathsf{K}' \subseteq \mathsf{A}_\Sigma$ denoted by $\mathsf{K}'_\mho$ "in which case $\mho$, being an implication scheme for (the pre-variety generated by) K, providing this is quasi-equational, includes a finite one, by the Compactness Theorem for ultra-multiplicative classes of algebras [14]"|, and so implicative quasi-varieties, being thus finitely so, are exactly those in the original sense of [20]. Then, by Remark 2.4 therein, quasi-equational/finitely implicative pre-varieties are finitely disjunctive.

Given any $\mathfrak{A} \in \mathsf{A}_\Sigma$, let $\mho^{\mathfrak{A}} : \wp(A)^2 \to \wp(A), \langle X, Y\rangle \mapsto \{\langle \phi^{\mathfrak{A}}_0[x_i/a_i, x_{2+i}/b_i]_{i\in 2}, \phi^{\mathfrak{A}}_1[x_i/a_i, x_{2+i}/b_i]_{i\in 2}\rangle \mid \bar{\phi} \in \mho, \bar{a} \in X, \bar{b} \in Y\}$.

Given any $\tau \in \mathrm{Tm}^3_\Sigma$, put

$$\begin{aligned}
\mho^{\supset}_\tau &\triangleq \{\tau \approx (\tau[x_2/x_3])\},\\
\mho^{\partial\supset}_\tau &\triangleq \{(\tau[x_0/x_{2+k}, x_1/x_{3-k}, x_2/(\tau[x_2/x_{2+k}])]) \approx x_{2+k} \mid k \in 2\},\\
\mho^{\vee}_\tau &\triangleq \{(\tau[x_0/\tau, x_1/(\tau[x_2/x_3])]) \approx (\tau[x_0/\tau, x_1/(\tau[x_2/x_3]), x_2/x_3])\},
\end{aligned}$$

in which case $\mho^{\vee}_\tau$ is defined by $\mho^{\supset}_\tau$ according to [20, Remark 2.4].

Remark 3.1. Given any [dual] discriminator $\tau \in \mathrm{Tm}^3_\Sigma$ for a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, $\mho^{([\partial]\supset)/\vee}_\tau$ is a finite implicative/disjunctive system for K. In particular, any [dual] discriminator pre-variety is finitely both implicative and disjunctive. □

This enables us to build easily an example of a non-quasi-equational finitely both implicative and disjunctive pre-variety well-justifying the generic framework of pre-varieties we follow here:

Example 3.2. Let $\Sigma = \{\neg, \nabla, \tau\}$, where $\neg$ and ∇ are unary, while τ is ternary, $\mathfrak{A}$ the Σ-algebra such that $A \triangleq \omega$, $\tau(x_0, x_1, x_2)$ is a (dual) discriminator for $\mathfrak{A}$ and, for all $a \in A$, $\nabla^{\mathfrak{A}}(a) \triangleq \min(a, 1)$, whereas $\neg^{\mathfrak{A}}(a) \triangleq \max(0, a-1)$. Then, by Remark 3.1, the pre-variety P generated by $\mathfrak{A}$, being (dual) τ-discriminator, is finitely both implicative and disjunctive. Let us show, by contradiction, that it is not a quasi-variety. For suppose it is a quasi-variety. By induction on any

$n \in \omega$, put $\neg^{0[+n+1]} x_i \triangleq [\neg\neg^n] x_i [= \neg^n \neg x_i]$, where $i \in 2$, and set $\varepsilon^n \triangleq (\nabla(\neg^n x_0) \approx \nabla(\neg^n x_1))$. Then, given any $N \subseteq \omega$, set $\varepsilon_N \triangleq \{\varepsilon^n | n \in N\}$. Note that the Σ-implication $\varepsilon_\omega \to (x_0 \approx x_1)$ is true in $\mathfrak{A}$, and so in P. Hence, by Remark 2.3, there is some $N \in \wp_\omega(\omega)$ such that the Σ-quasi-identity $\varepsilon_N \to (x_0 \approx x_1)$ is true in $\mathsf{P} \ni \mathfrak{A}$. However, $\mathfrak{A} \models \varepsilon_N[x_i/(i+m+1)]_{i\in 2}$, where $m \triangleq (\bigcup N) \in \omega$, though $(m+1) \neq (m+2)$. This contradiction means that P is not a quasi-variety. □

3.1. **Implicativity versus REDPRC and relative semi-simplicity.**

Lemma 3.3. *Let $\mho \subseteq \mathrm{Eq}^4_\Sigma$ be an implication scheme for a [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$, $\mathfrak{A} \in \mathsf{P}$, $\bar{a}, \bar{b} \in A^2$ and $\theta \triangleq \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\bar{a})$. Suppose $\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i, x_{2+i}/b_i]_{i\in 2}$. Then, $\bar{b} \in \theta$.*

Proof. As (3.1) is true in $\mathsf{P} \ni (\mathfrak{A}/\theta) \models (\bigwedge \mho)[x_i/\nu_\theta(a_i), x_{2+i}/\nu_\theta(b_i)]_{i\in 2}$, while $\bar{a} \in \theta = (\ker \nu_\theta)$, we get $\bar{b} \in \theta$. □

Corollary 3.4. *Let $\mho \subseteq \mathrm{Eq}^4_\Sigma$ be an implication/REDPC scheme for a [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$. Then, $\mathsf{P}_\mho \subseteq / = (\mathrm{Si}_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma)$. In particular, any implicative [pre-]variety is [relatively] both semi-simple and subdirectly representable.*

Proof. Consider any non-one-element $\mathfrak{A} \in \mathsf{P}_\mho$ and $\vartheta \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$, in which case there is some $\bar{a} \in (\vartheta \setminus \Delta_A) \neq \varnothing$, and so, for any $\bar{b} \in A^2$, $\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i, x_{2+i}/b_i]_{i\in 2}$. Then, "by Lemma 3.3"/ $\bar{b} \in \vartheta$, in which case $\vartheta = A^2$, and so $\mathfrak{A}$ is [P-]simple. Conversely, for any $\mathsf{A} \in \mathrm{Si}_{[\mathsf{P}]}(\mathsf{P})$, $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) = \{\Delta_A, A^2\}$, in which case, for all $\bar{a} \in A^4$, as $\langle a_2, a_3 \rangle \in A^2$, we have $(\forall \theta \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) : (a_0 \; \theta \; a_1) \Rightarrow (a_2 \; \theta \; a_3)) \Leftrightarrow ((a_0 = a_1) \Rightarrow (a_2 = a_3))$, and so $\mathfrak{A}$ is $\mho$-implicative, whenever $\mho$ is an REDP[R]C scheme for $\mathsf{P} \ni \mathfrak{A}$. □

Theorem 3.5. *Any $\mho \subseteq \mathrm{Eq}^4_\Sigma$ is an identity congruence implication scheme for a[n equational] pre-variety $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ if[f] it is an REDPC one.*

Proof. The "if" part is immediate. [Conversely, if $\mho$ is an identity congruence implication scheme for K, then, by induction on construction of any $\varphi \in \mathrm{Tm}^\omega_\Sigma$, we conclude that K satisfies the Σ-identities in $\mho[x_{2+i}/(\varphi[x_0/x_i])]_{i\in 2}$, in which case, by Mal'cev Lemma [13] (cf. [6, Lemma 2.1]), for any $\mathfrak{A} \in \mathsf{A}$, $\bar{a} \in A^2$ and $\bar{b} \in \mathrm{Cg}^{\mathfrak{A}}(\bar{a})$, we have $\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i, x_{2+i}/b_i]_{i\in 2}$, and so Lemma 3.3 completes the argument]. □

This, by Lemma 3.3 and the Compactness Theorem for ultra-multiplicative classes of algebras (cf., e.g., [14]), immediately yields:

Corollary 3.6. *Any quasi-variety with REDPRC scheme $\mho$ has a finite one $\subseteq \mho$.*

Theorem 3.7. *Let $\mho \subseteq \mathrm{Eq}^4_\Sigma$. Then, any [(not necessarily) quasi-equational pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is $\mho$-implicative iff it is [relatively (both subdirectly-representable and)] semi-simple with REDP[R]C scheme $\mho$, in which case $((\mathrm{SI} \,|\, \mathrm{Si})_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma) = \mathsf{P}_\mho$.*

Proof. If P is $\mho$-implicative, that is, is the pre-variety generated by $\mathsf{P}_\mho$, then, for any $\mathfrak{A} \in \mathsf{P}$ and $\bar{a} \in A^4$ such that $\mathfrak{A} \not\models (\bigwedge \mho)[x_i/a_i]_{i\in 4}$, by (2.8), there are some $\mathfrak{B} \in \mathsf{P}_\mho$ and $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\mathfrak{B} \not\models (\bigwedge \mho)[x_i/h(a_i)]_{i\in 4}$, that is, $h(a_{0|2}) = | \neq h(a_{1|3})$, in which case, by (2.4), $\langle a_{0|2}, a_{1|3} \rangle \in | \not\in (\ker h) \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A})$, and so Remark 2.4, Lemma 3.3 and Corollary 3.4 complete the argument. □

3.1.1. *Implicativity versus filtrality.*

Definition 3.8. Given any $n \in \omega$, a $\mho \subseteq \mathrm{Eq}^{2\cdot(n+1)}_\Sigma$ is called a(n)/an *"restricted equationally definable n-generated [relative] congruence (n-REDG[R]C) scheme"/ "(equational) n-multiple{-premise} implicative system" for* a "[pre-]variety"/ $\mathsf{K} \subseteq \mathsf{P}_\Sigma$, if for each $\mathfrak{A} \in \mathsf{K}$ and every $\bar{a} \in (A^2)^{n+1}$, $(\forall \theta \in (\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A})/\{\Delta_A\}) : ((\bar{a} \restriction n) \in \theta^n) \Rightarrow (a_n \in \theta)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_{i+j}/\pi_j(a_i)]_{i\in(n+1), j\in 2})$. □

Given any $\mho \subseteq \mathrm{Eq}^4_\Sigma$, by induction on any $n \in \omega$, define $\mho_n \subseteq \mathrm{Eq}^{2\cdot(n+1)}_\Sigma$ by $\mho_0 \triangleq \{x_0 \approx x_1\}$ and $\mho_{n+1} \triangleq (\bigcup\{\mho[x_{2+i}/\varphi_i]_{i\in 2} \mid \bar{\varphi} \in (\mho_n[x_j/x_{j+2}]_{j\in(2\cdot(n+1))})\})$.

Lemma 3.9. *For any [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ with a REDP[R]C scheme $\mho \subseteq \mathrm{Eq}^4_\Sigma$ and any $n \in \omega$, $\mho_n$ is an n-REDPGRC scheme for P.*

Proof. By induction on n. For consider any $\mathfrak{A} \in \mathsf{P}$, in which case Δ_A is the least [P-]congruence of $\mathfrak{A}$, and so $\mho_0$ is a 0-REDPGRC scheme for P. Now, assume $\mho_n$ is an n-REDPGRC scheme for P and consider any $\bar{a} \in (A^2)^{n+2}$, in which case, by the right alternative of (2.6) with $\mathfrak{B} = (\mathfrak{A}/\theta) \in \mathsf{P}$ and $h = \nu_\theta \in \hom^{\mathrm{S}}(\mathfrak{A},\mathfrak{B})$, where $\theta \triangleq \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\bar{a}[n]) \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A})$, as $\theta = (\ker h)$, we have $(a_{n+1} \in \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\bar{a}[n{+}1]) = \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\theta\cup\{a_n\})) \Leftrightarrow (h_*(a_{n+1}) \in \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(\{h_*(a_n)\})) \Leftrightarrow (\mathfrak{B} \models (\bigwedge\mho)[x_{(2\cdot i)+j}/h(\pi_j(a_{n+i}))]_{i,j\in 2}) \Leftrightarrow (\mho^{\mathfrak{A}}(\{a_n\},\{a_{n+1}\}) \subseteq \theta) \Leftrightarrow (\mathfrak{A} \models (\bigwedge\mho_{n+1})$ $[x_{k+l}/\pi_l(a_k)]_{k\in(n+2),l\in 2})$, and so $\mho_{n+1}$ is an $(n+1)$-REDPGRC scheme for P. □

Corollary 3.10. *Let $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ be a ([quasi-]equational) [pre-]variety with an REDPC scheme $\mho \subseteq \mathrm{Eq}^4_\Sigma$, $I \in \Upsilon$, $\overline{\mathfrak{A}} \in \mathsf{P}^I$, $\mathfrak{B}$ a subalgebra of $\prod_{i\in I}\mathfrak{A}_i$ and $X \in \wp_\omega(B^2)$. Then, $\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(X) = (B^2 \cap (\bigcap_{i\in I}(\pi_i{\restriction}B)^{-1}_*[\mathrm{Cg}^{\mathfrak{A}_i}_{[\mathsf{P}]}((\pi_i{\restriction}B)_*[X])]))$. (In particular, $\mathrm{Co}_{[\mathsf{P}]}$ $(\mathfrak{B}) = \{B^2(\bigcap_{i\in I}(\pi_i{\restriction}B)^{-1}_*[\theta_i]) \mid \bar{\theta} \in (\prod_{i\in I}\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}_i))\}$.)*

Proof. Take any bijection $\bar{a}$ from $n \triangleq |X| \in \omega$ onto X, in which case, by Lemma 3.9, for all $\bar{b} \in B^2$, we have $(\bar{b} \in \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(X)) \Leftrightarrow (\mathfrak{B} \models (\bigwedge\mho_n)[x_{j+k}/\pi_k(a_j); x_{(2\cdot n)+l}/$ $b_l]_{j\in n;k,l\in 2}) \Leftrightarrow (\forall i \in I : \mathfrak{A}_i \models (\bigwedge\mho_n)[x_{j+k}/\pi_i(\pi_k(a_j)); x_{(2\cdot n)+l}/\pi_i(b_l)]_{j\in n;k,l\in 2}) \Leftrightarrow$ $(\forall i \in I : (\pi_i{\restriction}B)_*(\bar{b}) \in \mathrm{Cg}^{\mathfrak{A}_i}_{[\mathsf{P}]}((\pi_i{\restriction}B)_*[X])) \Leftrightarrow (\bar{b} \in (B^2 \cap (\bigcap_{i\in I}(\pi_i{\restriction}B)^{-1}_*[\mathrm{Cg}^{\mathfrak{A}_i}_{[\mathsf{P}]}((\pi_i{\restriction}$ $B)_*[X])])))$ (and so, for every $\eta \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$, since, by Remark 2.3, $\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}$ is finitary, while, for each $i \in I$, $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}_i)$, being inductive, contains $\vartheta_i \triangleq (\bigcup_{Y\in\wp_\omega(\eta)}\mathrm{Cg}^{\mathfrak{A}_i}_{[\mathsf{P}]}((\pi_i{\restriction}$ $B)_*[Y]))$, because $\{\mathrm{Cg}^{\mathfrak{A}_i}_{[\mathsf{P}]}((\pi_i{\restriction}B)_*[Z]) \mid Z \in \wp_\omega(\eta)\} \subseteq \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}_i)$ is upward-directed, as $\wp_\omega(\eta)$ is so, we get $\eta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(\eta) = (\bigcup\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}[\wp_\omega(\eta)]) = (B^2(\bigcap_{i\in I}(\pi_i{\restriction}B)^{-1}_*[\vartheta_i]))$, (2.5) then completing the argument).

□

Theorem 3.11. *Any [quasi-]equational/ [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is implicative iff it is [relatively] /"both subdirectly-representable and" (subdirectly) /"finitely|principally" filtral.*

Proof. First, assume P is {both quasi-equational and} $\mho$-implicative, for some $\mho \subseteq \mathrm{Eq}^4_\Sigma$, in which case, by Corollary 3.4, P is [relatively] both subdirectly-representable and semi-simple, while $\mathsf{K} \triangleq \mathsf{P}_\mho \supseteq \mathrm{Si}_{[\mathsf{P}]}(\mathsf{P})$ is both abstract and hereditary, whereas $\mathsf{P} = \mathbf{ISP}\mathsf{K} = \mathbf{IP}^{\mathrm{SD}}\mathbf{S}\mathsf{K}$. Consider any set I, any $\overline{\mathfrak{A}} \in (\mathrm{Si} \parallel \mathrm{SI})_{[\mathsf{P}]}(\mathsf{P})^I$, any subalgebra $\mathfrak{B}$ of $\prod\overline{\mathfrak{A}}$, any $\{Y \subseteq\}X \subseteq_\omega \{Z \subseteq\}B^2$ and any $\bar{b} \in B^2$. Let $I_X \triangleq (I\cap(\bigcap\varepsilon_I^{-1}[X]))\{\subseteq$ $I_Y\} \subseteq I$, in which case $\mathrm{Fi}(I) \ni \mathcal{F}_X \triangleq \wp(I_X, I)\{\supseteq \mathcal{F}_Y$, and so $\mathcal{G}_Z \triangleq \{\mathcal{F}_W \mid W \in \wp_\omega(Z)\} \subseteq \mathrm{Fi}(I)$ is upward-directed. Then, $\mathrm{Fi}(I)$, being inductive, contains $\mathcal{H}_Z \triangleq (\bigcup\mathcal{G}_Z)$.} Take any bijection $\bar{a}$ from $n \triangleq |X| \in \omega$ onto X, in which case $\mho_n$ is both an n-multiple implicative system for K, and, by Theorem 3.7 and Lemma 3.9, an n-REDG[R]C scheme for P. Then, as $(\mathrm{img}\prod\overline{\mathfrak{A}}) \subseteq \mathrm{Si}_{[\mathsf{P}]}(\mathsf{P}) \subseteq \mathsf{K}$, $(\bar{b} \in$ $\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(X)) \Leftrightarrow (\mathfrak{B} \models (\bigwedge\mho_n)[x_{j+k}/\pi_k(a_j); x_{(2\cdot n)+l}/b_l]_{j\in n;k,l\in 2}) \Leftrightarrow (\forall i \in I : \mathfrak{A}_i \models$ $(\bigwedge\mho_n)[x_{j+k}/\pi_i(\pi_k(a_j)); x_{(2\cdot n)+l}/\pi_i(b_l)]_{j\in n;k,l\in 2}) \Leftrightarrow (\forall i \in I : (i \in I_X) \Rightarrow (i \in$ $\varepsilon_I(\bar{b}))) \Leftrightarrow (I_X \subseteq \varepsilon_I(\bar{b})) \Leftrightarrow (\varepsilon_I(\bar{b}) \in \mathcal{F}_X) \Leftrightarrow (\bar{b} \in \theta^B_{\mathcal{F}_X})$, in which case $\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(X) = \theta^B_{\mathcal{F}_X}$ {and so, by Remark 2.3 and (2.11), $\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(Z) = \theta^B_{\mathcal{H}_Z}$}.

Conversely, assume P is [relatively] /"both subdirectly-representable and" subdirectly principally filtral, in which case, by "Remark 2.4 as well as"/ Footnote 1, Lemma 2.5 and Corollary 2.10, P is [relatively] both subdirectly-representable and semi-simple with abstract and non-trivially-hereditary $\mathsf{K} \triangleq (\mathrm{Si} \parallel \mathrm{SI})_{[\mathsf{P}]}(\mathsf{P})$. Let $I \triangleq \{\theta \in \mathrm{Co}_{\mathsf{K}}(\mathfrak{Tm}^4_\Sigma) \mid (x_0\ \theta\ x_1) \Rightarrow (x_2\ \theta\ x_3)\}$, $\overline{\mathfrak{A}} \triangleq \langle \mathfrak{A}/i\rangle_{i\in I} \in \mathsf{K}^I$, $\mathfrak{D} \triangleq (\prod \overline{\mathfrak{A}})$, $h \triangleq (\prod_{i\in I} \nu_i)$ and $\bar{a} \triangleq \langle h(v_j)\rangle_{j\in 4}$, in which case, by (2.2) and (2.7), $h \in \hom(\mathfrak{Tm}^4_\Sigma, \mathfrak{D})$, while $\mathfrak{B} \triangleq (\mathfrak{D}{\upharpoonright}(\mathrm{img}\, h))$ is a subdirect product of $\overline{\mathfrak{A}}$, whereas $h \in \hom^{\mathrm{S}}(\mathfrak{Tm}^4_\Sigma, \mathfrak{B})$, and so $\vartheta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(\langle a_0, a_1\rangle) = \theta^B_{\mathcal{F}}$, for some $\mathcal{F} \in \mathrm{Fi}(I)$. Then, $\langle a_0, a_1\rangle \in \vartheta$, in which case $\varepsilon_I(\langle a_2, a_3\rangle) \supseteq \varepsilon_I(\langle a_0, a_1\rangle) \in \mathcal{F}$, and so $\varepsilon_I(\langle a_2, a_3\rangle) \in \mathcal{F}$, i.e., $\langle a_2, a_3\rangle \in \vartheta$. Let $\mho \triangleq (\ker h) \subseteq \mathrm{Eq}^4_\Sigma$. Consider any $\mathfrak{C} \in \mathsf{K}$ and $g \in \hom(\mathfrak{Tm}^4_\Sigma, \mathfrak{C})$. Then, providing $\mho \subseteq \eta \triangleq (\ker g) \ni \langle x_0, x_1\rangle$, by the Homomorphism Theorem, $f \triangleq (h^{-1} \circ g) \in \hom(\mathfrak{B}, \mathfrak{C})$, in which case, by (2.5), $\langle a_0, a_1\rangle \in \zeta \triangleq (\ker f) = f^{-1}_*[\Delta_C] \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$, and so $\langle a_2, a_3\rangle \in \vartheta \subseteq \zeta$. In that case, $\langle x_2, x_3\rangle \in \eta$. Now, assume $(\langle x_0, x_1\rangle \in \eta) \Rightarrow (\langle x_2, x_3\rangle \in \eta)$, in which case $\mho \subseteq \eta$, i.e., $\mathfrak{C} \models (\bigwedge \mho)[g]$, whenever $\eta = \mathrm{Eq}^4_\Sigma$. Otherwise, by the $\{\}$-optional version of the right alternative of (2.4), $\eta \in I$, in which case, by (2.1), $\mho \subseteq \eta$, i.e., $\mathfrak{C} \models (\bigwedge \mho)[g]$, and so $\mho$ is an implicative system for K. Thus, P, being [relatively] subdirectly-representable, is $\mho$-implicative. □

Corollary 3.12. *Any finitely implicative pre-variety is relatively both subdirectly-representable and filtral.*

Proof. Any implicative system $\mho \subseteq_\omega \mathrm{Eq}^4_\Sigma$ for any $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ is so for $\mathbf{P}^{\mathrm{U}}\mathsf{K}$, in which case $\mathbf{ISPP}^{\mathrm{U}}\mathsf{K} \supseteq \mathbf{ISP}\mathsf{K}$ is $\mho$-implicative, and so Theorem 3.11 ends the proof. □

Whether the converse holds remains an open problem.

3.1.2. *Generic identity equivalence implication schemes for distributive lattice expansions.* Here, it is supposed that $\Sigma_+ \subseteq \Sigma$. Given any $\mathfrak{A} \in \mathsf{A}_\Sigma$, $X \subseteq A$ and $\Omega \subseteq \mathrm{Tm}^1_\Sigma$, we have $\Omega^{\mathfrak{A}}_X : A \to \wp(\Omega), a \mapsto \{\varphi \in \Omega \mid \varphi^{\mathfrak{A}}(a) \in X\}$.

Given any $\bar\varphi \in (\mathrm{Tm}^1_\Sigma)^*$ with $x_0 \in \Xi \triangleq (\mathrm{img}\, \bar\varphi)$, $\iota \in \Omega \in \wp(V_1, \Xi)$, $i \in 2$ and $\Delta \in \wp(\Xi)$, let $\varepsilon^{i,\iota}_{\bar\varphi,\Delta} \triangleq ((\wedge_+ \langle (\bar\varphi \cap \Delta) * ((\bar\varphi \cap \Delta) \circ [x_0/x_1]), \iota(x_{2+i})\rangle) \lessapprox (\vee_+ \langle (\bar\varphi \setminus \Delta) * ((\bar\varphi \setminus \Delta) \circ [x_0/x_1]), \iota(x_{3-i})\rangle)) \in \mathrm{Eq}^4_\Sigma$ and $\mho^{\bar\varphi}_\Omega \triangleq \{\varepsilon^{i,\iota}_{\bar\varphi,\Delta} \mid i \in 2, \iota \in \Omega, \Delta \in \wp(\Xi)\} \in \wp_\omega(\mathrm{Eq}^4_\Sigma)$.

Lemma 3.13. *Let $\mathfrak{A}$ be a Σ-algebra with (distributive) lattice Σ_+-reduct, $\bar\varphi \in (\mathrm{Tm}^1_\Sigma)^*$ with $x_0 \in \Xi \triangleq (\mathrm{img}\, \bar\varphi)$ and $\Omega \in \wp(V_1, \Xi)$. Then, $\mho^{\bar\varphi}_\Omega$ is an identity reflexive symmetric (transitive implication) scheme for $\mathfrak{A}$.*

Proof. Clearly, for all $j \in 2$, $\iota \in \Xi$ and $\Delta \in \wp(\Xi)$, there are some $\phi, \psi, \xi \in \mathrm{Tm}^3_\Sigma$ such that $(\varepsilon^{j,\iota}_{\bar\varphi,\Delta}[x_3/x_2]) = ((\phi \wedge \xi) \lessapprox (\psi \vee \xi))$, in which case this is satisfied in lattice Σ-expansions, and so in $\mathfrak{A}$. Likewise, there are then some $\bar\eta, \bar\zeta \in (\mathrm{Tm}^2_\Sigma)^+$ with $((\mathrm{img}\, \bar\eta) \cap (\mathrm{img}\, \bar\zeta)) \neq \varnothing$ such that $(\varepsilon^{j,\iota}_{\bar\varphi,\Delta}[x_{2+i}/x_i]_{i\in 2}) = ((\wedge_+ \bar\eta) \lessapprox (\vee_+ \bar\zeta))$, in which case this is satisfied in lattice Σ-expansions, and so in $\mathfrak{A}$. Furthermore, $(\mho^{\bar\varphi}_\Omega[x_2/x_3, x_3/x_2]) = \mho^{\bar\varphi}_\Omega$. (Next, since the Σ_+-quasi-identity $\{(x_0 \wedge x_1) \lessapprox (x_2 \vee x_3), (x_0 \wedge x_3) \lessapprox (x_2 \vee x_4)\} \to ((x_0 \wedge x_1) \lessapprox (x_2 \vee x_4))$, being satisfied in distributive latices, is so in $\mathfrak{A}$, so are logical consequences of its substitutional Σ-instances $(\mho^{\bar\varphi}_\Omega \cup (\mho^{\bar\varphi}_\Omega[x_{2+i}/x_{3+i}]_{i\in 2})) \to \Psi$, where $\Psi \in (\mho^{\bar\varphi}_\Omega[x_3/x_4])$. Finally, consider any $a \in A$ and $\bar b \in (A^2 \setminus \Delta_A)$, in which case, by the Prime Ideal Theorem, there are some $k \in 2$ and some prime filter F of $\mathfrak{A}$ such that $b_k \in F \not\ni b_{1-k}$, and so, as $\Delta \triangleq \Xi^{\mathfrak{A}}_F(a) \in \wp(\Xi)$ and $x_0 \in \Omega$, $\mathfrak{A} \not\models (\bigwedge \mho^{\bar\varphi}_\Omega)[x_i/a, x_{2+i}/b_i]_{i\in 2}$, for $\mathfrak{A} \not\models \varepsilon^{k,x_0}_{\bar\varphi,\Delta}[x_i/a, x_{2+i}/b_i]_{i\in 2}$.) □

This, by Corollary 3.4, immediately yields:

Corollary 3.14. *Let $\mathfrak{A}$ be a non-one-element Σ-algebra with distributive lattice Σ_+-reduct, $\bar\varphi \in (\mathrm{Tm}^1_\Sigma)^*$ with $x_0 \in \Xi \triangleq (\mathrm{img}\, \bar\varphi)$ and $\Omega \in \wp(V_1, \Xi)$. Suppose $\mho^{\bar\varphi}_\Omega$ is an implicative system for $\mathfrak{A}$. Then, $\mathfrak{A}$ is simple.*

3.1.2.1. Equality determinants versus implicativity. Recall that a *(logical) Σ-matrix* is any pair $\mathcal{A} = \langle\mathfrak{A}, D\rangle$ with a Σ-algebra $\mathfrak{A}$ and a $D \subseteq A$, in which case an $\Omega \subseteq \mathrm{Tm}^1_\Sigma$ is called an *equality/identity determinant for* $\mathcal{A}$, if $\Omega^{\mathfrak{A}}_D$ is injective (cf. [19]), and so one *for* a class M of Σ-matrices, if it is so for each member of M.

Theorem 3.15. *Let* M *be a class of* Σ*-matrices and* $\bar{\varphi} \in (\mathrm{Tm}^1_\Sigma)^*$ *with* $x_0 \in \Xi \triangleq (\mathrm{img}\,\bar{\varphi})$. *Suppose, for all* $\mathcal{A} \in \mathsf{M}$, $\pi_0(\mathcal{A})\restriction\Sigma_+$ *is a distributive lattice with set of its prime filters* $\pi_1[\mathsf{M} \cap \pi_0^{-1}[\{\pi_0(\mathcal{A})\}]]$. *Then,* Ξ *is an equality determinant for* M *iff* $\mho^{\bar{\varphi}}_{V_1}$ *is an implicative system for* $(\mathbf{IS}_{[>1]}\{\mathbf{P}^{\mathrm{U}}\})\pi_0[\mathsf{M}]$ *([in which case its members are simple]).*

Proof. Let $\mathcal{A} = \langle\mathfrak{A}, D\rangle \in \mathsf{M}$, $\bar{a} \in A^2$ and, for any $\bar{b} \in A^2$, $h_{\bar{b}} \triangleq [x_i/a_i, x_{2+i}/b_i]_{i\in 2}$. First, assume Ξ is an equality determinant for M. Consider any $\bar{b} \in A^2$. Assume $\mathfrak{A} \not\models \varepsilon^{j,x_0}_{\bar{\varphi},\Delta}[h_{\bar{b}}]$, for some $j \in 2$ and $\Delta \subseteq \Xi$, in which case, by the Prime Ideal Theorem, $\exists \mathcal{B} = \langle\mathfrak{A}, D'\rangle \in \mathsf{M} : \forall k \in 2 : \Delta = \Xi^{\mathfrak{A}}_{D'}(a_k)$, and so $a_0 = a_1$. Then, by Lemma 3.13 with $\Omega = \Xi$, $\mho^{\bar{\varphi}}_{V_1}$ is an implicative system for $\mathfrak{A}$. Conversely, assume $\mho^{\bar{\varphi}}_{V_1 r}$ is an implicative system for $\mathfrak{A}$ and $\Delta \triangleq \Xi^{\mathfrak{A}}_D(a_0) = \Xi^{\mathfrak{A}}_D(a_1)$. Take any $\bar{b} \in (D \times (A \setminus D)) \neq \varnothing$, in which case, as $\Delta \subseteq \Xi \ni x_0$, $\mathfrak{A} \not\models \varepsilon^{0,x_0}_{\bar{\varphi},\Delta}[h_{\bar{b}}]$, for D is a prime filter of $\mathfrak{A}\restriction\Sigma_+$, and so $a_0 = a_1$. (Finally, Corollary 3.14 completes the argument.) □

3.2. **Disjunctivity.** Unless otherwise specified, fix any $\mho \subseteq \mathrm{Eq}^4_\Sigma$.

3.2.1. *Disjunctivity versus finite subdirect irreducibility and congruence-distributivity.*

Lemma 3.16. *Any* $\mho$*-disjunctive /finite non-one-element* $\mathfrak{A} \in \mathsf{A}_\Sigma$ *is finitely/ subdirectly-irreducible. In particular, any disjunctive pre-variety is (relatively) finitely subdirectly-representable.*

Proof. Consider any $\theta, \vartheta \in (\mathrm{Co}(\mathfrak{A})\setminus\{\Delta_A\})$ and take any $(\bar{a}|\bar{b}) \in ((\theta|\vartheta)\setminus\{\Delta_A\}) \neq \varnothing$, in which case the Σ-identities in $\mho[x_{1|3}/x_{0|2}]$, being true in $\mathfrak{A}$, are so in $\mathfrak{A}/(\theta|\vartheta)$ (in particular, under $[x_{0|2}/\nu_{\theta|\vartheta}((a|b)_0), x_{(2|0)+i}/\nu_{\theta|\vartheta}((b|a)_i)]_{i\in 2}$), and so $\Delta_A \not\supseteq \{\langle\phi^{\mathfrak{A}}[x_i/a_i, x_{2+i}/b_i]_{i\in 2}, \phi^{\mathfrak{A}}[x_i/a_i, x_{2+i}/b_i]_{i\in 2}\rangle \mid (\phi \approx \psi) \in \mho\} \subseteq (\theta \cap \vartheta)$. Then, $(\theta \cap \vartheta) \neq \Delta_A$. Thus, induction on the cardinality of finite subsets of $\mathrm{Co}(\mathfrak{A})$ ends the proof. □

Lemma 3.17. *Let* $\mathsf{P} \subseteq \mathsf{A}_\Sigma \ni \mathfrak{A}$ *be a* $\mho$*-disjunctive pre-variety and* $X, Y, Z \subseteq A^2$. *Then,* $\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\mho^{\mathfrak{A}}(X, Y) \cup Z) = (\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(X \cup Z) \cap \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(Y \cup Z))$.

Proof. In that case, P is generated by $\mathsf{K} \triangleq \mathsf{P}_\mho = \mathbf{IS}\mathsf{K}$, so, by Remark 2.2 and (2.8), $\mathrm{Co}_{\mathsf{K}}(\mathfrak{A})$ is a basis of $\mathrm{Co}_{\mathsf{P}}(\mathfrak{A})$. Then, for any $\theta \in \mathrm{Co}_{\mathsf{K}}(\mathfrak{A})$, $\mathfrak{A}/\theta$ is $\mho$-disjunctive, in which case $(\mho^{\mathfrak{A}}(X, Y) \cup Z) \subseteq \theta$ iff either $(X \cup Z) \subseteq \theta$ or $(Y \cup Z) \subseteq \theta$, and so, for any $\bar{a} \in A^2$, $(\bar{a} \in \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\mho^{\mathfrak{A}}(X, Y) \cup Z)) \Leftrightarrow (\forall\theta \in \mathrm{Co}_{\mathsf{K}}(\mathfrak{A}) : ((\mho^{\mathfrak{A}}(X, Y) \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta)) \Leftrightarrow ((\forall\theta \in \mathrm{Co}_{\mathsf{K}}(\mathfrak{A}) : (X \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta))\&(\forall\theta \in \mathrm{Co}_{\mathsf{K}}(\mathfrak{A}) : ((Y \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta)) \Leftrightarrow (\bar{a} \in (\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(X \cup Z) \cap \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(Y \cup Z)))$, as required. □

Corollary 3.18. *Any* $\mho$*-disjunctive [pre-]variety* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ *is [relatively] congruence-distributive, and so is any [quasi-equational/finitely] implicative one.*

Proof. Then, by Lemma 3.17, for any $\mathfrak{A} \in \mathsf{P}$ and $\theta, \vartheta, \eta \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{A})$, we have $(\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\theta \cup \eta) \cap \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\vartheta \cup \eta)) = \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\mho^{\mathfrak{A}}(\theta, \vartheta) \cup \eta) = \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\mho^{\mathfrak{A}}(\theta, \vartheta)) \cup \eta) = \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}((\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\theta) \cap \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\vartheta)) \cup \eta) = \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}((\theta \cap \vartheta) \cup \eta)$, as required. □

Lemma 3.19. *Let* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ *be a* $\mho$*-implicative pre-variety and* $\mho'$ *a disjunctive system for* $\mathsf{P}_\mho$. *Then, every* $\mho'$*-disjunctive member of* P *is* $\mho$*-implicative.*

Proof. In that case, $\mho$, being is an identity implication scheme for $\mathsf{P}_\mho$, is so for $\mathsf{P} = \mathbf{ISP}\mathsf{P}_\mho$, while the Σ-identities in $\bigcup\{\mho'[x_{2+i}/\varphi_i]_{i\in 2} \mid \bar{\varphi} \in \mho\}$, being true in $\mathsf{P}_\mho$, are so in P, and so $\mho'$-disjunctive members of P are $\mho$-implicative, as required. □

Corollary 3.20. *For any $\mho$-disjunctive [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$, $\mathsf{P}_\mho = (\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1})$. In particular, any [quasi-equational/finitely] implicative [pre-]variety is [relatively] finitely semi-simple.*

Proof. Then, any one-element Σ-algebra is $\mho$-disjunctive, while, for any $\mathfrak{A} \in \mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$ and $\bar{a}, \bar{b} \in (A^2 \setminus \Delta_A)$, since $\mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\bar{a}|\bar{b}) \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$, whereas, by Lemma 3.17, $(\mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\bar{a}) \cap \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\bar{b})) = \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\mho^{\mathfrak{A}}(\bar{a}|\bar{b}))$, we have $\mho^{\mathfrak{A}}(\bar{a}|\bar{b}) \neq \Delta_A = \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\Delta_A)$, i.e., $\mathfrak{A} \not\models (\bigwedge \mho)[x_i/a_i, x_{2+i}/b_i]_{i\in 2}$, in which case $\mathfrak{A}$ is $\mho$-disjunctive, because the Σ-identities in $\bigcup_{j\in 2} \mho[x_{(2\cdot j)}/x_{(2\cdot j)+1}]$, being true in $\mathsf{P}_\mho$, are so in $\mathbf{ISP}\mathsf{P}_\mho = \mathsf{P} \ni \mathfrak{A}$, and so Lemmas 3.4, 3.16, 3.19 and [20, Remark 2.4] complete the argument. □

Theorem 3.21. *Any [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is disjunctive iff it is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1}$ being "a universal (infinitary) model class"/hereditary.*

Proof. The "only if" part is by Lemma 3.2.1 and Corollary 3.20. Conversely, assume P is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with hereditary $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1}$, in which case, by Remark 2.4, it is [relatively] finitely-subdirectly-representable, while, by (2.5), $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma) \cap \wp(\theta, \mathrm{Eq}^4_\Sigma)$, where $\theta \triangleq (\mathrm{Eq}^4_\Sigma \cap (\bigcap \mathrm{Co}_{\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})}(\mathfrak{Tm}^4_\Sigma))) \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma)$, is distributive, for $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma/\theta)$ is so. Let $\forall j \in 2 : \vartheta_j \triangleq \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\theta \cup \{\langle x_{2\cdot j}, x_{(2\cdot j)+1}\rangle\}) \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma) \cap \wp(\theta, \mathrm{Eq}^4_\Sigma)) \ni \mho \triangleq (\vartheta_0 \cap \vartheta_1) \subseteq \mathrm{Eq}^4_\Sigma$. Consider any $\mathfrak{A} \in \mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$ and any $\bar{a} \in A^4$. Let $h \in \hom(\mathfrak{Tm}^4_\Sigma, \mathfrak{A})$ extend $\{\langle x_i, a_i\rangle \mid i \in 4\}$, in which case $\mathfrak{B} \triangleq (\mathfrak{A}{\restriction}(\mathrm{img}\, h)) \in (\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1})$, and so $((\{\langle a_0, a_1\rangle, \langle a_2, a_3\rangle\} \cap \Delta_A) \neq \varnothing) \;\&|\Leftrightarrow (\mathfrak{A} \models \Phi^4_\mho[h{\restriction}V_4])$, unless $\mathfrak{B} \in \mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$. Otherwise, by (2.5) and the Homomorphism Theorem, $\theta \subseteq \eta \triangleq (\ker h) \in \mathrm{MI}^\omega(\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma))$, in which case we have:

$$(\mathfrak{A} \models \Phi^4_\mho[h{\restriction}V_4]) \Leftrightarrow ((\vartheta_0 \cap \vartheta_1) = \mho \subseteq \eta) \Leftrightarrow (\eta = \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup (\vartheta_0 \cap \vartheta_1)) =$$
$$(\mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_0) \cap \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_1)) \Leftrightarrow (\exists j \in 2 : \eta = \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_j)) \Leftrightarrow$$
$$(\exists j \in 2 : \vartheta_j \subseteq \eta) \Leftrightarrow (\exists j \in 2 : \langle x_{2\cdot j}, x_{(2\cdot j)+1}\rangle \in \eta) \Leftrightarrow (\exists j \in 2 : a_{2\cdot j} = a_{(2\cdot j)+1}),$$

and so $\mho$ is a disjunctive system for $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$. Thus, P, being [relatively] finitely-subdirectly-representable, is $\mho$-disjunctive, as required. □

This, by Remark 2.4 and Corollary 3.20 (as well as the Compactness Theorem for ultra-multiplicative classes; cf., e.g., [14]), immediately yields:

Corollary 3.22. *Any [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is (finitely) disjunctive iff it is [relatively] congruence-distributive with $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathsf{A}_\Sigma^{=1}$ being "a universal (first-order) model class"/"hereditary (and ultra-multiplicative)".*

This, in its turn, by Footnote 1, Corollary 2.11 and Lemma 2.12, immediately yields:

Corollary 3.23. *Any locally-finite [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is (finitely) disjunctive iff it is [relatively] congruence-distributive with $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathsf{A}_\Sigma^{=1}$ being "a universal {infinitary} model class"/hereditary.*

Finally, this, by the congruence-distributivity of lattice expansions (cf., e.g., [16]) and Corollary 2.7, immediately yields:

Corollary 3.24. *Suppose $\Sigma_+ \subseteq \Sigma$. Then, any finitely-generated variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ of lattice expansions with non-trivially-hereditary $\mathrm{SI}^{(\omega)}(\mathsf{V})$ is finitely disjunctive.*

This provides an immediate (though far from being constructive) insight into the finite disjunctivity of the finitely-generated variety of distributive/Stone|"De Morgan" lattices/algebras|algebras||lattices, a constructive one being given by [18, Example 1/2] and [19, Lemma 11].

3.2.1.1. Implicativity versus finite semi-simplicity and disjunctivity. By Footnote 1, Theorem 3.11, Corollaries 2.8, 2.11, 2.13, 3.20, 3.22, 3.23, Lemma 2.12 and [20, Remark 2.4], we eventually get:

Theorem 3.25. *Any locally-finite/ [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is implicative iff it is /finitely both disjunctive and [relatively] semi-simple iff it is [relatively] both congruence-distributive and semi-simple with $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathsf{A}_\Sigma^{=1}$ being "a universal /first-order model class"| "hereditary /"and ultra-multiplicative"".*

This, by the congruence-distributivity of lattice expansions (cf., e.g., [16]), Corollaries 2.7, 3.4 and Footnote 1, immediately yields:

Corollary 3.26. *Suppose $\Sigma_+ \subseteq \Sigma$. Then, any locally-finite variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ of lattice expansions is implicative iff it is semi-simple "and (finitely) disjunctive"| "with non-trivially-hereditary $(\mathrm{Si} \mid \mathrm{SI})(\mathsf{V})$".*

Corollary 3.27. *Suppose $\Sigma_+ \subseteq \Sigma$. Let $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ be a finite set of finite lattice expansions without non-simple non-one-element subalgebras and V the variety generated by K. Then, V is implicative with $(\mathrm{Si} \mid \mathrm{SI})(\mathsf{V}) = \mathbf{IS}_{>1}\mathsf{K}$.*

These provide an immediate /{though far from being constructive} insight into the not/ implicativity of (and so not/ REDPC for; cf. Theorem 3.7) the not/ semi-simple finitely-generated variety of Stone/distributive|"De Morgan" algebras/lattices|algebras||lattices /(cf. [8]|[21]||) /"a constructive one being given by Theorem 3.15 and [18, Example 1]|"Remark 4.3"".

Whether the /-alternative stipulations are necessary in Theorem 3.25 remains an open issue. On the other hand, the necessity of the "[relative] congruence-distributivity"//"lattice expansion" stipulation therein// as well as in Corollaries 3.22, 3.23, 3.24, 3.26, 3.27 and Theorem 3.21 is demonstrated by:

Example 3.28. Let $\Sigma = \{\wedge\}$ and SL the variety of semi-lattices, in which case, for any filter $F \neq A$ of any $\mathfrak{A} \in \mathsf{SL}$, χ^F_A is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{S}_2 \in \mathsf{SL}$ with $S_2 \triangleq 2$ and $\wedge^\mathfrak{A} \triangleq (\cap \restriction 2^2)$, and so, by (2.8), $\mathsf{SL} = \mathbf{IP}^{\mathrm{SD}}\mathfrak{S}_2$. Now, assume $|A| > 2$, in which case, providing $\mathfrak{A}$ is a chain, for any $\bar{a} \in A^3$ with $|\operatorname{img} \bar{a}| = 3$ such that $a_0 \leqslant^\mathfrak{A} a_1 \leqslant^\mathfrak{A} a_2$ and $i \in 2$, $\Delta_A \neq \theta_i \triangleq ([a_i, a_{i+1}]^2_\mathfrak{A} \cup \Delta_A) = \mathrm{Cg}^\mathfrak{A}(\{\langle a_i, a_{i+1}\rangle\}) \in \mathrm{Co}(\mathfrak{A})$, while $(\theta_0 \cap \theta_1) = \Delta_A$, and so $\mathfrak{A}$ is not finitely-sibdirectly-irreducible. Otherwise, take any $\bar{b} \in A^2$ such that $c \triangleq (b_0 \wedge^\mathfrak{A} b_1) \notin (\operatorname{img} \bar{b})$, in which case, for each $j \in 2$, $\vartheta_j \triangleq ((\bigcup\{[c \wedge^\mathfrak{A} d, b_j \wedge^\mathfrak{A} d]^2_\mathfrak{A} \mid d \in A\}) \cup \Delta_A) \supsetneq \Delta_A$ is symmetric and forms a subalgebra of $\mathfrak{A}^2$, and so the transitive closure $\eta_j = \mathrm{Cg}^\mathfrak{A}(\{\langle c, b_j\rangle\}) \supseteq \vartheta_j$ of ϑ_j is a congruence of $\mathfrak{A}$ distinct from Δ_A. By contradiction, prove that $(\eta_0 \cap \eta_1) \subseteq \Delta_A$. For suppose $(\eta_0 \cap \eta_1) \nsubseteq \Delta_A$. Take any $\bar{e} \in ((\eta_0 \cap \eta_1) \setminus \Delta_A) \neq \varnothing$, in which case, for all $k, l \in 2$, $\langle e_k, e_{1-k}\rangle \in (\theta_l \setminus \Delta_A)$, that is, there are some $m_l \in \omega$, $\bar{f}^l \in A^{m_l+2}$ and $\bar{g}^l \in A^{m_l+1}$ such that $f^l_0 = e_k$, $f^l_{m_l+1} = e_{1-k}$ and, for every $n \in (m_l + 1)$, $f^l_{n[+1]} \in [c \wedge^\mathfrak{A} g^l_n, b_l \wedge^\mathfrak{A} g^l_n]_\mathfrak{A}$, and so $e_k \leqslant^\mathfrak{A} c$, when taking $n = 0$, because $\{l, 1-l\} = 2$, while $e_k = f^{l|(1-l)}_0 \leqslant^\mathfrak{A} (b_{l|(1-l)} \wedge^\mathfrak{A} g^{l|(1-l)}_0) \leqslant^\mathfrak{A} b_{l|(1-l)}$. By induction on any $\ell \in (m_l + 2)$, show that $e_k \leqslant^\mathfrak{A} f^l_\ell$. The case $\ell = 0$ is by the equality $e_k = f^l_0$. Otherwise, $(m_l + 2) \ni (\ell - 1) < \ell$, in which case, by induction hypothesis, we have $c \geqslant^\mathfrak{A} e_k \leqslant^\mathfrak{A} f^l_{\ell-1} \leqslant^\mathfrak{A} (b_l \wedge^\mathfrak{A} g^l_{\ell-1}) \leqslant^\mathfrak{A} g^l_{\ell-1}$, and so we get

$e_k \leqslant^{\mathfrak{A}} (c \wedge^{\mathfrak{A}} g^l_{\ell-1}) \leqslant^{\mathfrak{A}} f^l_\ell$. In particular, $e_k \leqslant^{\mathfrak{A}} e_{1-k}$, when taking $\ell = (m_l + 1)$, since $f^l_{m_l+1} = e_{1-k}$. Then, $e_0 = e_1$, in which case this contradiction shows that $(\eta_0 \cap \eta_1) = \Delta_A$, and so $\mathfrak{A}$ is not finitely-sibdirectly-irreducible. Thus, by (2.10) as well as the simplicity of two-element algebras and absence of their proper non-one-element subalgebras, $((\mathrm{SI}^{(\omega)} \mid \mathrm{Si})(\mathsf{SL})\{\cup \mathsf{A}^{=1}_{\Sigma}\}) = (\mathbf{I}\mathfrak{S}_2\{\cup \mathsf{A}^{=1}_{\Sigma}\})$ is the class of {no-more-than-}two-element semi-lattices {that is, the universal first-order model subclass of SL relatively axiomatized by the single universal first-order sentence $\forall_{\imath \in 3} x_\imath ((x_2 \approx x_1) \vee (x_2 \approx x_0) \vee (x_1 \approx x_0))$}, while SL, being finitely-semi-simple and finitely-generated, is semi-simple and locally-finite. On the other hand, since $\mathrm{Fi}(2) = \{\wp(N, 2) \mid N \subseteq 2\}$, the set $\{\Delta_{2^2}, (2^2)^2\} \cup \{\ker(\pi_\jmath \restriction 2^2) \mid \jmath \in 2\}$ of filtral congruences of $\mathfrak{S}^2_2$ does not contain its congruence $\Delta_{2^2} \cup \{\langle \langle 0, \Bbbk\rangle, \langle 0, 1-\Bbbk\rangle\rangle \mid \Bbbk \in 2\}$, in which case, by Theorem 3.11, SL, not being filtral, is not implicative, and so, by Theorem 3.25, is neither congruence-distributive nor disjunctive. □

3.2.2. *Disjunctivity versus distributivity of lattices of sub-varieties.*

Lemma 3.29. *Let* K *be a class of* Σ*-algebras with a disjunctive system* $\mho \subseteq \mathrm{Eq}^4_\Sigma$ *as well as* R *and* S *are relative sub-varieties of* K*. Then, so is* $\mathsf{R} \cap \| \cup \mathsf{S}$*. In particular, relative sub-varieties of* K *form a distributive lattice.*

Proof. Take any $\mathfrak{I}, \mathfrak{J} \subseteq \mathrm{Tm}^\omega_\Sigma$ with $(\mathsf{R}|\mathsf{S}) = (\mathsf{K} \cap \mathrm{Mod}(\mathfrak{I}|\mathfrak{J}))$, in which case $(\mathsf{R} \cap \| \cup \mathsf{S})$ $= (\mathsf{K} \cap \mathrm{Mod}((\mathfrak{I} \cup \mathfrak{J}) \| \bigcup \{\mho[x_i/\phi_i, x_{2+i}/\psi_i]_{i \in 2} \mid (\bar\phi|\bar\psi) \in ((\mathfrak{I}|\mathfrak{J})[x_j/x_{(2 \cdot j)+(0|1)}]_{j \in \omega})\}))$, and so the distributivity of unions with intersections completes the argument. □

This, by (2.10), (2.9) and Lemma 3.16, immediately yields:

Corollary 3.30. *Let* K *be a [finite] class of finite* Σ*-algebras with a disjunctive system* $\mho \subseteq \mathrm{Eq}^4_\Sigma$ *and* P *the pre-variety generated by* K*. Suppose* P *is a variety. Then,* $\mathrm{SI}(\mathsf{P}) = \mathbf{IS}_{>1}\mathsf{K}$*, in which case* $\mathsf{S} \mapsto (\mathsf{S} \cap \mathbf{S}_{\{>1\}}\mathsf{K})$ *and* $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ *are inverse to one another isomorphisms between the lattices of sub-varieties of* P *and relative ones of* $\mathbf{S}_{\{>1\}}\mathsf{K}$*, and so they are distributive [and finite].*

Likewise, by (2.10), (2.9), Theorem 3.7 (as well as [20, Remark 2.4] and Lemma 3.29), we immediately have:

Corollary 3.31. *Let* K *be a [finite] class of [finite]* Σ*-algebras with a (finite) implicative system* $\mho \subseteq \mathrm{Eq}^4_\Sigma$ *and* P *the pre-variety generated by* K*. Suppose* P *is a variety. Then,* $(\mathrm{SI} \mid \mathrm{Si})(\mathsf{P}) = \mathsf{P}^{>1}_{\mho} = \mathbf{IS}_{>1}\mathsf{K}$*, in which case* $\mathsf{S} \mapsto (\mathsf{S} \cap \mathbf{S}_{\{>1\}}\mathsf{K})$ *and* $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ *are inverse to one another isomorphisms between the [finite] (distributive) lattices of sub-varieties of* P *and relative ones of* $\mathbf{S}_{\{>1\}}\mathsf{K}$*.*

4. Morgan-Stone lattices versus distributive ones

From now on, we deal with the signatures $\Sigma^{(-)}_{+[,01]} \triangleq (\Sigma_+(\cup\{\neg\})[\cup\{\bot, \top\}])$, [bounded] {distributive} lattices being supposed to be $\Sigma_{+[,01]}$-algebras with their variety denoted by $[\mathsf{B}]\{\mathsf{D}\}\mathsf{L}$ and the chain [bounded] distributive lattice with carrier $n \in (\omega \setminus 2)$ and the natural ordering on this denoted by $\mathfrak{D}_{n[,01]}$, in which case $\epsilon^n_2 \triangleq \{\langle 0, 0\rangle, \langle 1, n-1\rangle\}$ is an embedding of $\mathfrak{D}_{2[,01]}$ into $\mathfrak{D}_{n[,01]}$, while, for each $i \in 2$, $\epsilon^4_{3:i} \triangleq (\chi^{3\setminus(2-i)}_3 \times \chi^{3\setminus(1+i)}_3)$ is an embedding of $\mathfrak{D}_{3[,01]}$ into $\mathfrak{D}^2_{2[,01]}$. First, taking the Prime Ideal Theorem, (2.8), (2.10) and Corollary 3.13 into account, we immediately have the following well-known fact (cf. [8] as to REDPC for $[\mathsf{B}]\mathsf{DL}$):

Lemma 4.1. *Let* $\mathfrak{A} \in [\mathsf{B}]\mathsf{L}$ *and* $F \subseteq A$*. Suppose* F *is either a prime filter of* $\mathfrak{A}$ *or in* $\{\varnothing, A\}$*. Then, [unless* $F \in \{\varnothing, A\}$*]* $h \triangleq \chi^F_A \in \hom(\mathfrak{A}, \mathfrak{D}_{2[,01]})$ *[and* $h[A] = 2$*], in which case* $[\mathsf{B}]\mathsf{DL} = \mathbf{IP}^{\mathrm{SD}}\mathfrak{D}_{2[,01]}$*, and so* $[\mathsf{B}]\mathsf{DL}$ *is the semi-simple [pre-/quasi-]variety generated by* $\mathfrak{D}_{2[,01]}$ *with* $(\mathrm{Si} \mid \mathrm{SI})([\mathsf{B}]\mathsf{DL}) = \mathbf{I}\mathfrak{D}_{2[,01]}$ *and REDPC scheme* $\mho^{\langle x_0\rangle}_{V_1}$*.*

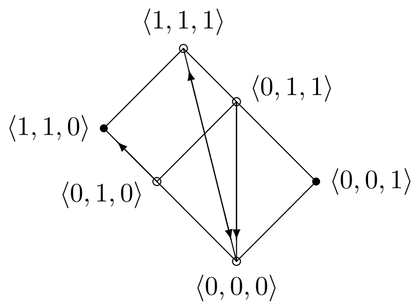

FIGURE 1. The Morgan-Stone lattice $\mathfrak{MS}_6$.

A *[bounded] (De) Morgan-Stone {(D)MS} lattice* is any $\Sigma^-_{+[,01]}$-algebra, whose $\Sigma_{+[,01]}$-reduct is a [bounded] distributive lattice and which satisfies the Σ^-_+-identities:

$$\neg(x_0 \wedge x_1) \approx (\neg x_0 \vee \neg x_1), \tag{4.1}$$

$$x_0 \lessapprox \neg\neg x_0, \tag{4.2}$$

in which case, by (4.1) [and (4.2)[$x_0/\top$]], it satisfies the Σ^-_+-quasi-identity [and the $\Sigma^-_{+[,01]}$-identity]:

$$(x_0 \lessapprox x_1) \to (\neg x_1 \lessapprox \neg x_0)[, \tag{4.3}$$

$$\neg\neg\top \approx \top], \tag{4.4}$$

and so the $\Sigma^-_{+[,01]}$-identities:

$$\neg(x_0 \vee x_1) \approx (\neg x_0 \wedge \neg x_1), \tag{4.5}$$

$$\neg\neg\neg x_0 \approx \neg x_0[, \tag{4.6}$$

$$\neg\bot \approx \top], \tag{4.7}$$

their variety being denoted by [B](D)MSL. Then, bounded Morgan-Stone lattices, satisfying the $\Sigma^-_{+,01}$-identity:

$$\neg\top \approx \bot, \tag{4.8}$$

are nothing but *(De) Morgan-Stone {MS} algebras* [2] ⟨cf. [23]⟩, their variety being denoted by (D)MSA. An $a \in A$ is called {a} *(negatively-)idempotent {element of* an $\mathfrak{A} \in \mathsf{MSL}$}, if $\{(\neg^{\mathfrak{A}})a\}$ forms a subalgebra of $\mathfrak{A}$, i.e., $\neg^{\mathfrak{A}}(\neg^{\mathfrak{A}})a = (\neg^{\mathfrak{A}})a$, with their set denoted by $\Im^{\mathfrak{A}}_{(\neg)}$, Morgan-Stone lattices with carrier of cardinality no less than 2({−1}) and with({out non-}negatively-)idempotent elements being said to be *({totally} negatively-)idempotent*.

Remark 4.2. By (4.1), (4.5), (4.6), Corollary 3.13 and Theorem 3.5, $\mho^{\langle x_0,\neg x_0,\neg\neg x_0\rangle}_{\{x_0,\neg x_0,\{\neg\neg x_0\}}$ is an REDPC scheme for [B]MS(L[/A]). □

4.1. **Subdirectly-irreducibles.** Let $\mathfrak{MS}_6$ be the Σ^-_+-algebra with $(\mathfrak{MS}_6{\restriction}\Sigma^-_+) \triangleq ((\mathfrak{D}^2_2{\restriction}(2^2 \setminus \{\langle 1,0\rangle\})) \times \mathfrak{D}_2)$ and $\neg^{\mathfrak{MS}_6}\bar{a} \triangleq \langle 1-a_2, 1-a_2, 1-a_1\rangle$, for all $\bar{a} \in MS_6$ (the Hasse diagram of its lattice reduct with its [non-]idempotent elements marked by [non-]solid circles and arrows reflecting action of its operation ¬ on its non-idempotent elements is depicted at Figure 1), in which case it is routine to check to be a Morgan-Stone lattice, and so are both $\mathfrak{MS}_5 \triangleq (\mathfrak{MS}_6{\restriction}(MS_6 \setminus \{\langle 0,0,1\rangle\}))$ and $\mathfrak{MS}_2 \triangleq (\mathfrak{MS}_5{\restriction}\{\langle i,1,0\rangle \mid i \in 2\})$ as well as, for each $j \in 2$, $\mathfrak{MS}_{4:j} \triangleq (\mathfrak{MS}_{5+j}{\restriction}(MS_{5+j} \setminus (((j+1) \times \{1\}) \times \{1-j\}))))$. Likewise, let $(\mathfrak{DM}|\mathfrak{S})_{4|3}$ be the Σ^-_+-algebra with $((\mathfrak{DM}|\mathfrak{S})_{4|3}{\restriction}\Sigma^-_+) \triangleq \mathfrak{D}^{2|}_{2|3}$ and $\neg^{(\mathfrak{DM}|\mathfrak{S})_{4|3}} \triangleq ((((\pi_1{\restriction}2) \circ (2^2 \setminus$

$\Delta_2)) \times ((\pi_0 \restriction 2) \circ (2^2 \setminus \Delta_2)))|\chi_3^1)$, in which case $\epsilon_{4|3}^{6|5} \triangleq ((((\pi_0 \restriction 2^2) \times (\pi_0 \restriction 2^2)) \times (\pi_1 \restriction 2^2))|(\epsilon_{3:0}^4 \times \chi_3^{3\setminus 1}))$ is an embedding of $(\mathfrak{DM}|\mathfrak{S})_{4|3}$ into $(\mathfrak{MS}|\mathfrak{MS})_{6|5}$. Finally, for any $n \in (\{3,4\}|\{2\})$, let $(\mathfrak{K}|\mathfrak{B})_n$ be the Σ_+^--algebra with $((\mathfrak{K}|\mathfrak{B})_n \restriction \Sigma_+^-) \triangleq \mathfrak{D}_n$ and $\neg^{(\mathfrak{K}|\mathfrak{B})_n} \triangleq \{\langle m, n-1-m\rangle \mid m \in n\}$, in which case $\epsilon_2^{3\|4}$ is an embedding of $\mathfrak{B}_2$ into $\mathfrak{K}_{3\|4}$, while, for every $l \in 2$, $\epsilon_{3:l}^4$ is an embedding of $\mathfrak{K}_3$ into $\mathfrak{DM}_4$, and so $\epsilon_{3:l}^4 \circ \epsilon_4^6$ is that into $\mathfrak{MS}_{4:(1-l)}$. Moreover, $\{MS_6, MS_5, MS_2, \mathrm{img}(\epsilon_2^3 \circ \epsilon_3^5)\} \cup (\bigcup\{\{MS_{4:k}, \mathrm{img}(\epsilon_{3:k}^4 \circ \epsilon_4^6)\} \mid k \in 2\})$ are exactly the carriers of members of $\mathbf{S}_{>1}\mathfrak{MS}_6$, in which case these are isomorphic to those of the skeleton $\mathsf{MS} \triangleq (\{\mathfrak{MS}_\ell \mid \ell \in \{6,5,2\}\} \cup \{\mathfrak{MS}_{4:\Bbbk} \mid \Bbbk \in 2\} \cup \{\mathfrak{DM}_4, \mathfrak{K}_3, \mathfrak{S}_3, \mathfrak{B}_2\})$, and so this is that of $\mathbf{IS}_{>1}\mathfrak{MS}_6$ with the embeddability *partial* ordering $\preceq$ between members of MS, for these are all finite. And what is more, $D_6 \triangleq (MS_6 \cap \pi_0^{-1}[\{1\}])$ is a prime filter of $\mathfrak{MS}_6 \restriction \Sigma_+$, while $\Omega \triangleq \{x_0, \neg x_0, \neg\neg x_0\}$ is an equality determinant for $\langle \mathfrak{MS}_6, D_6\rangle$, in which case, by [19, Lemma 11], $\mho_\Omega \triangleq \{(\tau(x_\imath) \wedge \rho(x_{2+\jmath})) \lessapprox (\tau(x_{1-\imath}) \vee \rho(x_{3-\jmath})) \mid \imath, \jmath \in 2, \tau, \rho \in \Omega\}$ is a disjunctive system for $\mathfrak{MS}_6$, and so, for $\mathbf{IS}\mathfrak{MS}_6$.

Remark 4.3. Elements of $\mathcal{PF}_4 \triangleq \{2^2 \cap \pi_i^{-1}[\{1\}] \mid i \in 2\}$ are exactly all prime filters of $\mathfrak{D}_2^2$, while $\{x_0, \neg x_0\}$ is an equality determinant for $\mathsf{M} \triangleq (\{\mathfrak{DM}_4\} \times \mathcal{PF}_4)$, in which case, by Theorem 3.15, $\mho_{V_1}^{\langle x_0, \neg x_0\rangle}$ is an implicative system for $\mathbf{IS}_{\{>1\}}\mathfrak{DM}_4$ {and so, by Corollary 3.14, its members are simple, as it is well-known but shown directly in a more cumbersome way}. □

Theorem 4.4. *For any prime filter F of the Σ_+-reduct of any $\mathfrak{A} \in \mathsf{MSL}$ there is an $h \in \hom(\mathfrak{A}, \mathfrak{MS}_6)$ with $(\ker h) \subseteq (\ker \chi_A^F)$, in which case MSL is the [pre-/quasi-]variety generated by $\mathfrak{MS}_6$ with REDPC scheme $\mho_\Omega^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}$, and so* $\mathrm{SI}(\mathsf{MSL}) = \mathbf{I}\mathsf{MS}$.

Proof. Let $f \triangleq \chi_A^F$, $G \triangleq (\neg^{\mathfrak{A}})^{-1}[(\neg^{\mathfrak{A}})^{-1}[F]]$, $H \triangleq (A \setminus (\neg^{\mathfrak{A}})^{-1}[F])$ and $h \triangleq (f \times \chi_A^G) \times \chi_A^H)$, in which case, by (2.1) and (4.6), $(\ker f) \supseteq (((\ker f) \cap (\ker \chi_A^G)) \cap (\ker \chi_A^H)) = (\ker h) \subseteq (\neg^{\mathfrak{A}} \circ h)$, while, by (4.1) and (4.5), $G|H$ is either a prime filter of $\mathfrak{A} \restriction \Sigma_+$ or in $\{\varnothing, A\}$, whereas, by (4.2), $F \subseteq G$, and so, by (2.2), $\pi_0(h(a)) \leqslant \pi_1(h(a))$, for all $a \in A$. Then, by (2.7), Lemma 4.1 and the Homomorphism Theorem, h is a surjective homomorphism from $\mathfrak{A}$ onto the Σ_+^--algebra $\mathfrak{B}$ with $(\mathfrak{B} \restriction \Sigma_+) \triangleq (\mathfrak{D}_2^3 \restriction h[A])$ as well as $\neg^{\mathfrak{B}} \triangleq (h^{-1} \circ \neg^{\mathfrak{A}} \circ h)$, in which case $B \subseteq MS_6$, since $\pi_0(h(a)) \leqslant \pi_1(h(a))$, for all $a \in A$, and so $\mathfrak{B} = (\mathfrak{MS}_6 \restriction h[A])$, as, for all $a \in A$, $(\neg^{\mathfrak{A}} a \in G) \Leftrightarrow (\neg^{\mathfrak{A}} a \in F) \Leftrightarrow (a \notin H)$, in view of (4.6), as well as $(\neg^{\mathfrak{A}} a \in H) \Leftrightarrow (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \notin F) \Leftrightarrow (a \notin G)$. Hence, $h \in \hom(\mathfrak{A}, \mathfrak{MS}_6)$ and $(\ker h) \subseteq (\ker f)$. Thus, the Prime Ideal Theorem, (2.8), Corollary 3.30 and Remark 4.2 complete the argument. □

The Σ_+^--reduct of any $\mathfrak{A} \in \mathsf{MS}$, being a finite lattice, has zero/unit a/b, in which case we have the bounded Morgan-Stone lattice $\mathfrak{A}_{01}$ with $(\mathfrak{A}_{01} \restriction \Sigma_+^-) \triangleq \mathfrak{A}$ and $(\bot/\top)^{\mathfrak{A}_{01}} \triangleq (a/b)$, and so, for all $\mathfrak{C} \in \mathsf{MS}_{01} \triangleq \{\mathfrak{B}_{01} \mid \mathfrak{B} \in \mathsf{MS}\}$ and $\mathfrak{D} \in \mathsf{MS}_{-2,01} \triangleq (\mathsf{MS}_{01} \setminus \{\mathfrak{MS}_{2,01}\})$, $((\mathfrak{D} \restriction \Sigma_+^-) \preceq (\mathfrak{C} \restriction \Sigma_+^-)) \Rightarrow (\mathfrak{D} \preceq \mathfrak{C})$. Then, since $\mathfrak{MS}_{2,01} \notin \mathsf{MSA} \supseteq (\mathbf{IS}\mathfrak{MS}_{6,01}) \supseteq \mathsf{MS}_{-2,01}$, while surjective lattice homomorphisms preserve lattice bounds (if any), whereas expansions by constants alone preserve congruences, by (2.8), (2.9) and Theorem 4.4, we immediately get:

Corollary 4.5. *Let $\mathsf{K} \triangleq (\varnothing|\{\mathfrak{MS}_{2,01}\}$. Then, $\mathsf{V} \triangleq (\mathsf{BMSL}|\mathsf{MSA})$ is the [pre-/quasi-]variety generated by $\{\mathfrak{MS}_{6,01}, \mathfrak{MS}_{2,01}\} \setminus \mathsf{K}$ with* $\mathrm{SI}(\mathsf{V}) = \mathbf{I}(\mathsf{MS}_{01} \setminus \mathsf{K})$ *and REDPC scheme $\mho_\Omega^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}$.*

This subsumes [2] and also yields a uniform insight into REDPC for Stone and De Morgan algebras, originally given by separate distinct schemes in [12, 21] and a bit enhanced in Corollary 4.7.

4.2. **The lattice of sub-varieties.** [Bounded/] Morgan-Stone lattices[/algebras], satisfying either of the following equivalent — in view of (4.2) — Σ_+^--identities:

$$(\neg\neg x_0(\vee\neg x_0)) \approx \| \lessapprox (x_0(\vee\neg x_0)), \tag{4.9}$$

are called *[bounded/] (nearly) {De} Morgan lattices[/algebraas]*, their variety being denoted by $[\mathsf{B}/](\mathsf{N})\{\mathsf{D}\}\mathsf{M}(\mathsf{L}[/\mathsf{A}])$. Likewise, those, satisfying the Σ_+^--identity:

$$(x_0 \wedge \neg x_0) \lessapprox x_1, \tag{4.10}$$

are nothing but *[bounded/] Stone lattices[/algebras]* [cf., e.g., [7]], their variety being denoted by $[\mathsf{B}/]\mathsf{S}(\mathsf{L}[/\mathsf{A}])$. Then, members of $[[\mathsf{B}/]\mathsf{B}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}]\mathsf{DM}(\mathsf{L}[/\mathsf{A}]) \cap [\mathsf{B}]\mathsf{S}(\mathsf{L}[/\mathsf{A}]))$ are exactly *[bounded/] Boolean lattices[/algebras]*. Further, [bounded/] Morgan-Stone lattices[/algebras], satisfying "either of the former"|"the latter" of the following Σ_+^--identities:

$$(\neg\neg x_0 \wedge \neg x_0) \quad \approx \| \lessapprox \quad (x_0 \wedge \neg x_0), \tag{4.11}$$

$$\neg\neg x_0 \quad \lessapprox \quad (x_0 \vee (\neg\neg x_1 \vee \neg x_1)), \tag{4.12}$$

"in which case they satisfy the $\Sigma^-_{+[,01]}$-quasi-identities [(4.8) and]:

$$((\neg x_0 \wedge x_1) \lessapprox (x_0 \vee x_2)) \leftarrow \| \rightarrow ((\neg x_0 \wedge x_1) \lessapprox ((\neg\neg x_0 \vee x_2)), \tag{4.13}$$

in view of [(4.7) and] (4.2)"| are said to be *quasi-|pseudo-strong*, their variety being denoted by $[\mathsf{B}/](\mathsf{Q}|\mathsf{P})\mathsf{SMS}(\mathsf{L}[/\mathsf{A}])$. Then, members of

$$[\mathsf{B}/]\mathsf{SMS}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}/]\mathsf{QSMS}(\mathsf{L}[/\mathsf{A}]) \cap [\mathsf{B}/]\mathsf{PSMS}(\mathsf{L}[/\mathsf{A}])) \supseteq ([\mathsf{B}/]\mathsf{DM}(\mathsf{L}[/\mathsf{A}]) \cup [\mathsf{B}/]\mathsf{S}(\mathsf{L}[/\mathsf{A}]))$$

are said to be *strong*, in which case, by (4.2) and the uniqueness of relative complements in distributive lattices:

$$([\mathsf{B}]\{\mathsf{Q}\}\mathsf{SMSL} \cap [\mathsf{B}]\mathsf{NDML}) = [\mathsf{B}]\mathsf{DML}. \tag{4.14}$$

Furthermore, *[bounded/] ((⌊⌈quasi-|pseudo-⌉strong⌋) {weakly} Kleene(⟨-Morgan⟩-Stone) lattices [/algebras]* are [bounded/] (⌊⌈quasi-|pseudo-⌉strong⌋) De-Morgan(-Stone) lattices[/algebras] satisfying the following Σ_+^--identity:

$$\mathcal{K}^{\{\mathrm{W}\}}_{\langle\mathrm{M}\rangle} \triangleq ((\langle\neg\neg x_2\wedge\rangle(x_0 \wedge \neg x_0)) \lessapprox (\langle x_2\vee\rangle(\neg x_1 \vee \{\neg\neg\}x_1))),$$

their variety being denoted by

$$[\mathsf{B}/](\lfloor\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rfloor)\{\mathsf{W}\}\mathsf{K}(\langle\mathsf{M}\rangle\mathsf{S})(\mathsf{L}[/\mathsf{A}]) \supseteq (\varnothing(\cup([\mathsf{B}/]\mathsf{S}(\mathsf{L}[/\mathsf{A}]))) \{\cup[\mathsf{B}/](\lfloor\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rfloor)\mathsf{K}(\langle\mathsf{M}\rangle\mathsf{S})(\mathsf{L}[/\mathsf{A}])\} (\langle[\mathsf{B}/]\mathsf{DM}(\mathsf{L}[/\mathsf{A}]) \cup [\mathsf{B}/](\lfloor\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rfloor)\{\mathsf{W}\}\mathsf{K}(\mathsf{S})(\mathsf{L}[/\mathsf{A}])\rangle))$$

{in view of (4.2)}. Likewise, members of

$$[\mathsf{B}/]\mathsf{NK}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}/]\{\mathsf{W}\}\mathsf{KS}(\mathsf{L}[/\mathsf{A}]) \cap [\mathsf{B}/]\mathsf{NDM}(\mathsf{L}[/\mathsf{A}]))$$

are called *[bounded/] nearly Kleene lattices[/algebras]*. Next, the variety of totally negatively-idempotent [bounded] Morgan-Stone lattices, being relatively axiomatized by the Σ_+^--identity:

$$\neg\neg x_0 \approx \neg x_0, \tag{4.15}$$

is denoted by $[\mathsf{B}]\mathsf{TNIMSL}$. Likewise, the variety of one-element [bounded/] Morgan-Stone lattices[/algebras], being (relatively) axiomatized by the Σ_+^--identity:

$$x_0 \approx x_1, \tag{4.16}$$

is denoted by $[\mathsf{B}/]\mathsf{OMS}(\mathsf{L}[/\mathsf{A}])$. Further, members of $[\mathsf{B}/](\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}])$, satisfying following Σ_+^--identity:

$$((\neg x_0 \wedge \neg\neg x_0) \wedge \neg\neg x_1) \lessapprox ((\neg x_0 \wedge x_0) \vee \neg x_1), \tag{4.17}$$

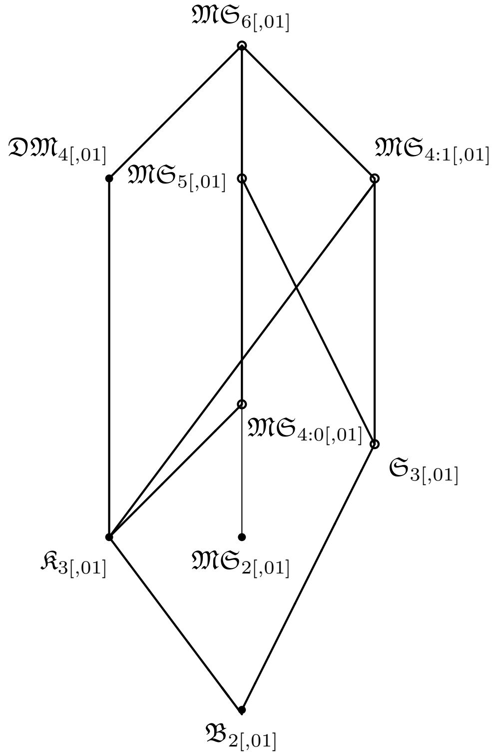

FIGURE 2. The poset $\langle \mathsf{MS}_{[01]}, \preceq \rangle$ [with merely thick lines].

are said to be *almost quasi-strong*, their variety being denoted by

$$[\mathsf{B}/]\mathsf{AQS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \supseteq ([\mathsf{B}/]\mathsf{QS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \cup ([\mathsf{B}]\mathsf{TNIMSL}[/\varnothing])).$$

Then, members of

$$[\mathsf{B}/]\mathsf{AS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}/]\mathsf{AQS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \cap$$
$$[\mathsf{B}/]\mathsf{PS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}])) \supseteq ([\mathsf{B}/]\mathsf{S}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \cup ([\mathsf{B}]\mathsf{TNIMSL}[/\varnothing]))$$

are said to be *almost strong*. Likewise, members of $[\mathsf{B}/](\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}])$, satisfying the following Σ^-_+-identity:

$$(\neg\neg x_0 \wedge \neg\neg x_1) \lessapprox (x_0 \vee \neg x_1), \tag{4.18}$$

are called *[bounded/] almost "De Morgan"|"{weakly} Kleene" lattices[/algebras]*, their variety being denoted by $[\mathsf{B}/]\mathsf{A}(\mathsf{DM}|\{\mathsf{W}\}\mathsf{K})(\mathsf{L}[/\mathsf{A}]) \supseteq ([\mathsf{B}/](\mathsf{DM}|\{\mathsf{W}\}\mathsf{K})(\mathsf{L}[/\mathsf{A}]) \cup ([\mathsf{B}]\mathsf{TNIMSL}[/\varnothing]))$. Finally, [bounded/] Morgan-Stone lattices[/algebras], satisfying the optional|non-optional version of the following Σ^-_+-identity:

$$(\neg x_0 \vee \lceil\neg\neg\rceil x_0) \gtrapprox x_1, \tag{4.19}$$

are called *[bounded/] almost Stone|Boolean lattices[/algebras]*, their variety being denoted by $[\mathsf{B}/]\mathsf{A}(\mathsf{S}|\mathsf{B})(\mathsf{L}[/\mathsf{A}])$.

Let[2]

$$\mathcal{MS}_{[01]}\lfloor(\mathfrak{A})\rfloor \triangleq (\{[(4.8),](4.9),((4.9)),(4.10),(4.11),(4.12),\mathcal{K},\mathcal{K}^{\mathrm{W}},$$
$$\mathcal{K}_{\mathrm{M}},\mathcal{K}^{\mathrm{W}}_{\mathrm{M}},(4.17),(4.18),(4.19),\lceil(4.19)\rceil,(4.15)\}\lfloor\cap\,\mathcal{E}(\mathfrak{A})\rfloor)$$

$\lfloor$where $\mathfrak{A} \in \mathsf{MS}_{[01]}\rfloor$.

---

[2]From now on, to unify equation environment references, those <not> incorporated into option brackets mean corresponding <non->optional versions of referred quasi-identities.

Lemma 4.6. *For any* $\mathfrak{A} \in \mathsf{MS}_{[01]}$, $\mathcal{MS}_{[01]}(\mathfrak{A})$ *is given by Table 1. In particular, the poset* $\langle \mathsf{MS}_{[01]}, \preceq \rangle$ *is given by Figure 2 with (non-)simple/*$\mho^{\langle x_0, \neg x_0 \lceil, \neg\neg x_0 \rceil \rangle}_{\{x_0 \lceil, \neg x_0 \lfloor, \neg\neg x_0 \rfloor \rceil\}}$*-implicative members marking (non-)solid circles-nodes [and merely thick lines].*

Proof. Clearly, for any line of Table 1, the identities of the second column of it are true in the algebra of the first one. Conversely,

$$\begin{aligned}
\mathfrak{MS}_{(5|6)[,01]} &\not\models \mathcal{K}^{|\mathrm{W}}_{\|\mathrm{M}}[x_i/\langle 1-\min(1,i), 1|\max(1-i,i-1), \min(1,i)\rangle]_{i\in(2\|3)},\\
\mathfrak{S}_{3[,01]} &\not\models ((((4.9))\|(4.9))|((4.18)\|(4.19)))[x_i/(1+i)]_{i\in(1|2)},\\
\mathfrak{DM}_{(4[,01]} &\not\models \mathcal{K}^{\{\mathrm{W}\}}[x_i/(\langle i,i,1-i\rangle]_{i\in 2},\\
\mathfrak{MS}_{4:1[,01]} &\not\models (4.12)[x_0/\langle 0,1,1\rangle, x_1/\langle 0,0,1\rangle],\\
\mathfrak{MS}_{4:0[,01]} &\not\models (4.17)[x_i/\langle i,1,i\rangle]_{i\in 2},\\
\mathfrak{K}_{3[,01]} &\not\models ((4.10)|(\lceil(4.19)\rceil\|(4.19)))[x_0/1, x_1/(0|2)],\\
(\mathfrak{B}|\mathfrak{MS})_{2[,01]} &\not\models (4.15|(4.9\|4.11))[x_0/(0|\langle 0,1,0\rangle)][,\\
\mathfrak{MS}_{2,01} &\not\models (4.8)].
\end{aligned}$$

Moreover, by Remark 4.2, $\mho^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}_{\Omega}$ is an REDPC scheme for $\mathsf{MSL} \supseteq \mathsf{MS}$, in which case, by Corollary 3.4, any simple member $\mathfrak{A}$ of it is $\mho^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}_{\Omega}$-implicative, and so all those members of MS, which are embeddable into $\mathfrak{A}$, being then $\mho^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}_{\Omega}$-implicative as well, are simple too. On the other hand,

$$\chi^{3\setminus 1}_3 = (\epsilon^5_3 \circ \pi_2) \in \hom(\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}), \tag{4.20}$$

in which case $(\ker \chi^{3\setminus 1}_3) \in (\mathrm{Co}(\mathfrak{S}_{3[,01]}) \setminus \{\Delta_3, 3^2\})$, and so $\mathfrak{S}_{3[,01]}$ is not simple. Likewise, $h \triangleq \{\langle \bar{a}, [\frac{a_0+a_1+a_2+1}{2}]\rangle \mid \bar{a} \in MS_{4:0}\} \in \hom(\mathfrak{MS}_{4:0[,01]}, \mathfrak{K}_{3[,01]})$, in which case $(\ker h) \in (\mathrm{Co}(\mathfrak{MS}_{4:0[,01]}) \setminus \{\Delta_{MS_{4:0}}, MS^2_{4:0}\})$, and so $\mathfrak{MS}_{4:0[,01]}$ is not simple. Thus, the fact that varieties are abstract and hereditary, the simplicity of two-element algebras, the equality $(4.11) = ((4.10)[x_0/\neg x_0, x_1/(x_0 \wedge \neg x_0)]$, Remark 4.3 and the truth of the identity $(4.9)|(\neg x_0 \approx \neg x_1)$ in $(\mathfrak{DM}|\mathfrak{MS})_{4|2}$ end the proof. □

Corollary 4.7. *Sub-varieties of* $[\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$ *form the non-chain distributive lattice with* $29[(+11)/(-9)]$ *elements, whose Hasse diagram with [both thick and] thin lines is depicted at Figure 3, any (non-)solid circle-node of it being marked by a (non-)semi-simple|filtral|*$\langle \mho^{\langle x_0, \neg x_0 \lceil, \neg\neg x_0 \rceil \rangle}_{\{x_0 \lceil, \neg x_0 \lfloor, \neg\neg x_0 \rfloor \rceil\}} - \rangle$*implicative variety* $\mathsf{V} \subseteq [\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$, *numbered from* $1[+(0/20)]$ *to* $29[+11]$ *according to Table 2 with* $\Bbbk \triangleq (9 \cdot (1[/0]))$ *[as well as* $\ell \triangleq (29 \cdot (0/1))$*] and* $\mathsf{MS}_{\mathsf{V}[,01]} \triangleq \max_{\preceq}((\mathsf{MS}_{[-2,01]}[\cup \mathsf{K}]) \cap \mathsf{V})$, *where* $\mathsf{K} \triangleq (\{\mathfrak{MS}_{2[,01]}\}[/\varnothing])$, *given by the third column, in which case* $\mathrm{SI}(\mathsf{V}) = \mathbf{IS}_{>1}\mathsf{MS}_{\mathsf{V}[,01]}$, *and so* V *is the (pre-*$\|$*quasi-)variety generated by* $\mathsf{MS}_{\mathsf{V}[,01]}$, *while*

Table 1. Identities of $\mathcal{MS}_{[01]}$ true in members of $\mathsf{MS}_{[01]}$.

$\mathfrak{MS}_{6[,01]}$	$\emptyset[\cup\{(4.8)\}]$
$\mathfrak{MS}_{5[,01]}$	$\{[(4.8),](4.12), \mathcal{K}^{\mathrm{W}}, \mathcal{K}^{\mathrm{W}}_{\mathrm{M}}\}$
$\mathfrak{MS}_{4:0[,01]}$	$\{[(4.8),]((4.9)), (4.12), \mathcal{K}, \mathcal{K}^{\mathrm{W}}, \mathcal{K}_{\mathrm{M}}, \mathcal{K}^{\mathrm{W}}_{\mathrm{M}}\}$
$\mathfrak{MS}_{4:1[,01]}$	$\{[(4.8),](4.11), \mathcal{K}, \mathcal{K}_{\mathrm{M}}, \mathcal{K}_{\mathrm{M}}, \mathcal{K}^{\mathrm{W}}_{\mathrm{M}}, (4.17)\}$
$\mathfrak{DM}_{4[,01]}$	$\mathcal{MS}_{[01]} \setminus \{\mathcal{K}, \mathcal{K}^{\mathrm{W}}, (4.10), (4.19), \lceil(4.19)\rceil, (4.15)\}$
$\mathfrak{MS}_{2[,01]}$	$\mathcal{MS}_{[01]} \setminus \{[(4.8),](4.9), (4.11), (4.10)\}$
$\mathfrak{K}_{3[,01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.10), (4.19), \lceil(4.19)\rceil, (4.15)\}$
$\mathfrak{S}_{3[,01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.9), ((4.9)), (4.18), (4.19), (4.15)\}$
$\mathfrak{B}_{2[,01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.15)\}$

$[\mathsf{B}]\mathsf{SMSL}$ *is that generated by* $\{\mathrm{SI}\}([\mathsf{B}]\mathsf{DML}\cup[\mathsf{B}]\mathsf{SL})$ *with REDPC scheme* $\mho^{\langle x_0,\neg x_0\rangle}_{\{x_0,\neg x_0\}}$, *whereas any disjunctive sub-pre-variety of* $[\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$ *is equational, and so is any quasi-equational//finitely implicative one.*

Proof. We use Lemma 4.6 tacitly. Then, the intersections of $\mathsf{MS}_{[-2,01]}[\cup\mathsf{K}]$ with the 29[(+11)/(−9)] sub-varieties of $[\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$ involved are exactly all lower cones of the poset $\langle\mathsf{MS}_{[-2,01]}[\cup\mathsf{K}],\preceq\rangle$, i.e., the sets appearing in the third column of Table 2 are exactly all anti-chains of the poset. So, (2.8), (2.9), (4.1), (4.5), Theorems 3.7, 3.11, 4.4, Corollaries 3.13, 4.5, Lemmas 3.16, 3.29, [20, Remark 2.4], the truth of the Σ^-_+-quasi-identities in $\{(\bigcup_{i\in 2}\{(x_2\wedge x_i)\lessapprox(x_{1-i}\vee x_3),(x_2\wedge\neg x_i)\lessapprox(\neg x_{1-i}\vee x_3)\})\to((x_2\wedge\neg\neg x_j)\lessapprox(\neg\neg x_{1-j}\vee x_3))\mid j\in 2\}$ in $\{\mathfrak{DM}_4,\mathfrak{S}_3\}$ and the fact that pre-varieties are abstract and hereditary complete the argument. □

It is in this sense that $[\mathsf{B}]\mathsf{SMSL}$ is the implicational/[quasi-]equational join of $[\mathsf{B}]\mathsf{DML}$ and $[\mathsf{B}]\mathsf{SL}$. Likewise, QSMSL is the greatest sub-variety of MSL not containing $\mathfrak{MS}_2$, in which case it is that containing the Σ^-_+-reduct of no member of $\mathsf{BMSL}\setminus\mathsf{MSA}$, and so it is in this sense that it is viewed as "an equational unbounded approximation of MSA" due to absence of any class of Σ^-_+-implications axiomatizing MSA relatively to BMSL, simply because any sub-pre-variety of MSL including $\mathsf{K}'\triangleq(\mathsf{MS}\setminus\{\mathfrak{MS}_2\})$ contains $\mathfrak{MS}_2\in\mathbf{S}\mathsf{K}'$ (this is why the node 30 at Figure 3 corresponds to no sub-variety of MSL). The finite lattice of its sub-quasi-varieties is found in the next Section. This task (as well as that solved in [17]) cannot be solved with using tools elaborated in [20] because of Proposition 5.11 therein. And what is more, despite of implicativity of {sub-varieties of} $[\mathsf{B}](\mathsf{A})\mathsf{DML}$ and Remark 3.1, we have:

Remark 4.8. Clearly, $\theta\triangleq(\Delta_3\cup(\{1\}\times 3))\subseteq(3^2\setminus(\{0,2\}^2\setminus\Delta_{\{0,2\}}))$ forms a subalgebra of $\mathfrak{K}^2_{3[,01]}$, in which case, if $\mathfrak{K}_{3[,01]}$ had a dual discriminator δ, then we would have $2=\delta^{\mathfrak{K}_{3[,01]}}(1,0,2)\;\theta\;\delta^{\mathfrak{K}_{3[,01]}}(0,0,2)=0$, and so, by Theorem 4.4 and Corollary 2.7, no sub-variety of $[\mathsf{B}]\mathsf{MSL}$ containing |"the non-simple subdirectly-irreducible" $\mathfrak{K}|\mathfrak{S}_{3[,01]}$ (viz., including $[\mathsf{B}](\mathsf{K}|\mathsf{S})\mathsf{L}$; cf. Corollary 4.7) is {dual} discriminator. □

Table 2. Maximal subdirectly-irreducibles of varieties of [bounded/] Morgan-Stone lattices[/algebras].

$1[+\ell]$	$[\mathsf{B}]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{6[,01]}\}[\cup\mathsf{K}]$
$2[+\ell]$	$[\mathsf{B}]\mathsf{PS}\langle\mathsf{WK}\rangle\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]},\mathfrak{DM}_{4[,01]}\}[\cup\mathsf{K}]$
$3\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{WK}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]},\mathfrak{MS}_{4:1[,01]}\lceil,\mathfrak{DM}_{4[,01]}\rceil\}[\cup\mathsf{K}]$
$5[+\ell]$	$[\mathsf{B}]\mathsf{PSWKS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]}\}[\cup\mathsf{K}]$
$6\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{K}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:i[,01]}\mid i\in 2\}\lceil\cup\{\mathfrak{DM}_{4[,01]}\}\rceil[\cup\mathsf{K}]$
$8\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{PSK}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]},\mathfrak{S}_{3[,01]}\lceil,\mathfrak{DM}_{4[,01]}\rceil\}[\cup\mathsf{K}]$
$10[+\ell]$	$[\mathsf{B}]\mathsf{NDM}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]},\mathfrak{DM}_{4[,01]}\}[\cup\mathsf{K}]$
$11[+\ell]$	$[\mathsf{B}]\mathsf{NK}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]}\}[\cup\mathsf{K}]$
12	$[\mathsf{B}]\mathsf{TNIMSL}$	$\{\mathfrak{MS}_{2[,01]}\}$
$22\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{QSMS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:1[,01]},\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$23\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{QS}\{\mathsf{W}\}\mathsf{KS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:1[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$24\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{SMS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]},\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$25\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{DM}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$26\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{S}\{\mathsf{W}\}\mathsf{KS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]},\mathfrak{K}_{3[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$27\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\{\mathsf{W}\}\mathsf{K}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{K}_{3[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$28\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$29\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{B}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{B}_{2[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
21	$[\mathsf{B}/]\mathsf{OMS}(\mathsf{L}[/\mathsf{A}])$	$\varnothing$

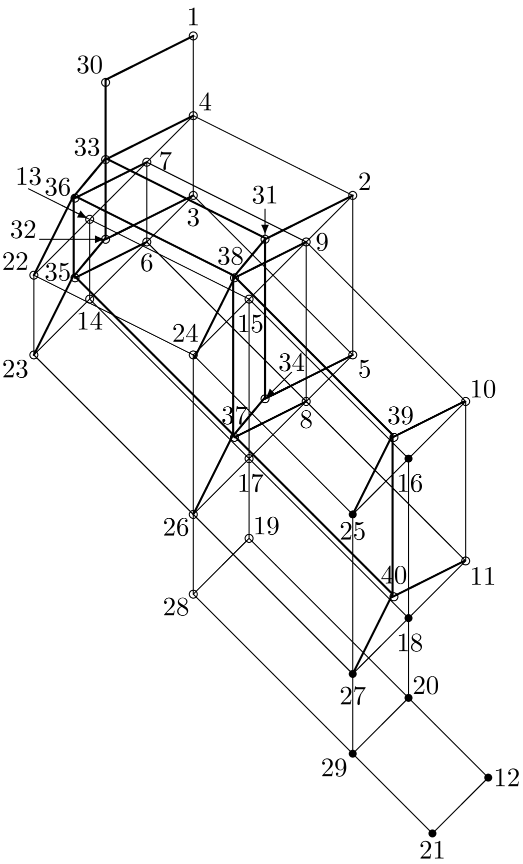

FIGURE 3. The lattice of varieties of [bounded/] Morgan-Stone lattices[/algebras].

On the other hand, the majority term μ_+ for the variety of lattices, being a dual discriminator for $\mathfrak{D}_2$, is that for $\{\mathfrak{B}_{2[,01]}, \mathfrak{MS}_{2[,01]}\}$, in which case, by Corollary 4.7, sub-varieties of [B]ABL are dual μ_+-discriminator, and so, by Remark 4.8, these are exactly all dual (μ_+-)discriminator sub-varieties of [B]MSL. Nevertheless, since $\neg x_0 \approx \top$ is true in $\mathfrak{MS}_{2,01}$, its isomorphic copy by $\pi_0 \restriction MS_2$ is term-wise-definitionally equivalent to $\mathfrak{D}_{2,01}$ generating the variety BDL (cf., e.g., [1] or Lemma 4.1), in its turn, being well-known (e.g., due to [4] {cf. [20, Lemma 2.10]} and existence of a three-element subdirect square of $\mathfrak{D}_{2,01}$ with carrier $2^2 \setminus \{\langle 0, 1\rangle\}$, though $3 \neq 1$ is odd), in which case $\mathfrak{MS}_{2[,01]}$ has no congruence-permutation term, for, otherwise, $\mathfrak{D}_{2,01}$ would have one, and so, by Corollaries 2.7 and 4.7, [B]BL is the only discriminator sub-variety of [B]MSL.

5. QUASI-VARIETIES OF QUASI-STRONG MORGAN-STONE LATTICES

Given any $\mathsf{K} \subseteq$ [B]MSL, (N)IK stands for the class of (non-)idempotent members of K (in which case it is the relative sub-quasi-variety of K, relatively axiomatized by the Σ_+^--quasi-identity:

$$(\neg x_0 \approx x_0) \to (x_0 \approx x_1), \tag{5.1}$$

and so a quasi-variety, whenever K is so).

Given any $\mathsf{K}', \mathsf{K}'' \subseteq$ [B]MSL, set $(\mathsf{K}' \otimes \mathsf{K}'') \triangleq \{\mathfrak{A} \times \mathfrak{B} \mid (\mathfrak{A}|\mathfrak{B}) \in (\mathsf{K}'|\mathsf{K}'')\}$.

Let $\bar{\boldsymbol{\mu}} \triangleq \langle \neg x_i \vee \neg\neg x_i \rangle_{i\in\omega}$. Then, [by induction on any $j \in \omega$] put $\boldsymbol{\iota}_{1[+j+1]} \triangleq ((\boldsymbol{\mu}_{0[+j+1]}[\vee\neg\boldsymbol{\iota}_{j+1}])[\wedge\boldsymbol{\iota}_{j+1}]) \in \mathrm{Tm}^{1[+j+1]}_{\Sigma^-_+}$.

Lemma 5.1. *Any (non-one-element finitely-generated)* $\mathfrak{A} \in [\mathsf{B}]\mathsf{MSL}$ *is non-idempotent if(f)* $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$, *in which case* $\mathsf{I}[\mathsf{B}]\mathsf{SMSL} \subseteq [\mathsf{B}]\mathsf{DML}$, *and so* $[\mathsf{B}]\mathsf{S}(\mathsf{M}|\mathsf{K})\mathsf{SL} = (\mathsf{NI}[\mathsf{B}]\mathsf{S}(\mathsf{M}|\mathsf{K})\mathsf{SL} \cup [\mathsf{B}](\mathsf{M}|\mathsf{K})\mathsf{L})$. *In particular,* $(\mathsf{NI}[\mathsf{B}]\mathsf{SMSL} \cup [\mathsf{B}]\mathsf{KL}) = (\mathsf{NI}[\mathsf{B}]\mathsf{SMSL} \cup [\mathsf{B}]\mathsf{SKSL})$, *while* $\mathsf{NIMS}_{[01]} = \{\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\}$, *whereas any variety* $\mathsf{V} \subseteq [\mathsf{B}]\mathsf{MSL}$ *with* $\mathsf{NIV} \nsubseteq [\mathsf{B}]\mathsf{OMSL}$ *contains* $\mathfrak{B}_{2[,01]}$.

Proof. The "if" part is by the fact that $\mathfrak{B}_{2[,01]}$ has no idempotent element. (Conversely, assume $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) = \varnothing$, in which case, by (4.20), $\hom(\mathfrak{A}, \mathfrak{S}_{3[,01]}) = \varnothing$, and so $(\hom(\mathfrak{A}, \{\mathfrak{MS}_{6[,01]}[, \mathfrak{MS}_{2,01}]\}) \cap (\mathrm{img}\, \epsilon^5_3)^A) = \varnothing$. Then, by (2.8), Theorem 4.4 [resp., Corollary 4.5] and the right alternative of the following claim, $\mathfrak{A}$, being non-one-element, is idempotent:)

Claim 5.2. *Let* $\mathfrak{B} \in [\mathsf{B}]\mathsf{MSL}$, $n \in (\omega \setminus 1)$, $\bar{b} \in B^n$, $\mathfrak{C} \in \{\mathfrak{MS}_{6[,01]}[, \mathfrak{MS}_{2,01}]\}$ *and* $h \in (\hom(\mathfrak{B}, \mathfrak{C}) \setminus (\varnothing|(\mathrm{img}\, \epsilon^5_3)^B))$. |*"Suppose* $\mathfrak{B}$ *is generated by* $\bar{b}$.*" Then,* $h(\neg^{\mathfrak{B}}\boldsymbol{\iota}^{\mathfrak{B}}_n(\bar{b})) \leqslant^{\mathfrak{C}} | = h(\boldsymbol{\iota}^{\mathfrak{B}}_n(\bar{b}))$, *in which case* $\neg^{\mathfrak{B}}\boldsymbol{\iota}^{\mathfrak{B}}_n(\bar{b}) \leqslant^{\mathfrak{B}} \boldsymbol{\iota}^{\mathfrak{B}}_n(\bar{b})$, *and so the* Σ^-_+-*identity* $\neg\boldsymbol{\iota}_n \lessapprox \boldsymbol{\iota}_n$ *of rank* n *is true in* $[\mathsf{B}]\mathsf{MSL}$.

Proof. In that case, by induction on any $l \in \omega \ni (n-1)$, we see that $h(\boldsymbol{\iota}^{\mathfrak{B}}_{l+1})(\bar{b}{\restriction}((l+1))$ is in $\{\langle \imath, \imath, \jmath\rangle \mid \langle \imath, \jmath\rangle \in (2^2 \setminus \langle 0, 0\rangle)\}$, for $h(\boldsymbol{\mu}^{\mathfrak{B}}_l[x_l/b_l])$ is so, and so $h(\neg^{\mathfrak{B}}\boldsymbol{\iota}^{\mathfrak{B}}_{l+1}(\bar{b}{\restriction}((l+1))) = \neg^{\mathfrak{C}} h(\boldsymbol{\iota}^{\mathfrak{B}}_{l+1}(\bar{b}{\restriction}((l+1))) \leqslant^{\mathfrak{C}} h(\boldsymbol{\iota}^{\mathfrak{B}}_{l+1}(\bar{b}{\restriction}((l+1)))$. |"In particular, as $(\mathrm{img}\, h) \nsubseteq (\mathrm{img}\, \epsilon^5_3)$, there is some $k \in n$ such that $h(b_k) \notin (\mathrm{img}\, \epsilon^5_3)$, in which case $h(\boldsymbol{\mu}^{\mathfrak{B}}_k[x_k/b_k]) \in \{\langle m, m, 1-m\rangle \mid m \in 2\}$, and so, by induction on any $\ell \in ((n+1) \setminus (k+1)) \ni n$, we eventually conclude that $h(\boldsymbol{\iota}^{\mathfrak{B}}_\ell(\bar{b}{\restriction}\ell))$, being then in $\{\langle \Bbbk, \Bbbk, 1-\Bbbk\rangle \mid \Bbbk \in 2\}$, is equal to $\neg^{\mathfrak{C}} h(\boldsymbol{\iota}^{\mathfrak{B}}_\ell(\bar{b}{\restriction}\ell)) = h(\neg^{\mathfrak{B}}\boldsymbol{\iota}^{\mathfrak{B}}_\ell(\bar{b}{\restriction}\ell))$." In this way, (2.8) and Theorem 4.4 [resp., Corollary 4.5] complete the argument. □

Finally, (2.8), (4.20), Corollary 4.7 and absence of proper subalgebras of $\mathfrak{B}_{2[,01]}$ complete the argument. □

Lemma 5.3. $\mathfrak{B}_{2[,01]}$ *is embedable into any quasi-strong [bounded] MS lattice* $\mathfrak{A}$.

Proof. [In that case, $\bot^{\mathfrak{A}} \neq \top^{\mathfrak{A}}$, and so, by (4.7) and (4.8), $\{\langle 0, \bot^{\mathfrak{A}}\rangle, \langle 1, \top^{\mathfrak{A}}\rangle\}$ is an embedding of $\mathfrak{B}_{2,01}$ into $\mathfrak{A}$.] In particular, the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\mathrm{img}\, \bar{a}$, where $\bar{a} \in ((A^2 \setminus \Delta_A)$, being finitely-generated, is a non-one-element finite quasi-strong MS lattice, in view of the local finiteness of the finitely-generated variety MSL (cf. Theorem 4.4), in which case $\mathfrak{C}$, being a finite lattice, has both zero/unit $b_{0/1}$, and so we get the non-one-element quasi-strong bounded MS lattice $\mathfrak{D}$ with $(\mathfrak{D}{\restriction}\Sigma^-_+) \triangleq \mathfrak{C}$ and $(\bot|\top)^{\mathfrak{D}} \triangleq b_{0|1}$. In this way, the []-optional case completes the argument. □

The stipulation "quasi-strong" here can be neither omitted nor replaced by the one "pseudo-strong" nor, even, weakened with replacing it by that "almost quasi-strong", when taking $\mathfrak{A} = \mathfrak{MS}_{2[,01]}$.

The above two lemmas, by (2.1), (2.7) with $I = 2$, (2.8), (2.10), Corollary 4.7, the locality of quasi-varieties, the quasi-equationality of finitely-generated pre-varieties, the simplicity of two-element algebras and the equality $\mathsf{NI}[\mathsf{B}]\mathsf{TNIMSL} = [\mathsf{B}]\mathsf{OMSL}$, immediately yield:

Corollary 5.4. *Let* $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{MSL}$ *and* $\mathsf{P} \triangleq \mathbf{PV}(\mathsf{K})$. *Suppose either* $\mathfrak{B}_{2[,01]} \in | \preceq (\mathsf{P}|\mathsf{K})$ *(more specifically, either* $[\mathsf{B}]\mathsf{OMSL} \nsupseteq (\mathsf{K}\|\mathsf{P}) \subseteq [\mathsf{B}]\mathsf{QSMSL}$ *or both* $[\mathsf{B}]\mathsf{OMSL} \nsupseteq \mathsf{NI}(\mathsf{K}\|\mathsf{P})$ *and* P *is equational) or* $\mathsf{IK} = \varnothing$. *Then,* $\mathsf{NIP} = \mathbf{PV}((\mathsf{IK} \otimes (\{\mathfrak{B}_{2[,01]}\} \cap (\mathsf{P}|(\mathbf{IS}\mathsf{K})))) \cup \mathsf{NIK})$, *in which case, for any variety* $\mathsf{V} \subseteq [\mathsf{B}]\mathsf{MSL}$ $\{$*such that* $[\mathsf{B}]\mathsf{BL} \subseteq \mathsf{V}$ *(i.e.,* $[\mathsf{B}]\mathsf{TNIMMSL} \nsupseteq \mathsf{V}$*)*$\}$, $\mathsf{NIV} = (\mathbf{P}/\!/\mathbf{Q})\mathbf{V}(\varnothing\{\cup((\mathsf{MS}_{\mathsf{V}[,01]} \setminus \{\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\}) \otimes$

$\{\mathfrak{B}_{2[,01]}\})\}\cup(\mathsf{MS}_{\mathsf{V}[,01]}\cap\{\mathfrak{S}_{3[,01]},\mathfrak{B}_{2[,01]}\}))$, *and so* $\mathsf{NI[B/]\{(PSM)|(WK\langle M\rangle)|(PSWK)\}S(L[/A])}$ *is the pre-//quasi-variety generated by* $(\{\mathfrak{MS}_{(6\{-1\})[,01]}\}\{\cup(\{\mathfrak{DM}_{4[,01]}\}|\{\mathfrak{MS}_{(4:1)[,01]}\langle,\mathfrak{DM}_{4[,01]}\rangle\}|\varnothing)\}[\cup(\{\mathfrak{MS}_{2,01}\}/\varnothing)])\otimes\{\mathfrak{B}_{2[,01]}\}$, *while* $\mathsf{NI[B/]\{(PS)|(\langle A\rangle\lfloor Q\rfloor S)\}(\{|M\|\}(K\lceil M\rceil))S(L[/A])}$ *is the one generated by* $((\{\mathfrak{MS}_{(4:i)[,01]} \mid i\in(2\{\setminus(\{1\}|(2\lfloor\cap 1\rfloor))\})\}\cup(\{|\{\mathfrak{DM}_{4[,01]}\}\|\}(\varnothing\cup(\varnothing|(\varnothing\cup(\{\mathfrak{K}_{3[,01]}\}\lceil\cap\varnothing\rceil)))\lceil\cup\{\mathfrak{DM}_{4[,01]}\}\rceil))\cup((\varnothing[\cup(\{\mathfrak{MS}_{2,01}\}/\varnothing)])|(\varnothing\langle\cup(\{\mathfrak{MS}_{2[,01]}\}[/\varnothing])\rangle)))\otimes\{\mathfrak{B}_{2[,01]}\})\cup(\{\mathfrak{S}_{3[,01]}\}|(\{\mathfrak{S}_{3[,01]}\}\lfloor\setminus\{\mathfrak{S}_{3[,01]}\}\rfloor))$, *whereas* $\mathsf{NI[B/]\{N\}(M|K)(L[/A])}$ *is that generated by* $((\{((\mathfrak{DM})|\mathfrak{K})_{(4|3)[,01]}\}\{\setminus(\varnothing|\{\mathfrak{K}_{3[,01]}\})\}\}\{\cup\{\mathfrak{MS}_{(4:0)[,01]}\}[\cup(\{\mathfrak{MS}_{2,01}\}/\varnothing)]\})\otimes\{\mathfrak{B}_{2[,01]}\}$. *In particular, any (non-one-element)* $\mathfrak{A}\in\mathsf{[B]MSL}$ *is non-idempotent if(f)* $\hom(\mathfrak{A},\mathfrak{B}_{2[,01]})\neq\varnothing$.

Corollary 5.5. $\mathsf{NI[B]MSL}\cup\mathsf{[B]TNIMSL}$ *is the sub-quasi-variety of* $\mathsf{[B]MSL}$ *relatively axiomatized by the* Σ^-_+*-quasi-identity:*

$$(\neg x_0\approx x_0)\to(x_0\approx\neg x_1) \tag{5.2}$$

and is the pre-/quasi-variety generated by $\{\mathfrak{MS}_{6[,01]}\times\mathfrak{B}_{2[,01]},\mathfrak{MS}_{2[,01]}\}$.

Proof. Clearly, (5.2) = $(5.1[x_1/\neg x_1])$ is true in both $\mathsf{NI[B]MSL}$ and $\mathfrak{MS}_{2[,01]}$. Conversely, any $\mathfrak{A}\in\mathsf{I[B]MSL}$, satisfying (5.2), has an idempotent element a, in which case, for any $b\in A$, as $\mathfrak{A}\models(5.2)[x_0/a,x_1/(\neg^{\mathfrak{A}})b]$, we have $\neg^{\mathfrak{A}}b=a(=\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b)$, and so $\mathfrak{A}\in\mathsf{[B]TNIMSL}$. Then, Corollaries 4.7 and 5.4 complete the argument. □

Likewise, we have:

Corollary 5.6. *For any (equational) /quasi-equational pre-variety* $\mathsf{P}\subseteq\mathsf{[B]MSL}$*, the class* $\mathsf{NIP}\cup(\mathsf{P}\cap\mathsf{[B]\{W\}KSL})$ *is the relative/ sub-quasi-variety of* P *relatively axiomatized by the* Σ^-_+*-quasi-identity:*

$$(\neg x_0\approx x_0)\to(x_0\lessapprox(\{\neg\neg\}x_1\vee\neg x_1)) \tag{5.3}$$

(and is the pre-|quasi-variety generated by $\mathsf{MS}_{\mathsf{P}\cap\mathsf{[B]\{W\}KSL})[,01]}\cup((\mathsf{MS}_{\mathsf{V}[,01]}\setminus\{\mathfrak{S}_{3[,01]},\mathfrak{B}_{2[,01]}\})\otimes\{\mathfrak{B}_{2[,01]}\})$*). In particular,* $\mathsf{NI[B](D\langle\|\lceil Q\rceil S\rangle)M\langle S\rangle L}\cup\mathsf{\langle\lceil Q\rceil S\rangle K\langle S\rangle L}$ *is the sub-quasi-variety of* $\mathsf{[B](D\langle\|\lceil Q\rceil S\rangle)M\langle S\rangle L}$ *relatively axiomatized by either version of* (5.3) *and is the pre-|quasi-variety generated by*

$$\{\mathfrak{DM}_{4[,01]}\times\mathfrak{B}_{2[,01]}\}\cup(\{\mathfrak{K}_{3[,01]}\langle,\mathfrak{S}_{3[,01]}\rangle\}\langle\lceil\cap\varnothing\rceil)\langle\lceil\cup\{\mathfrak{K}_{4:1[,01]}\}\rceil\rangle.$$

Proof. Clearly, (5.3) is satisfied in $\mathsf{NIP}\cup(\mathsf{P}\cap\mathsf{[B]\{W\}KSL})$. Conversely, consider any $\mathfrak{A}\in\mathsf{IP}$ satisfying (5.3) and any $a,b\in A$, in which case there is some $c\in A$ such that $\neg^{\mathfrak{A}}c=c$, and so, as $\mathfrak{A}(5.3)[x_0/c,x_1/(a|b)]$, we have $c\leqslant^{\mathfrak{A}}(\neg^{\mathfrak{A}}(a\|b)\vee^{\mathfrak{A}}\{\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\}(a\|b))$. Then, by (4.2), (4.3) and (4.5) {as well as (4.6)}, we get $(a\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}a)\leqslant^{\mathfrak{A}}c$, in which case $\mathfrak{A}\in(\mathsf{P}\cap\mathsf{[B]\{W\}KSL})$, and so Corollaries 4.7, 5.3 and 5.4 complete the argument. □

More generally, we, clearly, have:

Lemma 5.7. *For any* $\alpha\in(\infty\setminus 1)$*, any* $\mathcal{I}\subseteq(\wp(\mathrm{Eq}^{\alpha}_{\Sigma^-_{+[,01]}})\times\mathrm{Eq}^{\alpha}_{\Sigma^-_{+[,01]}})$ *and any pre-variety* $\mathsf{P}\subseteq\mathsf{[B]MSL}$*:*

$$(\mathsf{NIP}\cup(\mathsf{P}\cap\mathrm{Mod}(\mathcal{I})))=(\mathsf{P}\cap\mathrm{Mod}(\{(\{\neg x_\alpha\approx x_\alpha\}\cup\Gamma)\to\Phi\mid(\Gamma\to\Phi)\in\mathcal{I}\})).$$

This, by Corollaries 4.7 and 5.4, immediately yields:

Corollary 5.8. $(\mathsf{NI[B](A)QSMSL\{\langle\cap[B](A)QSKSL\rangle\}})\cup(\mathsf{[B](A)DML\{\cap[B](A)KL\}})$ *is the sub-quasi-variety of* $\mathsf{[B](A)QSMSL}$ *relatively axiomatized by the* Σ^-_+*-quasi-identity:*

$$(\neg x_{1(+1)}\approx x_{1(+1)})\to(((\neg\neg x_0(\wedge\neg\neg x_1))\approx(x_0(\vee\neg x_1))) \tag{5.4}$$

{collectively with the one $(\{\neg x_2\approx\neg x_2\}\langle\cap\varnothing\rangle)\to\mathcal{K}$*} and is the pre-/quasi-variety generated by* $\{\mathfrak{MS}_{4:1[,01]}\times\mathfrak{B}_{2[,01]},(,\mathfrak{MS}_{2[,01]})\}\cup((\{\mathfrak{DM}_{4[,01]}\}\{\cap\varnothing\})\{\cup\{\mathfrak{K}_{3[,01]}\}\cup(\{\mathfrak{DM}_{4[,01]}\times\mathfrak{B}_{2[,01]}\}\langle\cap\varnothing\rangle)\})$.

Definition 5.9 (cf. [17, Definition 4.6] for the non-otional case)**.** Members of any /quasi-equational $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{MSL}$, satisfying the Σ^{-}_{+}-quasi-identity of rank $2\{+1\}$:

$$\mathcal{R}^{(\mathrm{W})}_{\{\mathrm{M}\}} \triangleq ((\{\neg x_0 \lessapprox x_0, (x_0 \wedge \neg x_1) \lessapprox ((\neg x_0 \vee x_1)\}) \to ((\neg x_1\{\wedge\neg\neg x_2\}) \lessapprox ((\neg\neg)x_1\{\vee x_2\}))$$

are called *(weakly-)*$\{$*Morgan-*$\}$*regular*, their relative/ sub-quasi-variety of K being denoted by $((\mathsf{W})\{\mathsf{M}\}\mathsf{RK}$. □

Given any [bounded] Morgan-Stone lattice $\mathfrak{A}\lfloor\in [\mathsf{B}]\{\langle\lceil\mathsf{Q}\|\mathsf{P}\rceil\mathsf{S}\rangle\}(\mathsf{W})\mathsf{K}\{\mathsf{S}\}\mathsf{L}\rfloor$, by (4.1), (4.3) and (4.5) $\lfloor$as well as ((4.2) and) $\mathcal{K}^{(\mathrm{W})}\rfloor$, $(\mathfrak{I}|\mathfrak{F})^{\mathfrak{A}}_{(\mathrm{W})} \triangleq \{a \in A \mid (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})a(\leqslant \mid \geqslant)^{\mathfrak{A}}\neg^{\mathfrak{A}}a\} \supseteq \{b(\wedge|\vee)^{\mathfrak{A}}\neg^{\mathfrak{A}}b \mid b \in A\} \neq \varnothing$, for $A \neq \varnothing$, is $\lfloor$an|a ideal|filter of $\mathfrak{A}\restriction\Sigma_+\rfloor$ such that $\neg^{\mathfrak{A}}[(\mathfrak{I}|\mathfrak{F})^{\mathfrak{A}}_{(\mathrm{W})}] \subseteq (\mathfrak{F}|\mathfrak{I})^{\mathfrak{A}}_{(\mathrm{W})}$ $\lfloor$in which case $\mathfrak{R}^{\mathfrak{A}}_{(\mathrm{W})} \triangleq ((\mathfrak{F}^{\mathfrak{A}}_{(\mathrm{W})} \times \{1\}) \cup (\mathfrak{I}^{\mathfrak{A}}_{(\mathrm{W})} \times \{0\}))$ forms a subalgebra of $\mathfrak{A} \times \mathfrak{B}_{2[,01]}$ such that, for every $\bar{d} \in \mathfrak{R}^{\mathfrak{A}}_{(\mathrm{W})}$, $(d_1 = 1) \Rightarrow (d_0 \in \mathfrak{F}^{\mathfrak{A}}_{(\mathrm{W})})$, and so, by Corollary 4.7 and Lemma 5.1, the *(weak) regularization* $\mathfrak{R}_{(\mathrm{W})}(\mathfrak{A}) \triangleq ((\mathfrak{A} \times \mathfrak{B}_{2[,01]})\restriction\mathfrak{R}^{\mathfrak{A}}_{(\mathrm{W})})$ *of* $\mathfrak{A}$ is in $\mathsf{NI}(\mathsf{W})\mathsf{R}[\mathsf{B}]\{\langle\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rangle\}(\mathsf{W})\mathsf{K}\{\mathsf{S}\}\mathsf{L}\rfloor$. Then, $(\pi_0\restriction\mathfrak{R}^{\mathfrak{S}_{3[,01]}}) \in \hom(\mathfrak{R}(\mathfrak{S}_{3[,01]}), \mathfrak{S}_{3[,01]})$ is bijective, so, by Corollary 4.7, $\mathfrak{S}_{3[,01]} \in \mathsf{R}[\mathsf{B}]\mathsf{SKSL}$. Likewise, $(\epsilon^4_2\|\{\langle i, \langle\chi^{4\setminus 3}_4(i) + \chi^{4\setminus 1}_4(i), \chi^{4\setminus 2}_4(i)\rangle\rangle \mid i \in 4\}) \in \hom((\mathfrak{B}\|\mathfrak{K})_{(2\|4)[,01]}, \mathfrak{K}_{4[,01]}\|\mathfrak{R}(\mathfrak{K}_{3[,01]}))$ is injective‖bijective, so, by Corollary 4.7, $(\mathfrak{B}\|\mathfrak{K})_{(2\|4)[,01]} \in \mathsf{R}[\mathsf{B}]\mathsf{KL}$.

Lemma 5.10. *Any ((weakly)* $\{$*Morgan-*$\}$*regular [bounded/] MS lattice[/algebra]* $\mathfrak{A}$ *is a [bounded/] (weakly) Kleene-*$\{$*Morgan-*$\}$*Stone lattice[/algebra]. In particular,* $\mathsf{R}[\mathsf{B}]\mathsf{QSMSL} = \mathsf{R}[\mathsf{B}]\mathsf{QSKSL}$.

Proof. Consider any $a, b\{, c\} \in A$. Let $d \triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}a)$ and $e \triangleq ((b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b) \wedge^{\mathfrak{A}} d)$, in which case, by (4.5), we have $\neg^{\mathfrak{A}}d = (\neg^{\mathfrak{A}}a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a) \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}a \leqslant^{\mathfrak{A}} d$, and so, by (4.1) and (4.5), we get $(d \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}e) = ((d \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}}b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b)) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}d) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}d \vee^{\mathfrak{A}} (b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b)) \wedge^{\mathfrak{A}} d) = (\neg^{\mathfrak{A}}d \vee^{\mathfrak{A}} e)$. Then, since $\mathfrak{A} \models \mathcal{R}^{(\mathrm{W})}_{\{\mathrm{R}\}}[x_0/d, x_1/e\{, x_2/c\}]$, by (4.1), (4.2) and (4.5) (as well as (4.6)), we eventually get $((a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a)\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a)\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) = (\neg^{\mathfrak{A}}d\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) \leqslant^{\mathfrak{A}} (((\neg^{\mathfrak{A}}b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}d)\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) = (\neg^{\mathfrak{A}}e\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})e\{\vee^{\mathfrak{A}}c\}) = ((((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b) \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})d)\{\vee^{\mathfrak{A}}c\}) \leqslant^{\mathfrak{A}} (((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b)\{\vee^{\mathfrak{A}}c\})$, as required. □

Corollary 5.11. $(\varnothing\{\cup\{\mathfrak{S}_{3[,01]}\langle, \mathfrak{DM}_{4[,01]}\rangle(, \mathfrak{MS}_{2[,01]})\}\}) \subseteq (\mathsf{W})\langle\mathsf{M}\rangle\mathsf{R}[\mathsf{B}]\mathsf{M}\{\mathsf{S}\}\mathsf{L} \subseteq (\mathsf{NI}[\mathsf{B}]\mathsf{K}\{\langle\mathsf{M}\rangle\mathsf{S}\}\mathsf{L}(\cup[\mathsf{B}]\mathsf{TNIMSL})\langle\cup[\mathsf{B}](\mathsf{A})\mathsf{DML}\rangle)$. *In particular,* $[\mathsf{B}](\mathsf{A})\langle\{\mathsf{S}\}\mathsf{M}\rangle\{\mathsf{S}\}\mathsf{L} \subseteq (\mathsf{W})\langle\mathsf{M}\rangle\mathsf{RSMSL}$.

Proof. The first inclusion is immediate. For proving the second one, consider any $\mathfrak{A} \in (\mathsf{W})\langle\mathsf{M}\rangle\mathsf{R}[\mathsf{B}]\mathsf{M}\{\mathsf{S}\}\mathsf{L}$ and any $a, b, c\langle, d(, e)\rangle \in A$ such that $\neg^{\mathfrak{A}}a = a$, in which case, as $\mathfrak{A} \models ((4.1)|\mathcal{R}^{(\mathrm{W})}_{\langle\mathrm{R}\rangle})[x_0/a, x_1/(c|(a \wedge^{\mathfrak{A}} c))\langle|, x_2/d\rangle]$ (and $\mathfrak{A} \models (4.5)[x_0/\neg^{\mathfrak{A}}a, x_1/\neg^{\mathfrak{A}}c]$), we have $(\neg^{\mathfrak{A}}c\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}c)\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) = (\neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} c)\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})(a \wedge^{\mathfrak{A}} c)\langle\vee^{\mathfrak{A}}d\rangle) = ((a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c)\langle\vee^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c\langle\vee^{\mathfrak{A}}d\rangle)$, and so, as $\mathfrak{A} \models (4.2(\|4.6))[x_0/b]$, we get both $(b\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})\neg^{\mathfrak{A}}b\langle\vee^{\mathfrak{A}}d\rangle) = (\neg^{\mathfrak{A}}b\langle\vee^{\mathfrak{A}}d\rangle)$, when taking $c = \neg^{\mathfrak{A}}b$, and $(\neg^{\mathfrak{A}}b\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})b\langle\vee^{\mathfrak{A}}d\rangle)$, when taking $c = b$. Then, as, by Lemma 5.10, $\mathcal{K}^{(\mathrm{W})}_{\langle\mathrm{M}\rangle}$, being true in $\mathfrak{A}$, is so under $[x_0/(a\|(\neg^{\mathfrak{A}})b), x_1/(b\|a)\langle, x_2/d\rangle]$ $\langle$and $\mathfrak{A} \models (4.2)[x_0/d]\rangle$, we have both $((\neg^{\mathfrak{A}})b\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} (a\langle\vee^{\mathfrak{A}}d\rangle)$ $\langle$in which case, when taking $b = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d$ (resp., $b = \neg^{\mathfrak{A}}e$), we get $(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}e)) \leqslant^{\mathfrak{A}} (a \vee^{\mathfrak{A}} d(\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}e))\rangle$ and $(a\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}})b\langle\vee^{\mathfrak{A}}d\rangle)$ $\langle$in which case, when taking $b = d$ (resp., $b = e$), we get $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}e)) \leqslant^{\mathfrak{A}} (d(\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}e))\rangle$, and so eventually get $a = (\neg^{\mathfrak{A}})b$ $\langle$resp., $(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}e)) \leqslant^{\mathfrak{A}} (d(\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}e))$, since the

Σ_+-quasi-identity $\{(x_0 \wedge x_1) \lessapprox x_2, x_1 \lessapprox (x_0 \vee x_2)\} \to (x_1 \lessapprox x_2)$ is true in distributive lattices⟩. This, by Corollary 4.7 (and 5.5) ⟨as well as 5.8⟩, ends the proof. □

Before pursuing, note that, for each $i \in 2$,

$$\epsilon_i^4 \triangleq (\{\langle 3\cdot(1-i), \langle 1-i, 1, 1-i, 1-i\rangle\rangle\} \cup \{\langle j, \langle \epsilon_{3:0}^4(j-i), i\rangle\rangle \mid j \in ((3+i)\setminus i)\}),$$

being an isomorphism from $\mathfrak{D}_{4[,01]}$ onto $\mathfrak{MS}_{4:i[,01]}\restriction\Sigma_{+[,01]}$, is the one from $\mathfrak{K}_{4:i[,01]} \triangleq (\epsilon_i^4)^{-1}[\mathfrak{MS}_{4:i[,01]}]$ onto $\mathfrak{MS}_{4:i[,01]}$, in which case $\epsilon_i^8 \triangleq (\epsilon_i^4 \times \Delta_2)$ is that from $\mathfrak{K}_{4:i[,01]} \times \mathfrak{B}_{2[,01]}$ onto $\mathfrak{MS}_{4:i[,01]} \times \mathfrak{B}_{2[,01]}$, and so $\epsilon_i^5 \triangleq (\epsilon_i^8 \restriction \Re^{\mathfrak{K}_{4:i[,01]}})$ is so from $\Re(\mathfrak{K}_{4:i[,01]})$ onto $\Re(\mathfrak{MS}_{4:i[,01]})$, the former [bounded] MS lattices being preferably used below due to their having more transparent representation/notation of elements than those of the latter ones. Likewise, $\epsilon_i^5 \triangleq \{k+l \mid \langle k,l\rangle \in \Re^{\mathfrak{K}_{4:i[,01]}}\}$, being clearly injective, is an isomorphism from $\Re(\mathfrak{K}_{4:1[,01]})$ onto $\mathfrak{K}_{5:1[,01]} \triangleq \Re(\mathfrak{K}_{4:1[,01]})$ with $(\mathfrak{K}_{5:1[,01]}\restriction\Sigma_{+[,01]}) = \mathfrak{D}_{5[,01]}$.

Theorem 5.12. *Let* $\lfloor \mathsf{Q} \rfloor \mathsf{V} \triangleq \lfloor (\mathsf{W})\{\lceil \mathsf{M} \rceil\}\mathsf{R} \rfloor [\mathsf{B}/]\{\langle \mathsf{Q} \| \mathsf{P} \rangle \mathsf{S}\}(\mathsf{W})\mathsf{K}\{\lceil \mathsf{M} \rceil \mathsf{S}\}((\mathsf{L}[/\mathsf{A}])$ *and* $\mathsf{K} \triangleq (\mathsf{MS}_{\mathsf{V}[,01]} \cap \{\mathfrak{S}_{3[,01]}\lceil, \mathfrak{DM}_{4[,01]}\rceil(, \mathfrak{MS}_{2[,01]})\})$*. Then,* QV *is the pre-//quasi-variety generated by* $\Re_{(\mathrm{W})}[\mathsf{MS}_{\mathsf{V}[,01]} \setminus \mathsf{K}] \cup \mathsf{K}$*,* $\lceil \mathsf{M} \rceil \mathsf{R}[\mathsf{B}/]\{\langle \mathsf{Q} \| \mathsf{P} \rangle \mathsf{S}\}\mathsf{K}\{\lceil \mathsf{M} \rceil \mathsf{S}\}(\mathsf{L}[/\mathsf{A}])$ *being the one generated by* $\{\mathfrak{K}_{4\{\langle +1:1\|0\rangle\}[,01]}\{\lceil, \mathfrak{DM}_{4[,01]}\rceil\}\}\{\cup(\{\mathfrak{S}_{3[,01]}\}\langle\cap\varnothing\rangle)\langle\cup(\varnothing\| (\{\mathfrak{S}_{3[,01]}\}[\cup(\{\Re(\mathfrak{MS}_{2,01})\}/\varnothing)]))\rangle\}$*.*

Proof. Consider any finitely-generated

$$\mathfrak{A} \in (\mathsf{Q} \setminus ([\mathsf{B}]\mathsf{OMSL}(\cup[\mathsf{B}]\mathsf{TNIMSL})\{\lceil \cup [\mathsf{B}](\mathsf{A})\mathsf{DML}\rceil\}).$$

Take any $\bar{a} \in A^+$ such that $\mathfrak{A}$ is generated by $\operatorname{img} \bar{a}$. Let $n \triangleq (\operatorname{dom} \bar{a}) \in (\omega \setminus 1)$ and $b \triangleq \boldsymbol{\iota}_n^{\mathfrak{A}}(\bar{a})$, in which case, by the left alternative of Claim 5.2, we have $\neg^{\mathfrak{A}} b \leqslant^{\mathfrak{A}} b$. Consider any $\mathfrak{B} \in \mathsf{K}' \triangleq (\{\mathfrak{MS}_{6[,01]}\}[\cup(\{\mathfrak{MS}_{2,01}\}/\varnothing)])$ and $h \in \hom(\mathfrak{A}, \mathfrak{B})$ ⌈such that $(\operatorname{img} h) \not\subseteq (\operatorname{img} \epsilon_4^6)$, in which case, for some $i \in n$, $h(a_i) \notin (\operatorname{img} \epsilon_4^6)$, and so $\pi_0(h(a_i)) = 0 = (1 - \pi_0(h(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a_i)))$⌉. Let $(I|J) \triangleq \{j \in n \mid h(a_j) \notin (\mathfrak{F}|\mathfrak{I})_{(\mathrm{W})}^{\mathfrak{B}}\}$, $(\imath|\jmath) = |(I|J)|$ and $\bar{\Bbbk}|\bar{\ell}$ any bijection from $\imath|\jmath$ onto $I|J$. We prove, by contradiction, that there is some $g \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$ such that $g[\operatorname{img}((\bar{\Bbbk}|\bar{\ell}) \circ \bar{a})] = \{0|1\}$. For suppose that, for every $g \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$, there is either some $i' \in \imath$ or some $j' \in \jmath$ such that $g(a_{(\Bbbk|\ell)_{i'|j'}})) = (1|0)$, in which case, as, by Lemma 5.1 and Corollary 5.11, $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$, we have $(I \cup J) \neq \varnothing$, and so we are allowed to put $c \triangleq (\vee_+^{\mathfrak{A}}((\bar{\Bbbk} \circ \bar{a}(\circ\neg^{\mathfrak{A}} \circ \neg^{\mathfrak{A}})) * (\bar{\ell} \circ \bar{a} \circ \neg^{\mathfrak{A}})))$. Then, $\pi_{0\wr 2}(h((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c)) = 0$, in which case (by (4.6)) $\pi_0(h(\neg^{\mathfrak{A}} c)) = 1$, and so $(\neg^{\mathfrak{A}} c\lceil\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a_i\rceil) \not\leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c\lceil\vee^{\mathfrak{A}} a_i\rceil)$, for $(h \circ \pi_0) \in \hom(\mathfrak{A}\restriction\Sigma_+, \mathfrak{D}_2)$. Now, consider any $\mathfrak{C} \in \mathsf{K}'$, $f \in \hom(\mathfrak{A}, \mathfrak{C})$ and the following complementary cases:

- $(\operatorname{img} f) \subseteq (\operatorname{img} \epsilon_3^5)$,
  in which case, by (4.20), $e \triangleq (f \circ (\epsilon_3^5)^{-1} \circ \chi_3^{3\setminus 2}) \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$, and so, by the assumption to be disproved, $\pi_{1\wr 2}(f(c)) = e(c) = 1$. Then, $f(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} c) = \langle 0,0,0\rangle \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$.
- $(\operatorname{img} f) \not\subseteq (\operatorname{img} \epsilon_3^5)$,
  in which case, by the right alternative of Claim 5.2, $f(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} c) \leqslant^{\mathfrak{C}} f(b) = f(\neg^{\mathfrak{A}} b) \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$.

Thus, anyway, $f(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} c) \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$, in which case, by (2.8) and Theorem 4.4 [resp., Corollary 4.5], $(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} c) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$, and so $\mathfrak{A} \not\models \mathcal{R}_{\lceil \mathrm{M} \rceil}^{(\mathrm{W})}[x_0/b, x_1/c\lceil, x_2/a_i\rceil]$. This contradiction to the (weak) ⌈Morgan-⌉regularity of $\mathfrak{A}$ definitely shows that, for each $\mathfrak{D} \in ((\mathsf{MS}_{\mathsf{V}[,01]} \setminus \mathsf{K}) \subseteq \mathbf{IS}\mathsf{K}'$ and every $h' \in \hom(\mathfrak{A}, \mathfrak{D})$, there is some $g' \in \hom(\mathfrak{A}, \mathfrak{B}_2)$ such that $(\operatorname{img} f') \subseteq \Re_{(\mathrm{W})}^{\mathfrak{D}}$, where $f' \triangleq (h' \times g')$, in which case, by (2.7), $f' \in \hom(\mathfrak{A}, \Re_{(\mathrm{W})}(\mathfrak{D}))$, while, by (2.1), $(\ker f') \subseteq (\ker h')$, and so the locality of quasi-varieties, (2.8), (4.13), Corollaries 4.7 and 5.11 ⌈as well as the injectivity of ϵ_4^6⌉ complete the argument. □

This, by (2.9), Corollaries 4.7, 5.4 and Lemma 5.1, immediately yields:

Corollary 5.13. NIMR[B]QSMSL *is the pre-/quasi-variety generated by* $\{\mathfrak{K}_{5:1[,01]}, \mathfrak{DM}_{4[,01]} \times \mathfrak{B}_{2[,01]}\}$.

Corollary 5.14. MR[B]QS{W}KSL *is the pre-/quasi-variety generated by* $\{\mathfrak{K}_{5:1[,01]}, \mathfrak{K}_{3[,01]}\}$.

These, in their turn, by Corollaries 4.7, 5.4, 5.6, 5.8, 5.11 and (2.9), immediately yield:

Corollary 5.15. NIMR[B]QSKSL *is the pre-/quasi-variety generated by* $\{\mathfrak{K}_{5:1[,01]}, \mathfrak{K}_{3[,01]} \times \mathfrak{B}_{2[,01]}\}$.

Corollary 5.16. NIMR[B]QSMSL ∪ (MR)[B](QS){W}K(S)L *is the sub-quasi-variety of* MR[B]QSMSL *relatively axiomatized by either* $\{(5.4), (\neg x_2 \approx x_2) \to \mathcal{K}\}$ *or either version of* (5.3) *and is the pre-/quasi-variety generated by* $\{\mathfrak{K}_{5:1[,01]}, \mathfrak{K}_{3[,01]}, \mathfrak{DM}_{4[,01]} \times \mathfrak{B}_{2[,01]}\}$.

Thus, the apparatus of (weak) regularizations of [bounded] (weakly) Kleene-Stone lattices involved in proving Theorem 5.12 yields a more transparent and immediate insight/proof into/to [20, Proposition 4.7]. And what is more, it is involving $\boldsymbol{\iota}_n$ instead of $\wedge_+(\bar{\boldsymbol{\mu}}\restriction n)$, like therein, that has proved crucial for proving the ⌈⌉-optional version of Theorem 5.12 {though the former choice would suffice for proving the non-optional one, in its turn, sufficient within the framework of [B]SMSL; cf. the final equality therein}, in its turn, yielding axiomatizations of the quasi-equational joins of RQSKSL and all sub-quasi-varieties of DML not subsumed by RKL ⊆ RQSKSL (cf. [17] for latter ones), and so eventual finding the lattice of quasi-varieties of quasi-strong MS lattices, being equally due to the following series of "embedability" lemmas as well as "generation/axiomatization" corollaries presented above.

Lemma 5.17. $\mathfrak{K}_{4:1} \times \mathfrak{B}_2$ *is embedable into any*

$$\mathfrak{A} \in ((\mathsf{NIQSMSL} \cup \mathsf{DML}) \setminus \mathsf{MRQS\{K\}MSL}).$$

Proof. Then, there are some $a, b, c \in A$ such that $\neg^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} a$, $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} b)$ but $(\neg^{\mathfrak{A}} b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} \neg^{\mathfrak{A}} c) \nleqslant^{\mathfrak{A}} (b \vee^{\mathfrak{A}} c)$, in which case, by (4.2), (4.5) and (4.6), we have $((d|e)\|f) \triangleq (\neg^{\mathfrak{A}} \neg^{\mathfrak{A}} (a|b)\|(c \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} c \vee^{\mathfrak{A}} d) = \| \geqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} \neg^{\mathfrak{A}} (d|e)\|(\neg^{\mathfrak{A}} f/d))(\geqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} d|\,\|)$, while, applying (4.3) twice, by (4.1) and (4.5), we get $(d \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} e) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} d \vee^{\mathfrak{A}} e)$, whereas, by (2.8) and the []-()-non-optional version of Corollary 5.8, there are some $\mathfrak{C} \in \{\mathfrak{K}_{4:1} \times \mathfrak{B}_2, \mathfrak{DM}_4\}$ and $h \in \hom(\mathfrak{A}, \mathfrak{C})$ such that $(\neg^{\mathfrak{C}} h(b) \wedge^{\mathfrak{C}} \neg^{\mathfrak{C}} \neg^{\mathfrak{C}} h(c)) \nleqslant^{\mathfrak{C}} (h(b) \vee^{\mathfrak{C}} h(c))$, and so $\mathfrak{C} \triangleq (\mathfrak{K}_{4:1} \times \mathfrak{B}_2)$ and $h((a\|d)|(b\|e)|(c\|f)) = \langle 1|0|2, 1\rangle$, for $\neg^{\mathfrak{C}} h(a) \leqslant^{\mathfrak{C}} h(a)$ and $(h(a) \wedge^{\mathfrak{C}} \neg^{\mathfrak{C}} h(b)) \leqslant^{\mathfrak{C}} (\neg^{\mathfrak{C}} h(a) \vee^{\mathfrak{C}} h(b))$. In that case, using (4.1), (4.2), (4.5) and (4.6), it is routine checking that the mapping $g : (4 \times 2) \to A$, given by:

$$\begin{aligned} g(\langle 0|1, 1\rangle) &\triangleq ((d \wedge^{\mathfrak{A}} (e \vee^{\mathfrak{A}} (e|\neg^{\mathfrak{A}} d))) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} f), \\ g(\langle 3|1, 0\rangle) &\triangleq \neg^{\mathfrak{A}} g(\langle 0|1, 1\rangle), \\ g(\langle 0|3, 0|1\rangle) &\triangleq (g(\langle 0, 1\rangle)(\wedge|\vee)^{\mathfrak{A}} g(\langle 3, 0\rangle)), \\ g(\langle 2, 1\rangle) &\triangleq (((d \wedge^{\mathfrak{A}} e) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} (d \wedge^{\mathfrak{A}} e)) \wedge^{\mathfrak{A}} f), \end{aligned}$$

is a homomorphism from $\mathfrak{K}_{4:1} \times \mathfrak{B}_2$ to $\mathfrak{A}$ such that $(g \circ h) = \Delta_{4\times 2}$, and so it is injective, as required. □

Lemma 5.18. $\mathfrak{K}_4$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{NIQSMSL} \setminus \mathsf{SL}) \supseteq (\mathsf{RQSKSL} \setminus \mathsf{SL})$.

Proof. Then, there are some $a, b \in A$ such that $c \triangleq (a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \neq d \triangleq (b \wedge^{\mathfrak{A}} c) \leqslant^{\mathfrak{A}} c$, in which case, applying (4.1) and (4.3) [twice], we have $[\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} d \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}] c \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} c \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} d$, and so, by (4.2) and (4.11), we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d) = (c|d)$. In this way, as $c \neq d$, by (5.1), we have $\neg^{\mathfrak{A}} c \neq c$, in which case we get $\neg^{\mathfrak{A}} d \neq \neg^{\mathfrak{A}} c$, and so $\{\langle 0, d\rangle, \langle 1, c\rangle, \langle 2, \neg^{\mathfrak{A}} c\rangle, \langle 3, \neg^{\mathfrak{A}} d\rangle\}$ is an embedding of $\mathfrak{K}_4$ into $\mathfrak{A}$. Finally, Corollary 5.11 completes the argument. □

Lemma 5.19. $\mathfrak{K}_3$ *is embeddable into any idempotent quasi-strong MS lattice* $\mathfrak{A}$.

Proof. In that case, there are some $a, b \in A$ such that $\neg^{\mathfrak{A}} a = a \neq b$, and so, by (4.1), (4.3), (4.5), (4.6) and (4.11), $c \triangleq (a \wedge^{\mathfrak{A}} (b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b)) = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} c \leqslant^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} d \triangleq \neg^{\mathfrak{A}} c$, Then, $\neg^{\mathfrak{A}} d = c \neq a$, for, otherwise, we would have $b \geqslant^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} b$, the latter implying, by (4.2) and (4.3), $b \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} b \leqslant^{\mathfrak{A}} a$, in which case we get $d \neq a$, and so $\{\langle 0, c\rangle, \langle 1, a\rangle, \langle 2, d\rangle\}$ is an embedding of $\mathfrak{K}_3$ into $\mathfrak{A}$, as required. □

On the other hand, the stipulation "quasi-strong" here can be neither omitted nor replaced by the one "pseudo-strong" nor, even, weakened with replacing it by that "almost quasi-strong", when taking $\mathfrak{A} = \mathfrak{MS}_{2,01}$.

Lemma 5.20. $\mathfrak{DM}_4$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{QSMSL} \setminus (\mathsf{NIQSMSL} \cup \mathsf{QSKSL}))$.

Proof. In that case, by Corollary 5.6, there are some $a, b \in A$ such that $\mathfrak{A} \not\models \{(5.3)\}[x_0/a, x_1/b]$, and so (5.3) is not true in the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{a, \neg^{\mathfrak{A}} b\}$ under $[x_0/a, x_1/\neg^{\mathfrak{A}} b]$. On the other hand, by (4.1), (4.5), (4.6) and induction on construction of any $\varphi \in \mathrm{Tm}^2_{\Sigma^-_+}$, we have $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(a, \neg^{\mathfrak{A}} b) = \varphi^{\mathfrak{A}}(a, \neg^{\mathfrak{A}} b)$, in which case $\mathfrak{B}$ is a De Morgan lattice, and so $\mathfrak{DM}_4$, being embedable into $\mathfrak{B}$, in view of [17, Case 8 of Proof of Theorem 4.8], is so into $\mathfrak{A}$, as required. □

Lemma 5.21. $\mathfrak{K}_3 \times \mathfrak{B}_2$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{NIQSKSL} \setminus \mathsf{RQSKSL})$.

Proof. Then, by (4.1), (4.3), (4.5), (4.6) and (4.13), there are some $a, b \in A$ such that $(c|d) \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a|b)(\geqslant | \not\geqslant)^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d)$ and $(c \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} d) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} c \vee^{\mathfrak{A}} d)$, in which case, by (4.1), (4.5), (4.6) and induction on construction of any $\varphi \in \mathrm{Tm}^2_{\Sigma^-_+}$, we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(c, d) = \varphi^{\mathfrak{A}}(c, d)$, and so the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{c, d\}$ is a non-idempotent Kleene lattice such that $\mathfrak{B} \not\models \mathcal{R}[x_0/c, x_1/d]$. Thus, $\mathfrak{K}_3 \times \mathfrak{B}_2$, being embeddable into $\mathfrak{B}$, by [17, Case 4 of Proof of Theorem 4.8], is so into $\mathfrak{A}$. □

Lemma 5.22. $\mathfrak{DM}_4 \times \mathfrak{B}_2$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{NIQSMSL} \setminus \mathsf{QSKSL})$.

Proof. Then, as $\mathsf{QSWKSL} = \mathsf{QSKSL}$ (cf. Corollary 4.7), there are some $a, b \in A$ such that, by (4.2), $c \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \not\leqslant^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} b)$, in which case, by (4.1), (4.5) and (4.6), we have both $\neg^{\mathfrak{A}}(c|d)(\geqslant | \leqslant)^{\mathfrak{A}}(c|d) = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d)$, and so, by induction on construction of any $\varphi \in \mathrm{Tm}^2_{\Sigma^-_+}$, we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(c, d) = \varphi^{\mathfrak{A}}(c, d)$. Thus, the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{c, d\}$ is a non-idempotent De Morgan lattice such that $\mathfrak{B} \not\models \mathcal{K}[x_0/c, x_1/d]$, in which case $\mathfrak{DM}_4 \times \mathfrak{B}_2$ being embeddable into $\mathfrak{B}$, in view of the proof of [17, Lemma 4.10], is so into $\mathfrak{A}$, as required. □

Lemma 5.23. *Let* $\mathfrak{A} \in \mathsf{QSMSL}$ *and* $a \in A$. *Suppose* $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \neq a$. *Then,* $b \triangleq (\neg^{\mathfrak{A}} a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a) \leqslant^{\mathfrak{A}} c \triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \leqslant^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a)$, *while both* $\neg^{\mathfrak{A}} c = b = \neg^{\mathfrak{A}} d$ *and* $\neg^{\mathfrak{A}} b = d$, *whereas* $b \neq c \neq d$, *in which case* $\{\langle 0, b\rangle, \langle 1, c\rangle, \langle 2, d\rangle\}$ *is an embedding of* $\mathfrak{S}_3$ *into* $\mathfrak{A}$, *and so* $\mathfrak{S}_3$ *is embeddable into any member of* $(\mathsf{QSMSL} \setminus \mathsf{DML})$.

Proof. In that case, by (4.2), $b \leqslant^{\mathfrak{A}} c \leqslant^{\mathfrak{A}} d$, while, by (4.1), (4.5) and (4.6), both $\neg^{\mathfrak{A}} c = b = \neg^{\mathfrak{A}} d$ and $\neg^{\mathfrak{A}} b = d$, whereas $c \neq d$, for, otherwise, since $\mathfrak{A} \models (4.2|4.11)[x_0/a]$, $\{b, \neg^{\mathfrak{A}} a, a, \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a, d\}$ would be a pentagon of the distributive lattice $\mathfrak{A}\upharpoonright\Sigma_+$, and so $b \neq c$, for otherwise, we would have $c = b = \neg^{\mathfrak{A}} c = \neg^{\mathfrak{A}} b = d$. □

Lemma 5.24. $\mathfrak{K}_{4:1}$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{QSMSL} \setminus (\mathsf{NIQSMSL} \cup \mathsf{DML}))$.

Proof. In that case, by Corollary 5.8, there are some $a, e \in A$ such that $\mathfrak{A} \not\models (5.4)[x_0/a, x_1/e]$, i.e., $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \neq a$ and $\neg^{\mathfrak{A}} e = e$. Let $b, c, d \in A$ be as in Lemma 5.23, in which case $c \geqslant^{\mathfrak{A}} f \triangleq ((e \wedge^{\mathfrak{A}} c) \vee^{\mathfrak{A}} b) \geqslant^{\mathfrak{A}} b$, while $f \leqslant^{\mathfrak{A}} g \triangleq ((e \wedge^{\mathfrak{A}} d) \vee^{\mathfrak{A}} b)$, whereas, by (4.1) and (4.5), $\neg^{\mathfrak{A}} f = g = \neg^{\mathfrak{A}} g$, and so, since $\mathfrak{A} \models (4.11)[x_0/f]$, we get $f = g$ as well as $f \notin \{b, c\}$, for, otherwise, we would have $b = c$. Thus, $\{\langle 0, b\rangle, \langle 1, f\rangle, \langle 2, c\rangle, \langle 3, d\rangle\}$ is an embedding of $\mathfrak{K}_{4:1}$ into $\mathfrak{A}$, as required. □

Lemma 5.25. $\mathfrak{K}_{5:1}$ *is embeddable into an arbitrary* $\mathfrak{A} \in ((\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}) \setminus \mathsf{PSMSL})$.

Proof. Take any $a, e \in A$ such that $\mathfrak{A} \not\models (4.12)[x_0/a, x_1/e]$, in which case $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \nleq^{\mathfrak{A}} (a \vee^{\mathfrak{A}} f)$, where $f \triangleq (\neg^{\mathfrak{A}} e \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} e)) \geqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} f$, in view of (4.5), and so $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \neq a$. On the other hand, by Lemmas 4.6, 5.7, Corollary 5.4 and Theorem 5.12, $\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}$ is the pre-variety generated by $\mathsf{K} \triangleq \{\mathfrak{K}_{4:1} \times \mathfrak{B}_2, \mathfrak{DM}_4\}$, in which case, by (2.8), there are some $\mathfrak{C} \in \mathsf{K}$ and $h \in \hom(\mathfrak{A}, \mathfrak{C})$ such that $h(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a) \nleq^{\mathfrak{C}} h(a \vee^{\mathfrak{A}} f)$, and so $\mathfrak{C} = (\mathfrak{K}_{4:1} \times \mathfrak{B}_2)$, while $\pi_1(h(f)) = 1$, whereas $\pi_0(h((a|e)) = (2|1)$. Let $b, c, d \in A$ be as in Lemma 5.23 and $g \triangleq \{\langle 0, 0, b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} f\rangle, \langle 1, 0, \neg^{\mathfrak{A}} f\rangle, \langle 1, 1, f\rangle, \langle 2, 1, c \vee^{\mathfrak{A}} f\rangle, \langle 3, 1, d \vee^{\mathfrak{A}} f\rangle\} : \Re^{\mathfrak{K}_{4:1}} \to A$, in which case, for all $\bar{\imath}, \bar{\jmath} \in \Re^{\mathfrak{K}_{4:1}}$, $(\bar{\imath} \leqslant^{\mathfrak{D}_4} \bar{\jmath}) \Rightarrow (g(\bar{\imath}) \leqslant^{\mathfrak{A}} g(\bar{\jmath}))$ as well as $h(g(\bar{\imath})) = \bar{\imath}$, and so, since $\Re(\mathfrak{K}_{4:1})\upharpoonright\Sigma_+$ is a chain lattice, by (4.1), (4.5) and (4.6), g is an embedding of $\Re(\mathfrak{K}_{4:1}) \in \mathbf{I}(\mathfrak{K}_{5:1})$ into $\mathfrak{A}$, as required. □

Theorem 5.26. *Sub-pre/quasi-varieties of* $[(\mathsf{Q})\mathsf{S}]\mathsf{M}[\mathsf{S}]\mathsf{L}$ *form the non-chain lattice with* $8[+((3\cdot)7)]$ *elements, the Hasse diagram of which with [small (both non-solid and) solid as well as] large circles|nodes is depicted at Figure 4, in which case it is embedable into* $[\mathfrak{D}_{2(+3)}\times](\mathfrak{D}_5 \times \mathfrak{D}_3)$*, and so is distributive.*

Proof. We use Corollary 4.7 tacitly. First, by Corollaries 5.6, 5.8, 5.11 and 5.16, the 15(+14) subclasses of (Q)SMSL involved are quasi-equational. Next, by Corollary 5.4, $(\mathfrak{DM}_4 \times \mathfrak{B}_2) \in \mathsf{NIDML}$ is not in QSKSL, for $\mathfrak{DM}_4$ is not so, while $\pi_0\upharpoonright(2^2 \times \Delta_2)$ is a surjective homomorphism from the former onto the latter, in which case $\mathsf{NI}\{([\mathsf{MR}]\mathsf{Q})\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L} \subsetneq \mathsf{NI}\{([\mathsf{MR}]\mathsf{Q})\mathsf{S}\}\mathsf{M}\{\mathsf{S}\}\mathsf{L}$ as well as, by Corollaries 5.6, 5.11 and Lemma 5.1, $(\{\langle[(\mathsf{NI})|(\mathsf{MR})]\mathsf{Q}\rangle\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L}\{\langle[\cup(\mathsf{KL}|\varnothing)]\rangle\}) \subsetneq ((\{\langle\mathsf{Q}\rangle\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L}\{\langle[\cap\mathsf{KL}]\rangle\}) \cup \mathsf{NI}\{\langle[|(\mathsf{MR})]\mathsf{Q}\rangle\mathsf{S}\}\mathsf{M}\{\mathsf{S}\}\mathsf{L}) \subsetneq (\{\langle[(\mathsf{NI})|(\mathsf{MR})]\mathsf{Q}\rangle\mathsf{S}\}\mathsf{M}\{\mathsf{S}\}\mathsf{L}\{\langle[\cup(\mathsf{DML}|\varnothing)]\rangle\})$, for $\mathsf{DML} \ni \mathfrak{DM}_4 \not\models (5.3)[x_i/\langle i, 1-i\rangle]_{i\in 2}$. Likewise, $\mathfrak{K}_{5:1}$, being the isomorphic copy of $\Re(\mathfrak{K}_{4:1}) \in (\mathsf{M})\mathsf{RQSKSL}$ by ϵ^5_i, is not strong, for its homomorphic image $\mathfrak{K}_{4:1}$ by $(\epsilon^5_i)^{-1} \circ \pi_0$ is not so, in which case $\mathsf{RSKSL} \subsetneq \mathsf{RQSKSL}$ as well as, by Corollaries 5.4, 5.8, 5.11 and Lemma 5.1, both $([\mathsf{NI}]\mathsf{S}(\mathsf{K}|\mathsf{M})\mathsf{SL}[\{\cup(\varnothing|\mathsf{SKSL})\}]) \subsetneq ([\mathsf{NI}]\mathsf{MRQS}(\mathsf{K}|\mathsf{M})\mathsf{SL}[\{\cup(\varnothing|\mathsf{KL})\}]) \subsetneq (\mathsf{NIQS}(\mathsf{K}|\mathsf{M})\mathsf{SL} \cup ([(\varnothing\{\cup\mathsf{KL}\})\cap](\mathsf{K}|(\mathsf{DM}))\mathsf{L}))$, for $\mathsf{NIQSKSL} \ni (\mathfrak{K}_{4:1} \times \mathfrak{B}_2) \not\models \mathcal{K}_{\mathrm{M}}[x_i/\langle(2 \cdot \chi_3^{3\setminus 2}(i)) + (1 - \min(i, 1)), 1\rangle]_{i\in 3}$, and $(\mathsf{NIQS}(\mathsf{K}\|\mathsf{M})\mathsf{L} \cup (\langle KL\cap\rangle(\mathsf{K}\|(\mathsf{DM}))\mathsf{L})) \subsetneq (\langle\mathsf{NI}\rangle\mathsf{QS}(\mathsf{K}\|\mathsf{M})\mathsf{SL}\langle\cup\mathsf{QSKSL}\rangle)$, for $\mathsf{QSMSL} \ni \mathfrak{K}_{4:1} \not\models (5.4)[x_j/(2-j)]_{j\in 2}$. Furthermore, $\mathfrak{S}_3 \notin \mathsf{DML}$, so, by Corollary 5.11, $(\mathsf{KL} \cup \mathsf{NIDML}) \subsetneq (\mathsf{SKSL} \cup \mathsf{NISMSL})$, $[\mathsf{NI}]\mathsf{DML} \subsetneq [\mathsf{NI}]\mathsf{SMSL}$, $[\mathsf{NI}]\mathsf{KL} \subsetneq [\mathsf{NI}]\mathsf{SKSL}$ and $\mathsf{RKL} \subsetneq \mathsf{RSKSL}$, while, by Corollary 5.4, $\mathsf{NIDML} \supseteq \mathsf{NIKL} \ni (\mathfrak{K}_3 \times \mathfrak{B}_2) \not\models \mathcal{R}[x_i/\langle 1-i, 1\rangle]_{i\in 2}$, so, by Corollary 5.11, $\mathsf{R}\{(\mathsf{Q})\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L} \subsetneq \mathsf{NI}\{(\mathsf{MRQ})\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L}$, whereas $\mathsf{KL} \ni \mathfrak{K}_3 \not\models (5.1)[x_j/(1-j)]_{j\in 2}$, so, by Corollary 5.11,

$$\mathsf{NI}\{[\langle\mathsf{MR}\rangle\mathsf{Q}]\mathsf{S}\}(\mathsf{K}|\mathsf{M})\{\mathsf{S}\}\mathsf{L} \subsetneq (\mathsf{NI}\{[\langle\mathsf{MR}\rangle\mathsf{Q}]\mathsf{S}\}(\mathsf{K}|\mathsf{M})\{\mathsf{S}\}\mathsf{L} \cup (\{[\langle\mathsf{MR}\rangle\mathsf{Q}]\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L}\|\mathsf{KL})).$$

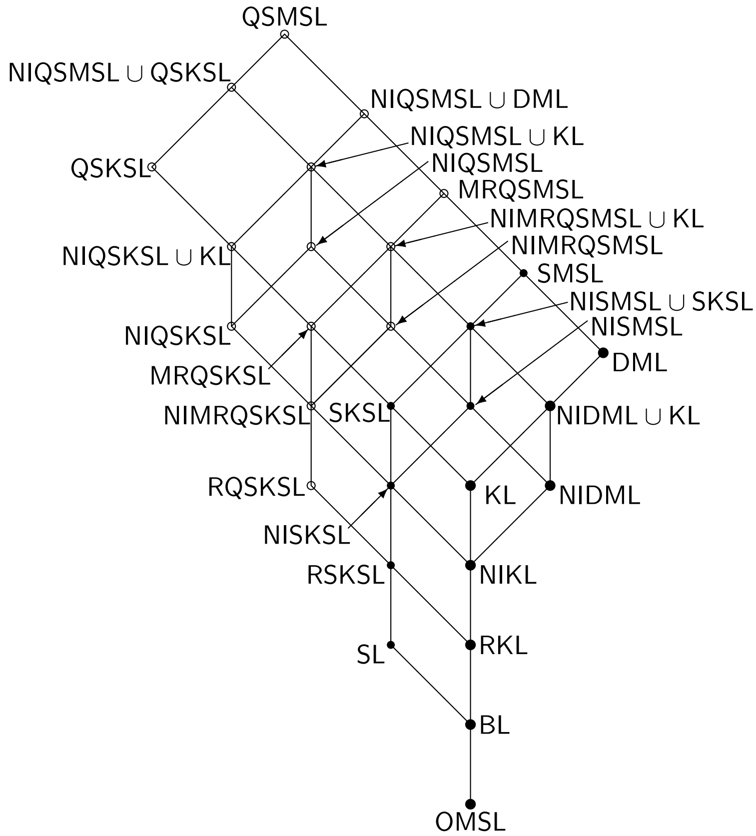

FIGURE 4. The lattice of pre-/quasi-varieties of quasi-strong Morgan-Stone lattices.

Finally, by Theorem 5.12, $\mathfrak{S}_3 \in \mathsf{RSKSL} \ni \mathfrak{K}_4 \not\models (4.10)[x_i/(1-i)]_{i\in 2}$, so both $\mathsf{SL} \subsetneq \mathsf{RSKSL}$ and $\mathsf{BL} \subsetneq \mathsf{RKL}$. Thus, the 29 quasi-varieties involved are pair-wise distinct and do form the poset depicted at Figure 4. Now, consider any pre-variety $\mathsf{P} \subseteq \mathsf{QSMSL}$ and the following complementary cases:

- $\mathsf{P} \subseteq \mathsf{SMSL}$.
  Consider the following complementary subcases:
  - $\mathsf{P} \subseteq \mathsf{DML}$.
    Consider the following exhaustive subcases:
    (i) $\mathsf{P} \not\subseteq (\mathsf{NIDML} \cup \mathsf{KL})$,
    in which case $\mathsf{P} \not\subseteq (\mathsf{NIQSMSL} \cup \mathsf{QSKSL})$, and so, by Lemma 5.20, $\mathsf{P} \ni \mathfrak{DM}_4$ is equal to DML.
    (ii) $\mathsf{P} \subseteq (\mathsf{NIDML} \cup \mathsf{KL})$ but neither $\mathsf{P} \subseteq (\mathsf{NIDML}|\mathsf{KL})$,
    in which case both $\varnothing \neq (\mathsf{P} \setminus (\mathsf{NIDML}|\mathsf{KL})) \subseteq (\mathsf{P} \cap (\mathsf{IQSMSL}|(\mathsf{NIQSMSL} \setminus \mathsf{QSKSL})))$, and so, by Lemma 5.19|5.22, both $(\mathfrak{K}_3|(\mathfrak{DM}_4 \times \mathfrak{B}_2)) \in \mathsf{P}$. Then, by Corollary 5.6, $\mathsf{P} = (\mathsf{NIDML} \cup \mathsf{KL})$.
    (iii) $\mathsf{NIKL} \not\supseteq \mathsf{P} \subseteq \mathsf{KL}$,
    in which case $\varnothing \neq (\mathsf{P} \setminus \mathsf{NIKL}) \subseteq (\mathsf{P} \cap \mathsf{IQSMSL})$, and so, by Lemma 5.19, $\mathsf{P} \ni \mathfrak{K}_3$ is equal to KL.
    (iv) $\mathsf{NIKL} \not\supseteq \mathsf{P} \subseteq \mathsf{NIDML}$,
    in which case $\varnothing \neq (\mathsf{P} \setminus \mathsf{NIKL}) \subseteq (\mathsf{P} \cap (\mathsf{NIQSMSL} \setminus \mathsf{QSKSL}))$, and

so, by Corollary 5.4 and Lemma 5.22, $\mathsf{P} \ni (\mathfrak{DM}_4 \times \mathfrak{B}_2)$ is equal to NIDML.

(v) $\mathsf{RKL} \not\supseteq \mathsf{P} \subseteq \mathsf{NIKL}$,
in which case $\varnothing \neq (\mathsf{P} \setminus \mathsf{RKL}) \subseteq (\mathsf{P} \cap (\mathsf{NIQSKSL} \setminus \mathsf{RQSKSL}))$, and so, by Corollary 5.4 and Lemma 5.21, $\mathsf{P} \ni (\mathfrak{K}_3 \times \mathfrak{B}_2)$ is equal to NIKL.

(vi) $\mathsf{BL} \not\supseteq \mathsf{P} \subseteq \mathsf{RKL}$,
in which case $\varnothing \neq (\mathsf{P} \setminus \mathsf{BL}) \subseteq (\mathsf{P} \cap (\mathsf{RQSKSL} \setminus \mathsf{SL}))$, and so, by Theorem 5.12 and Lemma 5.18, $\mathsf{P} \ni \mathfrak{K}_4$ is equal to RKL.

(vii) $\mathsf{OMSL} \not\supseteq \mathsf{P} \subseteq \mathsf{BL}$,
in which case $\varnothing \neq (\mathsf{P} \setminus \mathsf{OMSL}) \subseteq (\mathsf{P} \cap (\mathsf{QSMSL} \setminus \mathsf{OMSL}))$, and so, by Lemma 5.3, $\mathsf{P} \ni \mathfrak{B}_2$ is equal to BL.

(viii) $\mathsf{P} \subseteq \mathsf{OMSL}$,
in which case $\mathsf{P} = \mathsf{OMSL}$.

– $\mathsf{P} \not\subseteq \mathsf{DML}$,
in which case, by Lemma 5.23, $\mathfrak{S}_3 \in \mathsf{P}$, and so $\mathsf{SL} \subseteq \mathsf{P}$. Consider the following exhaustive subcases:

(1) $\mathsf{P} \not\subseteq (\mathsf{SKSL} \cup \mathsf{NISMSL})$,
in which case, by Lemma 5.20, $\mathfrak{DM}_4 \in \mathsf{P} \ni \mathfrak{S}_3$, and so $\mathsf{P} = \mathsf{SMSL}$.

(2) $\mathsf{P} \subseteq (\mathsf{SKSL} \cup \mathsf{NISMSL})$ but neither $\mathsf{P} \subseteq (\mathsf{SKSL}|\mathsf{NISMSL})$,
in which case neither $(\mathsf{SKSL}|\mathsf{NISMSL}) \supseteq (\mathsf{P} \cap (\mathsf{NISMSL}|\mathsf{SKSL}))$, and so, by Lemma 5.22|5.19 both $((\mathfrak{DM}_4 \times \mathfrak{B}_2)|\mathfrak{K}_3) \in \mathsf{P} \ni \mathfrak{S}_3$. Then, by Corollary 5.6, $\mathsf{P} = (\mathsf{SKSL} \cup \mathsf{NISMSL})$.

(3) $\mathsf{P} \subseteq \mathsf{NISMSL}$ but $\mathsf{P} \not\subseteq \mathsf{SKSL}$,
in which case, by Lemma 5.22, $(\mathfrak{DM}_4 \times \mathfrak{B}_2) \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Corollary 5.4, $\mathsf{P} = \mathsf{NISMSL}$.

(4) $\mathsf{P} \subseteq \mathsf{SKSL}$ but $\mathsf{P} \not\subseteq \mathsf{NISMSL}$,
in which case, by Lemma 5.19, $\mathfrak{K}_3 \in \mathsf{P} \ni \mathfrak{S}_3$, and so $\mathsf{P} = \mathsf{SKSL}$.

(5) $\mathsf{P} \subseteq \mathsf{NISKSL}$ but $\mathsf{P} \not\subseteq \mathsf{RSKSL}$,
in which case, by Lemma 5.21, $(\mathfrak{K}_3 \times \mathfrak{B}_2) \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Corollary 5.4, $\mathsf{P} = \mathsf{NISKSL}$.

(6) $\mathsf{P} \subseteq \mathsf{RSKSL}$ but $\mathsf{P} \not\subseteq \mathsf{SL}$,
in which case, by Lemma 5.18, $\mathfrak{K}_4 \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Theorem 5.12, $\mathsf{P} = \mathsf{RSKSL}$.

(7) $\mathsf{P} \subseteq \mathsf{SL}$,
in which case $\mathsf{P} = \mathsf{SL}$.

- $\mathsf{P} \not\subseteq \mathsf{SMSL}$.
Consider the following exhaustive subcases:

(a) neither $\mathsf{P} \subseteq (\mathsf{NIQSMSL} \cup (\mathsf{QSKSL}|\mathsf{DML}))$,
in which case, by Lemma 5.20|5.24, both $(\mathfrak{DM}_4|\mathfrak{K}_{4:1}) \in \mathsf{P}$, and so $\mathsf{P} = \mathsf{QSMSL}$.

(b) $\mathsf{P} \subseteq (\mathsf{NIQSMSL} \cup \mathsf{QSKSL})$ but neither $\mathsf{P} \subseteq (\mathsf{QSKSL}|(\mathsf{NIQSMSL} \cup \mathsf{KL}))$,
in which case both $\varnothing \neq (\mathsf{P} \setminus (\mathsf{QSKSL}|(\mathsf{NIQSMSL} \cup \mathsf{KL}))) \subseteq (\mathsf{P} \cap ((\mathsf{NIQSMSL} \setminus \mathsf{QSKSL})|(\mathsf{QSMSL} \setminus (\mathsf{NIQSMSL} \cup \mathsf{DML}))))$, and so, by Lemma 5.22|5.24, both $((\mathfrak{DM}_4 \times \mathfrak{B}_2)|\mathfrak{K}_{4:1}) \in \mathsf{P}$. Then, by Corollary 5.6, $\mathsf{P} = (\mathsf{NIQSMSL} \cup \mathsf{QSKSL})$.

(c) $\mathsf{P} \subseteq (\mathsf{NIQSMSL} \cup \mathsf{DML})$ but neither

$$\mathsf{P} \subseteq (\mathsf{MRQSMSL}|(\mathsf{NIQSMSL} \cup \mathsf{KL})),$$

in which case both $\varnothing \neq (\mathsf{P} \setminus (\mathsf{MRQSMSL}|(\mathsf{NIQSMSL} \cup \mathsf{KL}))) \subseteq (\mathsf{P} \cap (((\mathsf{NIQSMSL} \cup \mathsf{DML}) \setminus \mathsf{MRQSMSL})|(\mathsf{QSMSL} \setminus (\mathsf{NIQSMSL} \cup \mathsf{QSKSL}))))$, and so, by Lemma 5.17|5.20, both $((\mathfrak{K}_{4:1} \times \mathfrak{B}_2)|\mathfrak{DM}_4) \in \mathsf{P}$. Then, by Corollary 5.8, $\mathsf{P} = (\mathsf{NIQSMSL} \cup \mathsf{DML})$.

(d) $(\mathsf{NIQSMSL} \cup \mathsf{KL}) \not\supseteq \mathsf{P} \subseteq \mathsf{QSKSL}$,
in which case $\mathsf{P} \not\subseteq (\mathsf{NIQSMSL} \cup \mathsf{DML})$, and so, by Lemma 5.24, $\mathfrak{K}_{4:1} \in \mathsf{P}$. Then, $\mathsf{P} = \mathsf{QSKSL}$.

(e) $\mathsf{P} \subseteq (\mathsf{NIQSMSL} \cup \mathsf{KL})$ but neither

$$\mathsf{P} \subseteq ((\mathsf{NIQSKSL} \cup \mathsf{KL})|\mathsf{NIQSMSL}|(\mathsf{NIMRQSMSL} \cup \mathsf{KL})),$$

in which case ||"by Corollary 5.11" both

$$\varnothing \neq (\mathsf{P} \setminus ((\mathsf{NIQSKSL} \cup \mathsf{KL})|\mathsf{NIQSMSL}|(\mathsf{NIMRQSMSL} \cup \mathsf{KL}))) \subseteq (\mathsf{P} \cap ((\mathsf{NIQSMSL} \setminus \mathsf{QSKSL})|\mathsf{IQSMSL}|((\mathsf{NIQSMSL} \cup \mathsf{DML}) \setminus \mathsf{MRQSMSL}))),$$

and so, by Lemma 5.22|5.19|5.17, both $((\mathfrak{D}_4 \times \mathfrak{B}_2)|\mathfrak{K}_3|(\mathfrak{K}_{4:1} \times \mathfrak{B}_2)) \in \mathsf{P}$. Then, by Corollary 5.8, $\mathsf{P} = (\mathsf{NIQSMSL} \cup \mathsf{KL})$.

(f) $\mathsf{P} \subseteq (\mathsf{NIQSKSL} \cup \mathsf{KL})$ but neither $\mathsf{P} \subseteq (\mathsf{NIQSKSL}|\mathsf{MRQSKSL})$,
in which case both $\varnothing \neq (\mathsf{P} \setminus (\mathsf{NIQSKSL}|\mathsf{MRQSKSL})) \subseteq (\mathsf{P} \cap (\mathsf{IQSKSL}|((\mathsf{NIQSMSL} \cup \mathsf{DML}) \setminus \mathsf{MRQSMSL})))$, and so, by Lemma 5.19|5.17, both $(\mathfrak{K}_3|(\mathfrak{K}_{4:1} \times \mathfrak{B}_2)) \in \mathsf{P}$. Then, by Corollary 5.8, $\mathsf{P} = (\mathsf{NIQSKSL} \cup \mathsf{KL})$.

(g) $\mathsf{P} \subseteq \mathsf{NIQSMSL}$ but neither $\mathsf{P} \subseteq (\mathsf{NIQSKSL}|\mathsf{NIMRQSMSL})$,
in which case both $\varnothing \neq (\mathsf{P} \setminus (\mathsf{NIQSKSL}|\mathsf{NIMRQSMSL})) \subseteq (\mathsf{P} \cap ((\mathsf{NIQSMSL} \setminus \mathsf{QSKSL})|((\mathsf{NIQSMSL} \cup \mathsf{DML}) \setminus \mathsf{MRQSMSL})))$, and so, by Lemma 5.22|5.17, both $((\mathfrak{DM}_4|\mathfrak{K}_{4:1}) \times \mathfrak{B}_2)) \in \mathsf{P}$. Then, by Corollary 5.4, $\mathsf{P} = \mathsf{NIQSMSL}$.

(h) $(\mathsf{NIMRQSMSL} \cup \mathsf{KL}) \not\supseteq \mathsf{P} \subseteq \mathsf{MRQSMSL}$,
in which case "by Corollary 5.11"|"as $\mathsf{P} \not\subseteq \mathsf{SMSL}$" we have both $\varnothing \neq (\mathsf{P} \setminus ((\mathsf{NIMRQSMSL} \cup \mathsf{KL})|\mathsf{SMSL})) \subseteq (\mathsf{P} \cap ((\mathsf{QSMSL} \setminus (\mathsf{NIQSMSL} \cup \mathsf{QSKSL}))|((\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}) \setminus \mathsf{PSMSL})))$, and so, by Lemma 5.20|5.25, get both $(\mathfrak{DM}_4|\mathfrak{K}_{5:1}) \in \mathsf{P}$. Then, by Theorem 5.12, $\mathsf{P} = \mathsf{MRQSMSL}$.

(i) $\mathsf{NIMRQSKSL} \not\supseteq \mathsf{P} \subseteq \mathsf{NIQSKSL}$,
in which case $\mathsf{P} \not\subseteq \mathsf{MRQSMSL}$, and so, by Lemma 5.17, $(\mathfrak{K}_{4:1} \times \mathfrak{B}_2) \in \mathsf{P}$. Then, by Corollary 5.4, $\mathsf{P} = \mathsf{NIQSKSL}$.

(j) $\mathsf{P} \subseteq (\mathsf{NIMRQSMSL} \cup \mathsf{KL})$ but neither $\mathsf{P} \subseteq (\mathsf{MRQSKSL}|\mathsf{NIMRQSMSL})$,
in which case, by Corollary 5.11, ||"as $\mathsf{P} \not\subseteq \mathsf{SMSL}$" both

$$\varnothing \neq (\mathsf{P} \setminus (\mathsf{MRQSKSL}|\mathsf{NIMRQSMSL}|\mathsf{SMSL})) \subseteq (\mathsf{P} \cap ((\mathsf{NIQSMSL} \setminus \mathsf{QSKSL})| \mathsf{IQSMSL}|((\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}) \setminus \mathsf{PSMSL}))),$$

and so, by Lemma 5.22|5.19|5.25, both $((\mathfrak{DM}_4 \times \mathfrak{B}_2)|\mathfrak{K}_3|\mathfrak{K}_{5:1}) \in \mathsf{P}$. Then, by Corollary 5.16, $\mathsf{P} = (\mathsf{NIMRQSMSL} \cup \mathsf{KL})$.

(k) $\mathsf{NIMRQSKSL} \not\supseteq \mathsf{P} \subseteq \mathsf{MRQSKSL}$,
in which case |"as $\mathsf{P} \not\subseteq \mathsf{SMSL}$" both $\varnothing \neq (\mathsf{P} \setminus (\mathsf{NIMRQSKSL}|\mathsf{SMSL})) \subseteq (\mathsf{P} \cap (\mathsf{IQSMSL}|((\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}) \setminus \mathsf{PSMSL})))$, and so, by Lemma 5.19|5.25, both $(\mathfrak{K}_3|\mathfrak{K}_{5:1}) \in \mathsf{P}$. Then, by Corollary 5.14, $\mathsf{P} = \mathsf{MRQSKSL}$.

(l) $\mathsf{NIMRQSKSL} \not\supseteq \mathsf{P} \subseteq \mathsf{NIMRQSMSL}$,
in which case |"as $\mathsf{P} \not\subseteq \mathsf{SMSL}$" both $\varnothing \neq (\mathsf{P} \setminus (\mathsf{NIMRQSKSL}|\mathsf{SMSL})) \subseteq (\mathsf{P} \cap ((\mathsf{NIQSMSL} \setminus \mathsf{QSKSL})|((\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}) \setminus \mathsf{PSMSL})))$, and so, by Lemma 5.22|5.25, both $((\mathfrak{DM}_4 \times \mathfrak{B}_2)|\mathfrak{K}_{5:1}) \in \mathsf{P}$. Then, by Corollary 5.13, $\mathsf{P} = \mathsf{NIMRQSMSL}$.

(m) $\mathsf{RQSKSL} \not\supseteq \mathsf{P} \subseteq \mathsf{NIMRQSKSL}$,
in which case |"as $\mathsf{P} \not\subseteq \mathsf{SMSL}$" both $\varnothing \neq (\mathsf{P} \setminus (\mathsf{RQSKSL}|\mathsf{SMSL})) \subseteq (\mathsf{P} \cap$

$((\mathsf{NIQSKSL} \setminus \mathsf{RQSKSL})|((\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}) \setminus \mathsf{PSMSL})))$, and so, by Lemma 5.21|5.25, both $((\mathfrak{K}_3 \times \mathfrak{B}_2)|\mathfrak{K}_{5:1}) \in \mathsf{P}$. Then, by Corollary 5.15, $\mathsf{P} = \mathsf{NIMRQSKSL}$.

(n) $\mathsf{P} \subseteq \mathsf{RQSKSL}$,
in which case, as $\mathsf{P} \nsubseteq \mathsf{SMSL}$, by Lemma 5.25, $\mathfrak{K}_{5:1} \in \mathsf{P}$, and so, by Theorem 5.12, $\mathsf{P} = \mathsf{RQSKSL}$. □

This subsumes [17, Theorem 4.8] as well as, by Corollaries 4.7, 5.4, 5.6 and Theorem 5.12, immediately yields:

Corollary 5.27. *Any [pre-/-quasi-]variety* $\mathsf{P} \subseteq \mathsf{SMSL}$ *such that* $\mathsf{P} \nsubseteq \mathsf{DML}$ *is generated by* $(\mathsf{P} \cap \mathsf{DML}) \cup \mathsf{SL}$.

5.1. Relatively semi-simple quasi-varieties of quasi-strong Morgan-Stone lattices and algebras.

Lemma 5.28. *Let* $\mathsf{P} \subseteq [\mathsf{B}]\mathsf{QSMSL}$ *be a pre-variety. Then,* $(\mathrm{Si}_{\mathsf{P}}(\mathsf{P}) \cap \mathsf{NI}[\mathsf{B}]\mathsf{QSMSL}) \subseteq \mathbf{I}\mathfrak{B}_{2[0,1]} \subseteq [\mathsf{B}]\mathsf{BL} \subseteq [\mathsf{B}]\mathsf{KL} \subseteq [\mathsf{B}]\mathsf{DML}$.

Proof. Consider any $\mathfrak{A} \in (\mathrm{Si}_{\mathsf{P}}(\mathsf{P}) \cap \mathsf{NI}[\mathsf{B}]\mathsf{QSMSL})$, in which case $|A| > 1$, and so $\mathfrak{B}_{2[,01]}$, being embeddable into $\mathfrak{A}$, in view of Lemma 5.3, belongs to P. Then, by Corollary 5.4, there is some $h \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$, in which case, by (2.5), as $(\mathrm{img}\, h) = \Delta_2$ is not a singleton, $A^2 \neq (\ker h) = h_*^{-1}[\Delta_{B_2}] \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{A}) \subseteq \{A^2, \Delta_A\}$, and so h is injective, as required. □

Lemma 5.29. $h \triangleq \{\langle i, \min(2,i)\rangle \mid i \in 4\} \in \hom^{\mathrm{S}}(\mathfrak{K}_{4:1[,01]}, \mathfrak{K}_{3[,01]})$*,* $\ker h$ *being the only congruence of* $\mathfrak{K}_{4:1[,01]}$ *distinct from both* Δ_4 *and* 4^2.

Proof. Consider any $\theta \in (\mathrm{Co}(\mathfrak{K}_{4:1[,01]}) \setminus \{\Delta_4\})$ and take any $\bar{a} \in (\theta \setminus \Delta_A) \neq \varnothing$, in which case $(3 \cap (\mathrm{img}\, \bar{a})) \neq \varnothing$, and so we have the following exhaustive cases:

- there is some $i \in 2$ such that $a_i = 1 \neq a_{1-i}$,
  in which case $\langle 0|3, 1\rangle = \langle \neg^{\mathfrak{K}_{4:1[,01]}} \neg^{\mathfrak{K}_{4:1[,01]}} a_{1-i} (\wedge|\vee)^{\mathfrak{K}_{4:1[,01]}} \neg^{\mathfrak{K}_{4:1[,01]}} a_{1-i}, \neg^{\mathfrak{K}_{4:1[,01]}} \neg^{\mathfrak{K}_{4:1[,01]}} a_i (\wedge|\vee)^{\mathfrak{K}_{4:1[,01]}} \neg^{\mathfrak{K}_{4:1[,01]}} a_i \rangle \in \theta$, and so $\theta \ni \langle 0, 3\rangle$, being a congruence of $\mathfrak{D}_4$, is equal to 4^2.
- there is some $j \in 2$ such that $a_j = 2$ and $a_{1-j} \in \{0, 3\}$,
  in which case $\langle 3, 2\rangle = \langle \neg^{\mathfrak{K}_{4:1[,01]}} a_{1-j} \vee^{\mathfrak{K}_{4:1[,01]}} a_{1-j}, \neg^{\mathfrak{K}_{4:1[,01]}} a_j \vee^{\mathfrak{K}_{4:1[,01]}} a_j \rangle \in \theta$, and so $(\ker h) = (\Delta_4 \cup \{2,3\}^2) \subseteq \theta$. Then, by Remark 4.3, $\mathfrak{K}_{3[,01]}$ is simple, in which case, by (2.5), $(\ker h) = h_*^{-1}[\Delta_3] \in \max_{\subseteq}(\mathrm{Co}(\mathfrak{K}_{4:1[,01]}) \setminus 4^2)$, and so $\theta \in \{\ker h, 4^2\}$.
- there is some $k \in 2$ such that $a_k = 0$ and $a_{1-k} = 3$,
  in which case $\theta \ni \langle 0, 3\rangle$, being a congruence of $\mathfrak{D}_4$, is equal to 4^2.

Finally, by (2.4), $(\ker h) \in (\mathrm{Co}(\mathfrak{K}_{4:1[,01]}) \setminus \{\Delta_4, 4^2\})$, for $1 \neq 3 \neq 4$, as required. □

This, by (2.4) and the fact that $\epsilon^4_{3:0} \circ \epsilon^6_4 \circ (\epsilon^4_1)^{-1}$ is an embedding of $\mathfrak{K}_{3[,01]}$ into $\mathfrak{K}_{4:1[,01]}$, immediately yields:

Corollary 5.30. *Let* $\mathsf{P} \subseteq [\mathsf{B}]\mathsf{MSL}$ *be a pre-variety. Suppose* $\mathfrak{K}_{4:1[,01]} \in \mathsf{P}$*. Then,* $\mathfrak{K}_{4:1[,01]} \in (\mathrm{SI}_{\mathsf{P}}(\mathsf{P}) \setminus \mathrm{Si}_{\mathsf{P}}(\mathsf{P}))$.

Given any $\mathfrak{A} \in \mathsf{MSA} \supseteq \mathsf{BQSMSL}$, by (4.7) and (4.8), $(A \oplus 2) \triangleq ((A \times \{1\}) \cup \{\langle \bot^{\mathfrak{A}}, 0\rangle, \langle \top^{\mathfrak{A}}, 2\rangle\})$ forms a subalgebra of $\mathfrak{A} \times \mathfrak{K}_{3,01}$, in which case $(\mathfrak{A} \oplus 2) \triangleq ((\mathfrak{A} \times \mathfrak{K}_{3,01}) \restriction (A \oplus 2)) \in \mathsf{BQSMSL}$.

Lemma 5.31. *Let* $\mathsf{P} \subseteq \mathsf{BMSL}$ *be a pre-variety and* $\mathfrak{A} \in \mathsf{IMSA} \supseteq \mathsf{IBQSMSL}$ *(as well as* $h \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{K}_{3,01})$*), Suppose* $\mathrm{Co}(\mathfrak{A}) = \{\Delta_A, A^2(, \ker h)\}$ *and* $\mathfrak{B} \triangleq (\mathfrak{A} \oplus 2) \in \mathsf{P} \not\ni \mathfrak{A}$*. Then,* $\mathfrak{B} \in (\mathrm{SI}_{\mathsf{P}}(\mathsf{P}) \setminus \mathrm{Si}_{\mathsf{P}}(\mathsf{P}))$.

Proof. In that case, by the simplicity of $\mathfrak{K}_{3,01}$ (cf. Remark 4.3), Corollary 3.10 and Remark 4.2, $\mathrm{Co}(\mathfrak{B}) = \{\Delta_B, B^2, \ker(\pi_1{\restriction}B), \ker(\pi_0{\restriction}B)(, (\ker((\pi_0{\restriction}B) \circ h)) \cap (\ker(\pi_1{\restriction}B)), \ker((\pi_0{\restriction}B)\circ h))\}$, and so, as $\{\langle 0, \langle \bot^{\mathfrak{A}}, 0\rangle\rangle, \langle 1, \langle a, 1\rangle\rangle, \langle 2, \langle \top^{\mathfrak{A}}, 2\rangle\rangle\}$, where $a \in \Im^{\mathfrak{A}} \neq \varnothing$, is an embedding of $\mathfrak{K}_{3,01}$ into $\mathfrak{B}$, by (2.4) and the Homomorphism Theorem, $\mathrm{Co}_{\mathsf{P}}(\mathfrak{B}) = \{\Delta_B, B^2, \ker(\pi_1{\restriction}B)(, (\ker((\pi_0{\restriction}B) \circ h)) \cap (\ker(\pi_1{\restriction}B)), \ker((\pi_0{\restriction}B)\circ h))\}$, $(((\ker((\pi_0{\restriction}B)\circ h))\cap)(\ker(\pi_1{\restriction}B))) \neq B^2$ being then the least P-congruence of $\mathfrak{B}$ distinct from Δ_B, as required. □

Corollary 5.32. *Let* $\mathsf{P} \subseteq \mathsf{MSA} \supseteq \mathsf{BQSMSL}$ *be a relatively semi-simple pre-variety,* $\mathfrak{A} \in \mathsf{P}$*,* $\mathfrak{B} \in \{\mathfrak{K}_{4:1,01}, \mathfrak{DM}_{4,01}\}$ *and e an embedding of* $\mathfrak{B}{\restriction}\Sigma_+^-$ *into* $\mathfrak{A}{\restriction}\Sigma_+^-$*. Then,* $\mathfrak{B} \in \mathsf{P}$*.*

Proof. By contradiction. For suppose $\mathfrak{B} \notin \mathsf{P}$, in which case e is not an embedding of $\mathfrak{B}$ into $\mathfrak{A}$, and so, by (4.7) and (4.8), both $e((\bot|\top)^{\mathfrak{B}}) \neq (\bot|\top)^{\mathfrak{A}}$. Then, by (4.7) and (4.8), $((\pi_0{\restriction}(B \times \{1\})) \circ e) \cup \{\langle \bot^{\mathfrak{B}}, 0, \bot^{\mathfrak{A}}\rangle, \langle \top^{\mathfrak{B}}, 2, \top^{\mathfrak{A}}\rangle\}$ is an embedding of $\mathfrak{B} \oplus 2$ into $\mathfrak{A}$, in which case $(\mathfrak{B} \oplus 2) \in \mathsf{P}$, and so Lemmas 5.29, 5.31, the simplicity of $\mathfrak{DM}_{4,01}$ (cf. Remark 4.3), its idempotencity and that of $\mathfrak{K}_{4:1[,01]}$ contradict to the relative semi-simplicity of P, as required. □

Theorem 5.33. *Any relatively semi-simple relatively subdirectly-representable (more specifically, "relatively semi-simple quasi-equational"/implicative) pre-variety* $\mathsf{P} \subseteq [\mathsf{B}]\mathsf{QSMSL}$ *is a sub-variety of* $[\mathsf{B}]\mathsf{DML}$*, in which case it is* $\mho^{\bar{\varphi}}_{V_1|\Omega,\wp(\Omega)}$*-implicative, and so "{relatively} ⟨finitely-⟩semi-simple"//* $\lfloor\mho^{\bar{\varphi}}_{V_1\|\Omega,\wp(\Omega)}$*-*$\rfloor$ *implicative sub-{pre-}varieties of* $[\mathsf{B}]\mathsf{QSMSL}$ *are exactly sub-varieties of* $[\mathsf{B}]\mathsf{DML}$*.*

Proof. In that case, P is generated by $\mathsf{K} \triangleq \mathrm{Si}_{\mathsf{P}}(\mathsf{P})$. If there was some $\mathfrak{A} \in (\mathsf{P} \setminus (\mathsf{NI}[\mathsf{B}]\mathsf{QSMSL} \cup [\mathsf{B}]\mathsf{DML}))$, then, by Corollary 5.8 and Lemma 5.24 [as well as Corollary 5.32], $\mathfrak{K}_{4:1[,01]}$ would be in P, contrary to the relative semi-simplicity of P and Corollary 5.30. Hence, $\mathsf{K} \subseteq \mathsf{P} \subseteq (\mathsf{NI}[\mathsf{B}]\mathsf{QSMSL} \cup [\mathsf{B}]\mathsf{DML})$, in which case, by Lemma 5.28, $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{DML}$, and so $\mathsf{P} \subseteq [\mathsf{B}]\mathsf{DML}$. Consider the following complementary cases:

- $\mathsf{K} = \varnothing$,
  in which case $\mathsf{P} = [\mathsf{B}]\mathsf{OMSL}$.
- $\mathsf{K} \neq \varnothing$.
  Consider the following complementary subcases:
  - $\mathsf{K} \subseteq \mathsf{NI}[\mathsf{B}]\mathsf{QSMSL}$,
    in which case, by Footnote 1 and Lemma 5.28, $\mathsf{K} = \mathbf{I}\mathfrak{B}_{2[,01]}$, and so, by Corollary 4.7, $\mathsf{P} = [\mathsf{B}]\mathsf{BL}$.
  - $\mathsf{K} \nsubseteq \mathsf{NI}[\mathsf{B}]\mathsf{QSMSL}$.
    Consider the following complementary subcases:
    * $\mathsf{K} \subseteq ([\mathsf{B}]\mathsf{QSKSL} \cup \mathsf{NI}[\mathsf{B}]\mathsf{QSMSL})$,
      in which case $\mathsf{IK} \subseteq [\mathsf{B}]\mathsf{KL}$, and so, by Lemma 5.28, $\mathsf{P} \subseteq [\mathsf{B}]\mathsf{KL}$. Conversely, take any $\mathfrak{A} \in \mathsf{IK} \neq \varnothing$, in which case $(\mathfrak{A}[{\restriction}\Sigma_+^-]) \in \mathsf{IQSKSL}$, and so, by Lemma 5.19, there is an embedding e of $\mathfrak{K}_3$ into $\mathfrak{A}[{\restriction}\Sigma_+^-]$. Then, [as $a \triangleq e(\langle 0, 1\rangle) = \neg^{\mathfrak{A}} a$, by (4.7) and (4.8), $\{\langle 0, 0, \bot^{\mathfrak{A}}\rangle, \langle 0, 1, a\rangle, \langle 1, 1, \top^{\mathfrak{A}}\rangle\}$ is an embedding of $\mathfrak{K}_{3,01}$ into $\mathfrak{A}$, in which case] $\mathfrak{K}_{3[,01]} \in \mathsf{Q}$, and so, by Corollary 4.7, $\mathsf{P} = [\mathsf{B}]\mathsf{KL}$.
    * $\mathsf{K} \nsubseteq ([\mathsf{B}]\mathsf{QSKSL} \cup \mathsf{NI}[\mathsf{B}]\mathsf{QSMSL}$.
      Take any $\mathfrak{B} \in (\mathsf{K} \setminus ([\mathsf{B}]\mathsf{QSKSL} \cup \mathsf{NI}[\mathsf{B}]\mathsf{QSMSL})) \neq \varnothing$, in which case, by Lemma 5.20 [and Corollary 5.32], $\mathfrak{DM}_{4[,01]} \in \mathsf{P}$, and so, by Corollary 4.7, $\mathsf{P} = [\mathsf{B}]\mathsf{DML}$.

This, by Corollary 4.7 (and Remark/Corollary 2.4/3.4), completes the argument. □

Though, in view of Theorem 5.26, due to which pre-varieties of quasi-strong MS lattices are quasi-equational, the []-non-optional version of the right alternative of the ()-optional version of Theorem 5.33 is a particular case of Corollary 4.7, such is not the case for the []-optional one, because of the well-known existence of non-quasi-equational pre-varieties of De Morgan algebras.

6. Conclusions

Perhaps, the most acute problem remained open is the lattice of quasi-varieties of *all* MS lattices. Such equally concerns extension of Subsection 5.1 beyond quasi-strong MS lattices/algebras. After all, an interesting (though purely methodological) point remained open is to find equational proofs (like that of (4.14)) of the rather curious inclusions such as

$$[\mathsf{B}/]\mathsf{NDM}(\mathsf{L}[/\mathsf{A}]) \subseteq [\mathsf{B}/]\mathsf{PSMS}(\mathsf{L}[/\mathsf{A}]) \subseteq [\mathsf{B}/]\mathsf{WKMS}(\mathsf{L}[/\mathsf{A}])$$

and $[\mathsf{B}/]\mathsf{QSWKS}(\mathsf{L}[/\mathsf{A}]) \subseteq [\mathsf{B}/]\mathsf{QSKS}(\mathsf{L}[/\mathsf{A}])$ as well as

$$(\mathsf{NIMR}[\mathsf{B}]\mathsf{QSMSL} \cup \mathsf{MR}[\mathsf{B}](\mathsf{QS})\mathsf{KSL}) \subseteq (\mathsf{NIMR}[\mathsf{B}]\mathsf{QSMSL} \cup [\mathsf{B}]\mathsf{KL}),$$

just ensuing from Corollaries 4.7 and 5.16.

References

1. R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia (Missouri), 1974.
2. T.S. Blyth and J.C. Varlet, *On a common abstraction of De Morgan algebras and Stone algebras*, Proc. Roy. Soc. Edinburg A **94** (1983), 301–308.
3. A. I. Budkin and V. A. Gorbunov, *Implicative classes of algebras*, Algebra and Logic **12** (1973), 139–140.
4. A. L. Foster and A. F. Pixley, *Semi categorical algebras I*, Mathematische Zeitschrift **83** (1964), no. 2, 147–169.
5. T. Frayne, A.C. Morel, and D.S. Scott, *Reduced direct products*, Fundamenta Mathematicae **51** (1962), 195–228.
6. E. Fried, G. Grätzer, and R. Quackenbush, *Uniform congruence schemes*, Algebra Universalis **10** (1980), 176–189.
7. G. Grätzer, *General Lattice Theory*, Akademie-Verlag, Berlin, 1978.
8. G. Grätzer and E.T. Schmidt, *Ideals and congruence relations in lattices*, Acta. Math. Acad. Sci. Hungar. **9** (1958), 137–175.
9. B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. **21** (1967), 110–121.
10. J. A. Kalman, *Lattices with involution*, Transactions of the American Mathematical Society **87** (1958), 485–491.
11. A. I. Kogalovskiĭ, *On Birkgoff's Theorem*, Uspehi Mat. Nauk **20** (1965), 206–207, In Russian.
12. H. Lakser, *Principal congruences of pdeudocomplemented distributive lattices*, Proceedings of the American Mathematical Society **37** (1973), 32–37.
13. A. I. Mal'cev, *To a general theory of algebraic systems*, Mathematical Collection (New Seria) **35 (77)** (1954), 3–20, In Russian.
14. ———, *Algebraic systems*, Springer Verlag, New York, 1965.
15. G. C. Moisil, *Recherches sur l'algèbre de la logique*, Annales Scientifiques de l'Université de Jassy **22** (1935), 1–117.
16. A. F. Pixley, *Distributivity and permutability of congruence relations in equational classes of algebras*, Proceedings of the American Mathematical Society **14** (1963), no. 1, 105–109.
17. A. P. Pynko, *Implicational classes of De Morgan lattices*, Discrete mathematics **205** (1999), 171–181.
18. ———, *Sequential calculi for many-valued logics with equality determinant*, Bulletin of the Section of Logic **33** (2004), no. 1, 23–32.
19. ———, *A relative interpolation theorem for infinitary universal Horn logic and its applications*, Archive for Mathematical Logic **45** (2006), 267–305.
20. ———, *Subquasivarieties of implicative locally-finite quasivarieties*, Mathematical Logic Quarterly **56** (2010), no. 6, 643–658.

21. H.P. Sankappanavar, *A characterization of principal congruences of De Morgan algebras and its application*, Proceedings of IV Latin American Symposium on Mathematical Logic, Santiago, 1978 (Amsterdam) (A.I. Arruda, R. Chuaqui, and N.C.A. da Costa, eds.), North-Holland Publishing Company, 1980, pp. 341–349.
22. L. A. Skornyakov (ed.), *General algebra*, vol. 2, Nauka, Moscow, 1991, In Russian.
23. D. Ševčovič, *Free non-distributive Morgan-Stone algebras*, New Zealand Journal of Mathematics **94** (1996), 85–94.

Department of Digital Automata Theory (100), V.M. Glushkov Institute of Cybernetics, Glushkov prosp. 40, Kiev, 03680, Ukraine
Email address: pynko@i.ua