Four-Valued Expansions of Belnap's Logic: Inheriting Basic Peculiarities

Alexej Pynko

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FOUR-VALUED EXPANSIONS OF BELNAP'S LOGIC: INHERITING BASIC PECULIARITIES

ALEXEJ P. PYNKO

Abstract. The main results of the paper are that:

(1) any four-valued expansion L_4 of Belnap's four-valued logic B_4 (cf. [4]):
  (a) is defined by a unique expansion $\mathcal{M}_4$ of the four-valued matrix $\mathcal{DM}_4$ over the De Morgan truth lattice diamond $\{\mathsf{f},\mathsf{n},\mathsf{b},\mathsf{t}\}$ defining B_4 as such;
  (b) satisfies Variable Sharing Property iff it is has neither a theorem nor an inconsistent formula;
  (c) has no proper extension satisfying Variable Sharing Property;
  (d) is minimally four-valued;
  (e) is defined by no truth/false-singular matrix;
  (f) has an extension defined by an expansion of a consistent submatrix $\mathcal{B}$ of $\mathcal{DM}_4$ iff the underlying algebra of $\mathcal{B}$ is a subalgebra of the underlying algebra $\mathfrak{A}_4$ of $\mathcal{M}_4$;
  (g) is subclassical iff $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}_4$, in which case the logic of $\mathcal{M}_4{\upharpoonright}\{\mathsf{f},\mathsf{t}\}$ defines a unique classical extension of L_4 being also an extension of any inferentially consistent extension of L_4;
  (h) is [inferentially] maximal iff $\mathcal{M}_4$ has no proper consistent [truth-non-empty] submatrix;
  (i) is maximally paraconsistent iff $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}_4$ iff the proper axiomatic extension L_4^{EM} of L_4 relatively axiomatized by the *Excluded Middle law* axiom is either classical, if L_4 is subclassical, or inconsistent, otherwise, iff L_4^{EM} is not (maximally) paraconsistent iff L_4^{EM} is not an expansion of the logic of paradox $LP = B_4^{\mathrm{EM}}$ and, otherwise, providing L_4 is subclassical and either L_4 is [relatively] weakly implicative or $\mathfrak{A}_4$ is regular, L_4^{EM} has exactly two proper consistent extensions forming a chain, the greatest one being classical and relatively axiomatized by the *Modus ponens* rule for material implication, the least one being relatively axiomatized by the *Ex Contradictione Quodlibet* rule, both ones having same theorems as L_4^{EM} has, and so being non-axiomatic, while L_4^{EM} being the only proper consistent axiomatic extension of L_4, whenever $\mathfrak{A}_4$ is regular;
  (j) has no theorem/inconsistent formula iff $\{\mathsf{n}/\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}_4$;
  (k) [providing L_4 has a/no theorem] L_4 has the distributive lattice of its disjunctive [arbitrary/merely non-pseudo-axiomatic] extensions being dual isomorphic to the one of all lower cones of the set of all [truth-non-empty] consistent submatrices of $\mathcal{M}_4$ (in particular, to be found effectively, whenever the expanded signature is finite) and is a sublattice of the nine[six]-element non-chain distributive lattice of all disjunctive [non-pseudo-axiomatic] extensions of B_4;
  (l) has its proper disjunctive extension L_4^{R} relatively axiomatized by the *Resolution* rule that:
    (i) is paracomplete iff the carrier of the subalgebra of $\mathfrak{A}_4$ generated by $\{\mathsf{n}\}$ does not contain b;
    (ii) is not inferentially paracomplete iff it is inferentially either classical, if L_4 is subclassical, or inconsistent, otherwise, iff $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}_4$ iff L_4^{R} is not an expansion of Kleene's three-valued logic $K_3 = B_4^{\mathrm{R}}$;
  (m) has the entailment relation equal to the set of all inequalities identically true in $\mathfrak{A}_4$ iff L_4 is self-extensional iff it has the Property of Weak Contraposition iff the specular permutation on $\{\mathsf{f},\mathsf{n},\mathsf{b},\mathsf{t}\}$ retaining both f and t but permuting n and b is an endomorphism of $\mathfrak{A}_4$ iff the extension of L_4 relatively axiomatized by the *Modus Ponens[Ex Contradictione Quodlibet]* rule is defined by [the direct product of $\mathcal{M}_4$ and] $\langle\mathfrak{A}_4,\{\mathsf{t}\}\rangle$, in which case:
    (i) L_4 is subclassical;
    (ii) there is either no, if L_4 is maximally paraconsistent, or exactly one, otherwise, non-pseudo-axiomatic consistent non-classical proper self-extensional extension of L_4, any self-extensional extension of L_4 being disjunctive;
    (iii) $\{\mathsf{n}[,\mathsf{f},\mathsf{t}]\}$ forms a subalgebra of $\mathfrak{A}_4$ iff $\{\mathsf{b}[,\mathsf{f},\mathsf{t}]\}$ does so, in which case:
      (A) L_4 satisfies Variable Sharing Property iff it has no theorem/inconsistent formula;
      (B) L_4^{EM} is (maximally) paraconsistent iff L_4^{R} is inferentially paracomplete, in which case, providing $\mathfrak{A}_4$ is regular, L_4^{R} is maximally inferentially paracomplete, while any extension of L_4 is both paraconsistent and inferentially paracomplete iff it is a sublogic of $L_4^{\mathrm{EM}} \cap L_4^{\mathrm{R}}$.;
      (C) [providing L_4 has a/no theorem] disjunctive [arbitrary/merely non-pseudo-axiomatic] extensions of L_4 form the nine[six]-element non-chain distributive lattice isomorphic to that of B_4;
      (D) providing $\mathfrak{A}_4$ is regular [and L_4 has a/no theorem], [arbitrary/merely non-pseudo-axiomatic] extensions of $L_4^{\mathrm{EM}} \cap L_4^{\mathrm{R}}$ form the eleven[seven]-element non-chain distributive lattice, those of L_4^{R} being all disjunctive, proper ones being inferentially either classical or inconsistent, and so not inferentially paracomplete, in which case L_4^{R} is maximally (inferentially) paracomplete, as opposed to its implicative expansions;

(2) any three-valued (disjunctive/conjunctive) paraconsistent logic L_3 with subclassical negation:
  (a) is defined by a (unique disjunctive/conjunctive) *superclassical* matrix over $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, referred to as *characteristic* one of L_3;
  (b) is maximally paraconsistent iff either $\{\mathsf{b}\}$ does not form a subalgebra of the underlying algebra $\mathfrak{A}$ of any characteristic matrix of L_3 or there is a *ternary* b*-relative weak conjunction for* $\mathfrak{A}$, viz., a ternary formula φ such that $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f},\mathsf{t}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{t},\mathsf{f}) \neq \mathsf{t}$, in which case a characteristic matrix of L_3 is unique(;
  (c) has no proper paraconsistent disjunctive/conjunctive extension/, in which case it is maximally paraconsistent);
  (d) is minimally three-valued;
  (e) is subclassical if(f) $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of the underlying algebra of its characteristic matrix, in which case (L_3 is maximally paraconsistent, while)the logic of the restriction of its characteristic matrix on $\{\mathsf{f},\mathsf{t}\}$ defines a (unique) classical extension of L_3(/, being also an extension of any consistent extension of L_3);

(3) for every $n > 2$, there is a minimally n-valued maximally paraconsistent subclassical [both conjunctive and disjunctive] logic.

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1. Introduction

Perhaps, the principal value of *universal* mathematical investigations consists in discovering uniform transparent points behind particular results originally proved *ad hoc* with preferable covering new instances as well as in providing powerful generic tools enabling one, so to say, "to kill as much as possible birds with as less as possible stones". This thesis is the main methodological paradigm of the present study.

Belnap's "useful" four-valued logic (cf. [4]) arising as the logic of *first-degree entailment* in relevance logic R (*FDE*, for short) has been naturally expanded by additional connectives in [23]. The present paper pursues the study of such expansions with regard to certain *generic* aspects in addition to those of functional completeness and both sequential and equational axiomatizations comprehensively explored therein collectively with Paragraph 7.1.1.1 here and [28], respectively.

More precisely, we study how four-valued expansions of FDE (as well as their extensions) inherit certain *remarkable* features of FDE as such. This marks the *primary* framework of the paper. On the other hand, it is closely related to certain more (secondary) issues additionally studied here (especially because this study uses the generic tools initially elaborated for solving exactly the secondary tasks alone and only then applied to primary ones).

First of all, FDE satisfies Variable Sharing Property [1] in the sense that it satisfies the entailment $\phi \to \psi$ only if ϕ and ψ have a common propositional variable. This clarifies the items (1b,1c,1j) of the Abstract.

Moreover, the four-valued matrix defining FDE has four proper consistent submatrices, each defining a consistent proper extension of FDE. This explains the item (1f) of the Abstract.

In particular, FDE is *subclassical* in the sense that a definitional clone (viz., copy) of the classical logic is an extension of it. When exploring this peculiarity within the framework of expansions of FDE, we inevitably deal with formally miscellaneous classical logics as those which are defined by *classical* matrices, that is, consistent two-valued matrices with classical negation. In case such is conjunctive with respect to any (possibly, secondary) binary connective (in particular, is a model of an expansion of FDE), the logic defined by such a matrix is nothing but a definitional copy of the standard classical logic, because any two-valued operation is definable via the classical negation and conjunction. We equally follow this paradigm, when studying three-valued and n-valued paraconsistent logics. This clarifies the items (1g) and (2e) of the Abstract.

The four-valuedness typical of FDE and its expansions also implies their both *[inferential] paracompleteness* (viz., refuting the [inferential version of] *Excluded Middle law* axiom) and *paraconsistency* (viz., refuting the *Ex Contradictione Quodlibet* rule). It is this joint peculiarity of FDE that has predetermined its profound applications to Computer Science and Artificial Intelligence. This inevitably raises the issue of exploring how extensions of (four-valued expansions of) FDE retain such peculiarities (cf. the items (1i,1l) of the Abstract).

In this connection, the issue of strong [inferential] maximality typical of the classical logic in the sense of having no proper [inferentially] consistent extension becomes equally acute as for four-valued expansions of FDE. The thing is that [purely-]bilattice expansions of FDE with[out] truth and falsehood constant are [inferentially] maximal, as it ensues from the general characterization of the maximality of FDE expansions obtained here (cf. the item (1h) of the Abstract). Taking [25] into account, particular cases of such maximality have actually been proved in [23] *ad hoc*.

And what is more, four-valued expansions of FDE normally (but not at all generally) have three-valued paraconsistent/paracomplete extensions, defined by three-valued submatrices of characteristic four-valued matrices (cf. the items (1f,1i/l) of the Abstract), shown here to be relatively axiomatized by the *Excluded Middle law* axiom/ the *Resolution* rule in that case. Then, their defining three-valued paraconsistent submatrices appear to be conjunctive and *superclassical* in the sense of the reference [Pyn 95b] of [21], according to which any logic defined by such a matrix is *maximally* paraconsistent in the sense of having no proper paraconsistent extension (cf. the items (2b,2c) of the Abstract and historically the paragraph after Theorem 2.1 of [21]).[1] Particular cases of such three-valued maximal paraconsistency have been proved *ad hoc* in [21], [26] as well as in [33] taking [25] into account. On the other hand, as it follows from our characterization of the maximal paraconsistency (cf. the item (1i) of the Abstract), any (including constant-free purely) bilattice expansion is maximally paraconsistent, though is not subclassical, in view the item (1g) of the Abstract, as opposed to the expansion by classical (viz., Boolean) negation.

In this way, we conclude that the maximal paraconsistency is not at all a prerogative of three-valued logics. As a matter of fact, we argue that, for every $n > 2$, there is a *minimally* n-valued (in the sense of not being defined by a matrix with less than n values; cf. the items (1d,2d) of the Abstract in this connection) maximally paraconsistent subclassical logic (cf. the item (3) of the Abstract). In this connection, it is remarkable that existence of a non-minimally n-valued maximally paraconsistent subclassical logic has been actually due to [21], because the logic of paradox [18] is equally defined by an n-valued matrix. Among other things, such generic minimally n-valued example is defined by a false-singular matrix, as opposed to four-valued expansions of FDE (cf. the item (1e) of the Abstract).

Furthermore, FDE is disjunctive. This raises the problem of finding all disjunctive extensions of (four-valued expansions of) FDE (cf. the item (1k) of the Abstract). (Although, likewise, FDE is conjunctive, the conjunctivity is immediately inherited by extensions/expansions, so this point is just taken for granted.)

After all, a one more quite remarkable peculiarity of FDE is that its entailment relation is defined (semi)lattice-wise in the sense that FDE satisfies the entailment $\phi \to \psi$ iff the inequality $\phi \lessapprox \psi$ (viz., the equality $(\phi \wedge \psi) \approx \phi$) is identically true in the diamond non-Boolean De Morgan lattice, i.e, in the variety of De Morgan lattices. Within the framework of four-valued expansions of FDE, this property appears to be equivalent to the so-called *self-extensionality* (cf. Theorem 5.61(i)⇔(v)), profound study of which has been due to [22] that has provided a generic algebraic (more specifically, lattice-theoretic) approach

[1]Though being prepared and announced by 1995, the fundamental material of the both references [Pyn 95a] and [Pyn 95b] of [21] has never been published for a quarter of century, while certain quirky (mainly, Zionist) kleptomaniacs all over the world (like Avron & Co.; Tribus, Skura, at al.; Font, Jansana & Co. — including Prenosil, Albuquerque, Rivieccio et al.) have succeeded in plagiarizing it as well as other contributions (in particular, those announced in [21]) of mine that has been intentionally encouraged by the complicity on the part of certain equally dishonorable Zionist editors like D. Gabbay, R. Wojcicki, M. Fitting, Y. Shramko, H. Wansing and, especially, quite hypocritical J.M. Dunn. This is why we take the opportunity to eventually present them here to try to restore the genuine authorship.

to conjunctive non-pseudo-axiomatic self-extensional logics (cf. Section 4.1 therein) properly enhanced here by omitting the stipulation "non-pseudo-axiomatic". Recall that a propositional logic is said to be *self-extensional*, provided its interderivability relation is a congruence of the formula algebra, in which case any fragment of it is self-extensional as well (cf. [22]), while the converse is far from being generally valid. Any axiomatic extension of the intuitionistic logic as well as any inferentially consistent two-valued logic (including the classical one and its fragments) is self-extensional. This explains the meaning of the item (1m) of the Abstract.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set Theory, Lattice Theory, Universal Algebra, Model Theory and Mathematical Logic not specified here explicitly, to be found, e.g., in [3], [10], [12], [15] and [16]). Section 2 is a concise summary of basic issues underlying the paper, most of which have actually become a part of logical and algebraic folklore. Section 3 is devoted to certain key preliminary issues concerning false-singular matrices, disjunctivity, equality determinants and De Morgan lattices. In Section 5 we formulate and prove main results of the paper concerning solely four-valued expansions of FDE. Section 4/6 is mainly devoted to the issue of (especially, maximal) paraconsistency within three-valued/generic n-valued framework, respectively. Then, in Section 7, we exemplify Sections 3, 4 and 5 by applying them to both certain well-known three-valued paraconsistent logics and three general classes of FDE expansions, including those introduced in [23], with providing quick argumentations/refutations of their properties under consideration, and finding all disjunctive extensions of (first of all, self-extensional non-maximally paraconsistent) expansions of FDE as well extensions of (first of all, the unique proper non-classical self-extensional non-pseudo-axiomatic extension of regular self-extensional non-maximally paraconsistent) expansions of FDE (in particular, both FDE itself and its bounded version). Finally, Section 8 is a brief summary of principal contributions of the paper.

2. Basic issues

Notations like img, dom, ker, hom, π_i and Con and related notions are supposed to be clear.

2.1. **Set-theoretical background.** We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞.

Likewise, functions are viewed as binary relations, the left/right components of their elements being treated as their arguments/values, respectively. Then, to retain both the conventional prefix writing of functions and the fact that $(f \circ g)(a) = f(g(a))$, we have just preferred to invert the conventional order of relation composition components. In particular, given two binary relations R and Q, we put $R[Q] \triangleq (R \circ Q \circ R^{-1})$.

In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. A subset $T \subseteq S$ is said to be *proper*, if $T \neq S$. Further, given any equivalence relation θ on S, as usual, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq [a]_\theta \triangleq \theta[\{a\}]$, for all $a \in S$, in which case $\ker \nu_\theta = \theta$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component (viz., the value under argument s), where $s \in S$, being written as either t_s or t^s. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise, that is, $R \triangleq \{\langle \bar{a}, \bar{b} \rangle \in (A^S \times B^S) \mid \forall s \in S : a_s \; R \; b_s\}$. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any $\bar{f} \in (\prod_{s \in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a) \rangle_{s \in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, set $\Delta_S \triangleq \{\langle a, a \rangle | a \in S\}$, relations of such a kind being referred to as *diagonal*, and $S^+ \triangleq \bigcup_{i \in (\omega \setminus 1)} S^i$, elements of $S^* \triangleq (S^0 \cup S^+)$ being identified with ordinary finite tuples, the binary concatenation operation on which being denoted by $*$, as usual. In addition, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = (\operatorname{dom} \bar{a})$ of any $\bar{a} \in S^+$, put:

$$\diamond \bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a} \restriction (l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

In particular, given any $f : S \to S$ and any $n \in \omega$, set $f^n \triangleq (\circ \langle n \times \{f\}, \Delta_D \rangle) : S \to S$. Furthermore, given any $R \subseteq S^2$, $\mathrm{Tr}(R) \triangleq \{\langle \pi_0(\pi_0(\bar{r})), \pi_1(\pi_{l-1}(\bar{r})) \rangle | \bar{r} \in R^l, l \in (\omega \setminus 1)\}$ is the least transitive binary relation on S including R, referred to as the *transitive closure of* R. Finally, given any $T \subseteq S/f : S \to S/R \subseteq S^2$, an n-ary operation g on S, where $n \in \omega$, is said to be *T-idempotent/f-preserving/R-monotonic*, provided, for all $a \in T/\bar{b} \in A^n/\bar{c}, \bar{d} \in (A^n \cap R)$, it holds that $g(n \times \{a\}) = a/f(g(\bar{b})) = g(f \circ \bar{b})/ \; g(\bar{c}) \; R \; g(\bar{d})$, respectively.

In general, we use the following standard notations going back to [4]:

$$\mathsf{t} \triangleq \langle 1, 1 \rangle, \qquad \mathsf{f} \triangleq \langle 0, 0 \rangle,$$
$$\mathsf{b} \triangleq \langle 1, 0 \rangle, \qquad \mathsf{n} \triangleq \langle 0, 1 \rangle.$$

In addition, the mapping $\mu : 2^2 \to 2^2, \langle a, b \rangle \mapsto \langle b, a \rangle$ is said to be *mirror/specular*, in which case $\mu^{-1} = \mu$, so μ is bijective, i.e., a permutation on 2^2. Moreover, by $\sqsubseteq$ we denote the partial ordering on 2^2 defined by $(\vec{a} \sqsubseteq \vec{b}) \overset{\text{def}}{\iff} ((a_0 \leqslant b_0) \& (b_1 \leqslant a_1))$, for all $\vec{a}, \vec{b} \in 2^2$. Then, given any $B \subseteq 2^2$, $(\mu \restriction B)$-preserving/$(\sqsubseteq \cap B^2)$-monotonic n-ary operations on B, where $n \in \omega$, are referred to as *specular/regular*, respectively.

Let A be a set. An *anti-chain of* any $S \subseteq \wp(A)$ is any $N \subseteq S$ such that $\max(N) = N$. Likewise, a *lower cone of* S is any $L \subseteq S$ such that, for each $X \in L$, $(\wp(X) \cap S) \subseteq L$. This is said to be *generated by* a $G \subseteq L$, whenever $L = (G)_S^\triangledown \triangleq (S \cap \bigcup \{\wp(X) \mid X \in G\})$. (Clearly, in case S — in particular, A — is finite, the mappings $N \mapsto (N)_S^\triangledown$ and $L \mapsto \max(L)$ are inverse to one another bijections between the sets of all anti-chains and lower cones of S.) A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$. A subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. Further, any $X \in T \subseteq \wp(A)$ is said to be K*-meet-irreducible (in/of* T*)*, where $K \subseteq \infty$, provided it belongs to every $U \in \wp_K(T)$ such that $(A \cap \bigcap U) = X$ (in which case $X \neq A$, whenever $0 \in K$), the set of all them being denoted by $\mathrm{MI}^K(T)$.[2] A *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$, in which case the poset $\langle \mathcal{C}, \subseteq \cap\, \mathcal{C}^2 \rangle$ to be identified with $\mathcal{C}$ alone is a complete lattice with meet $A \cap \bigcap$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure) basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. An *operator over* A is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent]* {*transitive*} ⟨*inductive/finitary/compact*⟩, provided, for all $(B,)D \in \wp(A)$ ⟨resp., any upward-directed $U \subseteq \wp(A)$⟩, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)$ ⟨resp., $O(\bigcup U) \subseteq \bigcup O[U]$⟩. A *closure operator over* A is any monotonic idempotent transitive operator C over A, in which case $\operatorname{img} C$ is a closure system over A, determining C uniquely, because, for every closure basis $\mathcal{B}$ of $\operatorname{img} C$ (including $\operatorname{img} C$ itself) and each $X \subseteq A$, it holds that $C(X) = (A \cap \bigcap\{Y \in \mathcal{B} | X \subseteq Y\})$, called *dual to* C and vice versa. (Clearly, C is inductive iff $\operatorname{img} C$ is so.)

Remark 2.1. As a consequence of Zorn's Lemma, according to which any inductive non-empty set has a maximal element, given any inductive closure system $\mathcal{C}$, $\mathrm{MI}(\mathcal{C})$ is a closure basis of $\mathcal{C}$, and so is $\mathrm{MI}^K(\mathcal{C}) \supseteq \mathrm{MI}(\mathcal{C})$, where $K \subseteq \infty$. □

A *[dual] Galois retraction between* posets $\langle P, \leqq \rangle$ and $\langle Q, \lesssim \rangle$ is any couple $\langle f, g \rangle$ of anti-monotonic [resp., monotonic] mappings $f : P \to Q$ and $g : Q \to P$ such that $(g \circ f) = \Delta_P$ and $(f \circ g) \subseteq \lesssim^{[-1]}$, in which case case the former poset is said to be a *[dual] Galois retract of* the latter, while f is a dual embedding [resp., an embedding] of the former into the latter. (Galois retractions are exactly Galois connections with injective/surjective left/right component; cf. [25] and [33]. Moreover, dual Galois retractions between $\langle P, \leqq \rangle$ and $\langle Q, \lesssim \rangle$ are exactly Galois retractions between $\langle P, \leqq \rangle$ and $\langle Q, \lesssim^{-1} \rangle$.)

2.2. **Algebraic background.** Unless otherwise specified, abstract algebras are denoted by Fraktur letters (possibly, with indices/prefixes/suffixes), their carriers (viz., underlying sets) being denoted by corresponding Italic letters (with same indices/prefixes/suffixes, if any).

Let $\mathfrak{A}$ be an algebra. Then, $\mathrm{Con}(\mathfrak{A})$ is an inductive closure system over A^2, in which case $\mathfrak{A}$ is said to be *simple/congruence-distributive*, whenever the lattice $\mathrm{Con}(\mathfrak{A})$ is two-element/distributive. Next, $\mathfrak{A}$ is said to be *subdirectly irreducible*, provided $\Delta_A \in \mathrm{MI}(\mathrm{Con}(\mathfrak{A}))$, in which case $|A| > 1$. (Clearly, any simple algebra is subdirectly irreducible.)

A *(propositional) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with by default throughout the paper) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional) connectives.* Given any $\alpha \in \wp_{\infty[\setminus 1]}(\omega)$ [in case Σ has no nullary symbol], put $V_\alpha \triangleq \{x_\beta | \beta \in \alpha\}$ and $(\forall_\alpha) \triangleq (\forall V_\alpha)$. Then, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^\alpha_\Sigma$ freely-generated by the set V_α, elements of which being viewed as *(propositional) variables of rank* α, referred to as the *formula* Σ*-algebra of rank* α, its endomorphisms/elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ (viz., Σ-terms of rank α) being called *(propositional)* Σ*-substitutions/-formulas of rank* α. A Σ*-equation/identity of rank* α is then any couple of the form $\phi \approx \psi$, where $\phi, \psi \in \mathrm{Fm}^\alpha_\Sigma$, to be identified with the ordered pair $\langle \phi, \psi \rangle$, the set of all them being denoted by $\mathrm{Eq}^\alpha_\Sigma$. (In general, the reservation "of rank α" is normally omitted, whenever $\alpha = \omega$.) Given any $[m,]n \in \omega$, by $\sigma_{[m:]+n}$ we denote the Σ-substitution extending $[x_i/x_{i+n}]_{i \in (\omega[\setminus m])}$.

The *variety axiomatized by* a given $\mathcal{I} \subseteq \mathrm{Eq}^\omega_\Sigma$ is the class of all Σ-algebras satisfying each identity in $\mathcal{I}$. A $\theta \in \mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$ is said to be *fully invariant*, provided $\sigma[\theta] \subseteq \theta$, for every Σ-substitution σ, in which case θ is the set of all Σ-identities satisfied in the variety axiomatized by θ. Conversely, the set θ_V of all Σ-identities satisfied in a variety V (clearly, axiomatized by θ_V) is a fully invariant congruence of $\mathfrak{Fm}^\omega_\Sigma$. In this way, the closure system of all fully invariant congruences of $\mathfrak{Fm}^\omega_\Sigma$ is dual isomorphic to the lattice of all varieties of Σ-algebras (cf. [10]).

A class K of Σ-algebras is said to be *congruence-distributive*, whenever every member of it is so. In general, the class of all [non-one-element] subalgebras/homomorphic images/isomorphic copies of members of K is denoted by $(\mathbf{S}/\mathbf{H}/\mathbf{I})_{[>1]}\mathsf{K}$, respectively. Likewise, the class of all subdirectly irreducible members of K is denoted by $\mathrm{Si}(\mathsf{K})$. Finally, the variety *generated by* K (viz., the least one including K), being clearly axiomatized by the set of all Σ-identities true in K, is denoted by $\mathbf{V}(\mathsf{K})$. The variety $\mathbf{V}(\varnothing)$, constituted by all one-element Σ-algebras, is said to be *trivial.*

Let I be a set, $\overline{\mathfrak{A}}$ an I-tuple of Σ-algebras and $\mathfrak{B}$ a subalgebra of $\mathfrak{C} \triangleq \prod_{i \in I} \mathfrak{A}_i$. Given any *[prime] filter* $\mathcal{F}$ *on* I (viz., a non-empty [proper prime] filter of the lattice $\langle \wp(I), \cap, \cup \rangle$), we then have $\theta^B_\mathcal{F} \triangleq \{\langle \bar{a}, \bar{b} \rangle \in B^2 \mid \{i \in I \mid a_i = b_i\} \in \mathcal{F}\} \in \mathrm{Con}(\mathfrak{B})$, congruences of such a kind being referred to as *[prime] filtral* [in which case:

$$(\mathfrak{C}/\theta^C_\mathcal{F}) \in \mathbf{I}(\operatorname{img} \overline{\mathfrak{A}}), \tag{2.1}$$

whenever both $\operatorname{img} \overline{\mathfrak{A}}$ and all members of it are finite; cf., e.g., [7]].

Recall the following useful well-known facts:

Lemma 2.2. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* Σ*-algebras and* $h \in \hom(\mathfrak{A}, \mathfrak{B})$*. (Suppose* $(\operatorname{img} h) = B$*.) Then, for every* $\vartheta \in \mathrm{Con}(\mathfrak{B})$*,* $h^{-1}[\vartheta] \in \{\theta \in \mathrm{Con}(\mathfrak{A}) \mid (\ker h) \subseteq \theta\}$ *(whereas* $h[h^{-1}[\vartheta]] = \vartheta$*, while, conversely, for every* $\theta \in \mathrm{Con}(\mathfrak{A})$ *such that* $(\ker h) \subseteq \theta$*,* $h[\theta] \in \mathrm{Con}(\mathfrak{B})$*, whereas* $h^{-1}[h[\theta]] = \theta$*).*

Remark 2.3 (cf., e.g., Theorem 1.3 of [17]). In view of Remark 2.1, given any member $\mathfrak{A}$ of a variety V, $\Theta \triangleq \mathrm{MI}(\mathrm{Con}(\mathfrak{A}))$ is a basis of the inductive closure system $\mathrm{Con}(\mathfrak{A})$ over A^2, each $(\mathfrak{A}/\theta) \in \mathsf{V}$, where $\theta \in \Theta$, being subdirectly irreducible, in view of Lemma 2.2, in which case $\Delta_A = (A^2 \cap \bigcap \Theta)$, so $e \triangleq (\prod_{\theta \in \Theta} \nu_\theta) : A \to (\prod_{\theta \in \Theta}(A/\theta))$ is an embedding of $\mathfrak{A}$ into $\prod_{\theta \in \Theta}(\mathfrak{A}/\theta)$, and so is an isomorphism from $\mathfrak{A}$ onto the subdirect product $(\prod_{\theta \in \Theta}(\mathfrak{A}/\theta)) \upharpoonright (\operatorname{img} e)$ of the tuple $\langle \mathfrak{A}/\theta \rangle_{\theta \in \Theta}$ constituted by subdirectly irreducible members of V. In particular, $\mathsf{V} = \mathbf{V}(\mathrm{Si}(\mathsf{V}))$. □

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[2] In general, any mention of K is normally omitted, whenever $K = \infty$. Likewise, "finitely-/pairwise-" means "ω-/$\{2\}$-", respectively.

Lemma 2.4 (cf., e.g., the proof of Theorem 2.6 of [17])**.** *Let I be a set, $\overline{\mathfrak{A}}$ an I-tuple of Σ-algebras, $\mathfrak{B}$ a congruence-distributive subalgebra of $\prod_{i\in I}\mathfrak{A}_i$ and $\theta\in\mathrm{MI}(\mathrm{Con}(\mathfrak{B}))$. Then, there is some prime filter $\mathcal{F}$ on I such that $\theta^B_{\mathcal{F}}\subseteq\theta$.*

Then, combining (2.1), Lemmas 2.2, 2.4 and the Algebra Homomorphism Theorem, we get:

Corollary 2.5 (cf., e.g., Theorem 2.6 of [17])**.** *Let K be a finite class of finite Σ-algebras. Suppose $\mathsf{V}\triangleq\mathbf{V}(\mathsf{K})$ is congruence-distributive. Then, $\mathrm{Si}(\mathsf{V})\subseteq\mathbf{H}_{>1}\mathbf{S}_{>1}\mathsf{K}$. In particular, $\mathrm{Si}(\mathsf{V})=\mathbf{IS}_{>1}\mathsf{K}$, whenever every member of $\mathbf{S}_{>1}\mathsf{K}$ is simple, in which case every member of $\mathrm{Si}(\mathsf{V})$ is simple.*

And what is more, we also have:

Corollary 2.6 (Congruence filtrality)**.** *Let K be a finite class of finite Σ-algebras, I a set, $\overline{\mathfrak{A}}\in\mathsf{K}^I$ and $\mathfrak{B}$ a congruence-distributive subalgebra of $\mathfrak{C}\triangleq\prod_{i\in I}\mathfrak{A}_i$. Suppose every member of $\mathbf{S}_{>1}\mathsf{K}$ is simple. Then, each element of $\mathrm{Con}(\mathfrak{B})$ is filtral.*

Proof. Consider any $\theta\in\mathrm{MI}(\mathrm{Con}(\mathfrak{B}))$, in which case $\theta\neq B^2$. Then, by Lemma 2.4, there is some prime filter $\mathcal{F}$ on I such that $\mathrm{Con}(\mathfrak{B})\ni\vartheta\triangleq\theta^B_{\mathcal{F}}\subseteq\theta$, in which case we have $\eta\triangleq\theta^C_{\mathcal{F}}\in\mathrm{Con}(\mathfrak{C})$, while $B^2\neq\vartheta=(B^2\cap\eta)=\ker(\nu_\eta\restriction\Delta_B)$, and so, by the Algebra Homomorphism Theorem and (2.1), we get $(\mathfrak{B}/\vartheta)\in\mathbf{IS}_{>1}(\mathfrak{C}/\eta)\subseteq\mathbf{IS}_{>1}\mathbf{I}\mathsf{K}\subseteq\mathbf{IS}_{>1}\mathsf{K}$. Hence, by Lemma 2.2, we eventually get $\theta=\vartheta$. Thus, each element of $\mathrm{MI}(\mathrm{Con}(\mathfrak{B}))$ is filtral. In this way, Remark 2.1 and the fact that the set of all filters on I is a closure system over $\wp(I)$, while the mapping $\mathcal{F}\mapsto\theta^B_{\mathcal{F}}$ preserves intersections, complete the argument. ☐

By Corollary 2.6, we then immediately get:

Corollary 2.7 (Congruence inheritance)**.** *Let $\Sigma'\subseteq\Sigma$, K a finite class of finite Σ-algebras, I a set, $\overline{\mathfrak{A}}\in\mathsf{K}^I$ and $\mathfrak{B}$ a subalgebra of $\prod_{i\in I}\mathfrak{A}_i$. Suppose every member of $\mathbf{S}_{>1}(\mathsf{K}\restriction\Sigma')$ is simple and $\mathfrak{B}\restriction\Sigma'$ is congruence-distributive. Then, $\mathrm{Con}(\mathfrak{B})=\mathrm{Con}(\mathfrak{B}\restriction\Sigma')$.*

2.3. **Propositional logics and matrices.** A Σ-*rule* is any couple $\langle\Gamma,\varphi\rangle$, where $(\Gamma\cup\{\varphi\})\in\wp_\omega(\mathrm{Fm}^\omega_\Sigma)$, normally written in the standard sequent form $\Gamma\vdash\varphi$, φ/any element of Γ being referred to as the/a *conclusion/premise of* it. A *(substitutional)* Σ-*instance* of it is then any Σ-rule of the form $\sigma(\Gamma\vdash\varphi)\triangleq(\sigma[\Gamma]\vdash\sigma(\varphi))$, where σ is a Σ-substitution. As usual, Σ-rules without premises are called Σ-*axioms* and are identified with their conclusions. A[n] *[axiomatic]* Σ-*calculus* is any set $\mathcal{C}$ of Σ-rules [without premises], the set of all Σ-instances of its elements being denoted by $\mathrm{SI}_\Sigma(\mathcal{C})$. Then, $\Gamma\vdash\varphi$ is said to be *derivable in* $\mathcal{C}$, if there is a $\mathcal{C}$-*derivation of* it, i.e., a proof of φ (in the conventional proof-theoretical sense) by means of axioms and rules in $\Gamma\cup\mathrm{SI}_\Sigma(\mathcal{C})$.

A *(propositional)* Σ-*logic* is any closure operator C over $\mathrm{Fm}^\omega_\Sigma$ that is *structural* in the sense that $\sigma[C(X)]\subseteq C(\sigma[X])$, for all $X\subseteq\mathrm{Fm}^\omega_\Sigma$ and all $\sigma\in\hom(\mathfrak{Fm}^\omega_\Sigma,\mathfrak{Fm}^\omega_\Sigma)$, or, equivalently, $\operatorname{img}C$ is closed under inverse Σ-substitutions (we sometimes write $X\vdash_C Y$ for $C(X)\supseteq Y$). A(n) *(in)consistent set of* C is any $X\subseteq\mathrm{Fm}^\omega_\Sigma$ such that $C(X)\neq(=)\mathrm{Fm}^\omega_\Sigma$. Then, C is said to be *[inferentially] (in)consistent*, provided $\varnothing[\cup\{x_0\}]$ is a(n in)consistent set of C or, equivalently, in view of the structurality of C, $x_1\notin(\in)C(\varnothing[\cup\{x_0\}])$. A Σ-rule $\Gamma\vdash\varphi$ is said to be *satisfied in* C, provided $\varphi\in C(\Gamma)$, Σ-axioms satisfied in C being referred to as its *theorems*. A *[proper] extension of* C is any Σ-logic $C'\supseteq C$ [distinct from C], in which case C is said to be a *[proper] sublogic of* C'. Then, an extension C' of C is said to be *axiomatized by* a Σ-calculus $\mathcal{C}$ *relatively to* C, provided it is the least extension of C satisfying each rule of $\mathcal{C}$. The extension $\mathrm{Cn}_{\mathcal{C}}$ of the diagonal Σ-logic relatively axiomatized by $\mathcal{C}$ is said to be *axiomatized by* $\mathcal{C}$ and is referred to as the *consequence of* $\mathcal{C}$, in which case it is inductive and satisfies any Σ-rule iff this is derivable in $\mathcal{C}$. (Conversely, any inductive Σ-logic is axiomatized by the set of all Σ-rules satisfied in it.) An extension C' of C is said to be *axiomatic*, whenever it is relatively axiomatized by an axiomatic Σ-calculus $\mathcal{A}$, in which case, for all $X\subseteq\mathrm{Fm}^\omega_\Sigma$, it holds that:

$$C'(X)=C(X\cup\mathrm{SI}_\Sigma(\mathcal{A})). \tag{2.2}$$

Next, C is said to be *[inferentially] maximal*, whenever it is [inferentially] consistent and has no proper [inferentially] consistent extension. Further, C is said to be *[(dual-)weakly]* $\diamond$-*conjunctive* (cf. [22]), where $\diamond$ is a {possibly, secondary} binary connective of Σ, provided $C(\phi\diamond\psi)[(\subseteq)\supseteq]=C(\{\phi,\psi\})$, for all $\phi,\psi\in\mathrm{Fm}^\omega_\Sigma$. Next, C is said to *have the Property of Weak Contraposition with respect to* a unary $\wr\in\Sigma$ (cf. [20]), provided $(\psi\in C(\phi))\Rightarrow(\wr\phi\in C(\wr\psi))$, for all $\phi,\psi\in\mathrm{Fm}^\omega_\Sigma$. Likewise, C is said to be *[maximally]* $\wr$-*paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{x_0,\wr x_0\}\vdash x_1 \tag{2.3}$$

[and has no proper $\wr$-paraconsistent extension]. Furthermore, C is said to be *non-pseudo-axiomatic* (cf. [22]), provided $\bigcap_{k\in\omega}C(x_k)\subseteq C(\varnothing)$ (the converse inclusion always holds by the monotonicity of C). Likewise, it is said to be *purely-inferential/theorem-less*, provided $C(\varnothing)=\varnothing$ or, equivalently, $\varnothing\in(\operatorname{img}C)$. In addition, *Variable Sharing Property* (cf. [1]) is said to *hold/satisfied in* C, provided, for every $\alpha\in(\omega\setminus 1)$, all $\phi\in\mathrm{Fm}^\alpha_\Sigma$ and all $\psi\in\mathrm{Fm}^{\omega\setminus\alpha}_\Sigma$, $\psi\notin C(\phi)$, in which case C has neither a theorem nor an inconsistent formula. Finally, C is said to be *self-extensional* (cf. [22]), provided $\equiv_C\triangleq(\mathrm{Eq}^\omega_\Sigma\cap(\ker C))\in\mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$, in which case, by the sructurality of C, $\equiv_C$ is fully invariant, the corresponding variety being called the *intrinsic variety of* C and denoted by $\mathrm{IV}(C)$.

Remark 2.8. Given a Σ-logic C, we have the Σ-logic $C_{+/-0}$, defined by $C_{+/-0}(X)\triangleq C(X)$, for all non-empty $X\subseteq\mathrm{Fm}^\omega_\Sigma$, and $C_{+/-0}(\varnothing)\triangleq(\varnothing/(\bigcap_{k\in\omega}C(x_k)))$, being the greatest/least purely-inferential/non-pseudo-axiomatic sublogic/extension of C, called the *purely-inferential/non-pseudo-axiomatic version of* C, in which case $\equiv_C=\equiv_{C_{+/-0}}$. Then, the mappings $C\mapsto C_{+0}$ and $C\mapsto C_{-0}$ are inverse to one another isomorphisms between the posets of all non-pseudo-axiomatic and of all purely-inferential Σ-logics ordered by $\subseteq$. ☐

Remark 2.9 (cf. Theorem 4.8 of [22] for the "non-pseudo-axiomatic" case). Since any inductive non-pseudo-axiomatic conjunctive logic C'' is uniquely determined by $\equiv_{C''}$, while the conjunctivity is retained by extensions, in view of Remark 2.8, we conclude that, given any inductive non-pseudo-axiomatic/purely-inferential conjunctive self-extensional Σ-logic C, the mapping $C' \mapsto \mathrm{IV}(C')$ is a dual embedding of the poset of all inductive non-pseudo-axiomatic/purely-inferential self-extensional extensions of C into the lattice of all subvarieties of $\mathrm{IV}(C)$. $\square$

Since any logic is either purely-inferential or, otherwise, non-pseudo-axiomatic, Remark 2.9 actually enhances Theorem 4.8 of [22] beyond non-pseudo-axiomatic logics.

A *(logical)* Σ-*matrix* (cf. [13]) is any couple of the form $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $D^{\mathcal{A}} \subseteq A$ is called the *truth predicate of* $\mathcal{A}$, elements of which being referred to as *distinguished values of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters (possibly, with indices/prefixes/suffixes), their underlying algebras being denoted by corresponding Fraktur letters (with same indices/prefixes/suffixes, if any).) This is said to be *n-valued/truth[-non]-empty/(in)consistent/truth|false-singular*, where $n \in \omega$, provided $|A| = n/D^{\mathcal{A}} = [\neq]\varnothing/D^{\mathcal{A}} \neq (=)A/|(D^{\mathcal{A}}|(A \setminus D^{\mathcal{A}}))| \in 2/|D^{\mathcal{A}}| \in 2$. Next, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(*Σ*-)expansion of* $(\mathcal{A}{\restriction}\Sigma') \triangleq \langle \mathfrak{A}{\restriction}\Sigma', D^{\mathcal{A}} \rangle$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.) Finally, the Σ-matrix $\complement(\mathcal{A}) \triangleq \langle \mathfrak{A}, A \setminus D^{\mathcal{A}} \rangle$ is refereed to as *complementary to/of* $\mathcal{A}$.

A Σ-matrix $\mathcal{A}$ is said to be *finite/generated by* a $B \subseteq A$, whenever $\mathfrak{A}$ is so. Then, it is said to be K*-generated*, where $K \subseteq \infty$, whenever it is generated by some $B \in \wp_K(A)$.

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [15]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any Σ-rule $\Gamma \vdash \phi$ being viewed as the basic [or universal, depending upon the context] first-order Horn formula $[\forall_\omega]((\bigwedge \Gamma) \to \phi)$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$.

Given any $\alpha \in \wp_{\infty \setminus 1}(\omega)$ and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ defined by $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) \triangleq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap \{h^{-1}[D^{\mathcal{A}}] | \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}), h[X] \subseteq D^{\mathcal{A}}\}$, for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, in which case we have:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.4}$$

because $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}) = \{h{\restriction}\mathrm{Fm}^{\alpha}_{\Sigma} | h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})\}$, for any Σ-algebra $\mathfrak{A}$, as $A \neq \varnothing$. (Note that $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(\varnothing) = \varnothing$, whenever M has a truth-empty member. Moreover, using either the ultra-product technique (cf. [15]) or the topological one (cf. [13]), $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ is shown to be inductive, whenever both M and all members of it are finite.) Then, $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic called the one *of* M. A Σ-logic C is said to be K*-defined by* M, where $K \subseteq \infty$, provided $C(X) = \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)$, for all $X \in \wp_K(\mathrm{Fm}^{\omega}_{\Sigma})$. A Σ-logic is said to be *[minimally] n-valued*, where $n \in \omega$, whenever it is defined by an n-valued Σ-matrix [but by no m-valued one with $m \in n$]. A Σ-matrix $\mathcal{A}$ is said to be $\wr$*-paraconsistent*, where $\wr$ is a unary connective of Σ, whenever the logic of $\mathcal{A}$ is so. (Clearly, the logic of any class of matrices is [inferentially] consistent iff the class contains a consistent [truth-non-empty] member.)

Proposition 2.10. *Let* M *be a class of truth-non-empty* Σ*-matrices. Then, the logic of* M *is non-pseudo-axiomatic.*

Proof. Consider any $\varphi \in \bigcap_{k \in \omega} \mathrm{Cn}^{\omega}_{\mathsf{M}}(x_k)$, any $\mathcal{A} \in \mathsf{M}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, $\varphi \in \mathrm{Fm}^{k}_{\Sigma}$, for some $k \in \omega$. Choose any $a \in D^{\mathcal{A}} \neq \varnothing$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h{\restriction}(V_\omega \setminus \{x_k\})) \cup \{\langle x_k, a \rangle\}$. Then, $g(x_k) = a \in D^{\mathcal{A}}$, and so $h(\varphi) = g(\varphi) \in D^{\mathcal{A}}$. $\square$

Remark 2.11. Since the logic of any truth-empty matrix is both purely-inferential and inferentially inconsistent, taking Proposition 2.10 into account, given any class M of Σ-matrices, the purely-inferential/non-pseudo-axiomatic version of the logic of M is defined by $\mathsf{M} \cup / \setminus \mathsf{S}$, where S is any non-empty class of truth-empty Σ-matrices/ the class of all truth-non-empty members of M, respectively. $\square$

Example 2.12. Let $\mathcal{A}$ be a two-valued consistent truth-non-empty Σ-matrix and C the logic of $\mathcal{A}$. Then, $\equiv_C$ is the set of all Σ-identities true in $\mathfrak{A}$, i.e., in $\mathbf{V}(\mathfrak{A})$, in which case C is self-extensional, while $\mathrm{IV}(C) = \mathbf{V}(\mathfrak{A})$. $\square$

A Σ-matrix $\mathcal{A}$ is said to be a *model of* a Σ-logic C, provided $C \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}$, the class of all them being denoted by $\mathrm{Mod}(C)$. Next, $\mathcal{A}$ is said to be *[(dual-)weakly]* $\diamond$*-conjunctive*, where $\diamond$ is a {possibly, secondary} binary connective of Σ, provided $(\{a, b\} \subseteq D^{\mathcal{A}})[(\Rightarrow) \Leftarrow] \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, $\mathrm{Cn}^{\omega}_{\mathcal{A}}$ is [(dual-)weakly] $\diamond$-conjunctive. Then, $\mathcal{A}$ is said to be *[(dual-)weakly]* $\diamond$*-disjunctive*, whenever $\complement(\mathcal{A})$ is [(dual-)weakly] $\diamond$-conjunctive.

Given any [axiomatic] Σ-calculus $\mathfrak{C}$, members of $\mathrm{Mod}(\mathfrak{C}) \triangleq \mathrm{Mod}(\mathrm{Cn}_{\mathfrak{C}})$ are called its *models* as well. This fits well the above model-theoretic conventions, according to which, in particular, (given a class M of Σ-matrices) the class $(\mathsf{M} \cap) \mathrm{Mod}(\mathfrak{C})$ is referred to as the *(relative)* {*equality-free first-order strict*} *[positive] universal Horn model (sub)class (of* M *relatively) axiomatized by* $\mathfrak{C}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective] homomorphism from* $\mathcal{A}$ *[on]to* $\mathcal{B}$ is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and$]$ $D^{\mathcal{A}} \subseteq (=) h^{-1}[D^{\mathcal{B}}]$ ([in which case $\mathcal{B}$ is said to be a *strict homomorphic image of* $\mathcal{A}$]), the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$. Note that:

$$\hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B}) = \hom_{\mathrm{S}}(\complement(\mathcal{A}), \complement(\mathcal{B})). \tag{2.5}$$

And what is more, we have $(\forall h \in \hom(\mathfrak{A}, \mathfrak{B}) : [((\mathrm{img}\, h) = B) \Rightarrow](\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B}) \supseteq [=] \{h \circ g | g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\})$, so we get:

$$(\exists h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{B}} \subseteq [=] \mathrm{Cn}^{\alpha}_{\mathcal{A}}), \tag{2.6}$$

$$(\exists h \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.7}$$

for all $\alpha \in \wp_{\infty \setminus 1}(\omega)$. Then, $\mathcal{A} \neq \mathcal{B}]$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case we set $(\mathcal{B}{\restriction}A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from* $\mathcal{A}$ *into/onto* $\mathcal{B}$, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}/$ and called an *isomorphic copy of* $\mathcal{B}$.

Let $\mathcal{A}$ be a Σ-matrix. Elements of $\mathrm{Con}(\mathcal{A}) \triangleq \{\theta \in \mathrm{Con}(\mathfrak{A}) | \theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}\} \ni \Delta_A$ are called *congruences of* $\mathcal{A}$. Given any $\varnothing \neq \Theta \subseteq \mathrm{Con}(\mathcal{A}) \subseteq \mathrm{Con}(\mathfrak{A})$, $\mathrm{Tr}(\bigcup \Theta)$, being well-known to be a congruence of $\mathfrak{A}$, is then easily seen to be a congruence of $\mathcal{A}$. Therefore, $\partial(\mathcal{A}) \triangleq (\bigcup \mathrm{Con}(\mathcal{A})) \in \mathrm{Con}(\mathcal{A})$, in which case this is the greatest congruence of $\mathcal{A}$ (it is this fact that justifies using the symbol ∂), while $\mathrm{Con}(\mathcal{A}) = \{\theta \in \mathrm{Con}(\mathfrak{A}) | \theta \subseteq \partial(\mathcal{A})\}$. Then, $\mathcal{A}$ is said to be *simple*, provided $\partial(\mathcal{A}) = \Delta_A$, the class of all simple models of a Σ-logic C being denoted by $\mathrm{Mod}_*(C)$. Given any $\theta \in \mathrm{Con}(\mathfrak{A}[\mathcal{A}])$, we have the *quotient* Σ-matrix $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$, in which case $\nu_\theta \in \mathrm{hom}^{\mathrm{S}}_{[\mathrm{S}]}(\mathcal{A}, \mathcal{A}/\theta)$. The quotient $\Re(\mathcal{A}) \triangleq (\mathcal{A}/\partial(\mathcal{A}))$ is called the *reduction of* $\mathcal{A}$.

Corollary 2.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices and $h \in \mathrm{hom}^{(\mathrm{S})}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Then, for every $\vartheta \in \mathrm{Con}(\mathcal{B})$, $h^{-1}[\vartheta] \in \{\theta \in \mathrm{Con}(\mathcal{A}) \mid (\ker h) \subseteq \theta\}$ (whereas $h[h^{-1}[\vartheta]] = \vartheta$, while, conversely, for every $\theta \in \mathrm{Con}(\mathcal{A})$ such that $(\ker h) \subseteq \theta$, $h[\theta] \in \mathrm{Con}(\mathcal{B})$, whereas $h^{-1}[h[\theta]] = \theta$).*

Proof. With using Lemma 2.2. First, consider any $\vartheta \in \mathrm{Con}(\mathcal{B})$. Then, the fact that $h^{-1}[\vartheta][D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}$ is by the fact that $\vartheta[D^{\mathcal{B}}] \subseteq D^{\mathcal{B}}$, while $D^{\mathcal{A}} = h^{-1}[D^{\mathcal{B}}]$. (Conversely, consider any $\theta \in \mathrm{Con}(\mathcal{A})$ such that $\ker h \subseteq \theta$. Then, the fact that $(h[\theta])[D^{\mathcal{B}}] \subseteq D^{\mathcal{B}}$ is by the fact that $\theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}$, while $D^{\mathcal{A}} = h^{-1}[D^{\mathcal{B}}]$.) □

By Corollary 2.13, we immediately have:

Corollary 2.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices and $h \in \mathrm{hom}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Suppose $\mathcal{A}$ is simple. Then, h is injective.*

Proposition 2.15 (Matrix Homomorphism Theorem)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Σ-matrices, $f \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$ and $g \in \mathrm{hom}^{(\mathrm{S})}_{[\mathrm{S}]}(\mathcal{A}, \mathcal{C})$. Suppose $(\ker f) \subseteq (\ker g)$. Then, $h \triangleq (g \circ f^{-1}) \in \mathrm{hom}^{(\mathrm{S})}_{[\mathrm{S}]}(\mathcal{B}, \mathcal{C})$.*

Proof. The fact that $h \in \mathrm{hom}(\mathfrak{B}, \mathfrak{C})$ (and $h[B] = C$) is well-known due to the Algebra Homomorphism Theorem. Finally, we also have $h^{-1}[D^{\mathcal{C}}] = f[g^{-1}[D^{\mathcal{C}}]][=] \supseteq f[D^{\mathcal{A}}] = f[f^{-1}[D^{\mathcal{B}}]] = D^{\mathcal{B}}$, for $f[A] = B$, as required. □

Proposition 2.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices and $h \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Then, $\partial(\mathcal{A}) = h^{-1}[\partial(\mathcal{B})]$ and $\partial(\mathcal{B}) = h[\partial(\mathcal{A})]$.*

Proof. As $\Delta_B \in \mathrm{Con}(\mathcal{B})$, by Corollary 2.13, we have $\ker h = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, and so $\ker h \subseteq \partial(\mathcal{A})$, in which case, by Corollary 2.13, we get:

$$\begin{aligned} h^{-1}[\partial(\mathcal{B})] &\subseteq \partial(\mathcal{A}), \\ h[h^{-1}[\partial(\mathcal{B})]] &= \partial(\mathcal{B}), \\ h[\partial(\mathcal{A})] &\subseteq \partial(\mathcal{B}), \\ h^{-1}[h[\partial(\mathcal{A})]] &= \partial(\mathcal{A}). \end{aligned}$$

These collectively imply the equalities to be proved, as required. □

Since, for any equivalence θ on any set A, it holds that $\nu_\theta[\theta] = \Delta_{A/\theta}$, as an immediate consequence of Proposition 2.16, we also have:

Corollary 2.17. *Let $\mathcal{A}$ be a Σ-matrix. Then, $\Re(\mathcal{A})$ is simple.*

Proposition 2.18. *Let C be a Σ-logic and M a finite class of finite Σ-matrices. Suppose C is finitely-defined by M. Then, C is defined by M. In particular, C is inductive.*

Proof. In that case, $C' \triangleq \mathrm{Cn}^{\omega}_{\mathsf{M}} \subseteq C$, for C' is inductive, while $\equiv_C = \equiv_{C'}$. For proving the converse point-wise inclusion, it suffices to prove that $\mathsf{M} \subseteq \mathrm{Mod}(C)$. For consider any $\mathcal{A} \in \mathsf{M}$, any $\Gamma \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, any $\varphi \in C(\Gamma)$ and any $h \in \mathrm{hom}(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h[\Gamma] \subseteq D^{\mathcal{A}}$. Then, $\alpha \triangleq |A| \in (\wp_{\infty\setminus 1}(\omega) \cap \omega)$. Take any bijection $e : V_\alpha \to A$ to be extended to a $g \in \mathrm{hom}(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $e^{-1} \circ (h \restriction V_\omega)$ is extended to a Σ-substitution σ, in which case $\sigma(\varphi) \in C(\sigma[\Gamma])$, for C is structural, while $\sigma[\Gamma \cup \{\varphi\}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. For every $\mathcal{B} \in \mathsf{M}$, we have the equivalence relation $\theta^{\mathcal{B}} \triangleq \{\langle a, b \rangle \in B^2 \mid (a \in D^{\mathcal{B}}) \Leftrightarrow (b \in D^{\mathcal{B}})\}$ on B, in which case $B/\theta^{\mathcal{B}}$ is finite, for B is so. Moreover, as both α, M and all members of it are finite, we have the finite set $I \triangleq \{\langle h', \mathcal{B} \rangle \mid \mathcal{B} \in \mathsf{M}, h' \in \mathrm{hom}(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B})\}$, in which case, for each $i \in I$, we set $h_i \triangleq \pi_0(i)$, $\mathcal{B}_i \triangleq \pi_1(i)$ and $\theta_i \triangleq \theta^{\mathcal{B}_i}$. Then, by (2.4), we have $\theta \triangleq (\equiv_{C'} \cap \mathrm{Eq}^{\alpha}_{\Sigma}) = (\mathrm{Eq}^{\alpha}_{\Sigma} \cap \bigcap_{i \in I} h_i^{-1}[\theta_i])$, in which case, for every $i \in I$, $\theta \subseteq h_i^{-1}[\theta_i] = \ker(\nu_{\theta_i} \circ h_i)$, and so $g_i \triangleq (\nu_{\theta_i} \circ h_i \circ \nu_\theta^{-1}) : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to B_i$. In this way, $f \triangleq (\prod_{i \in I} g_i) : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to (\prod_{i \in I} B_i)$ is injective, for $(\ker f) = ((\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)^2 \cap \bigcap_{i \in I} (\ker g_i))$ is diagonal. Hence, $\mathrm{Fm}^{\alpha}_{\Sigma}/\theta$ is finite, for $\prod_{i \in I} B_i$ is so, and so is $(\sigma[\Gamma]/\theta) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)$. For each $c \in (\sigma[\Gamma]/\theta)$, choose any $\phi_c \in (\sigma[\Gamma] \cap \nu_\theta^{-1}[\{c\}]) \neq \varnothing$. Put $\Delta \triangleq \{\phi_c \mid c \in (\sigma[\Gamma]/\theta)\} \in \wp_\omega(\sigma[\Gamma])$. Consider any $\psi \in \sigma[\Gamma]$. Then, $\Delta \ni \phi_{[\psi]_\theta} \equiv_C \psi$, in which case $\psi \in C(\Delta)$, and so $\sigma[\Gamma] \subseteq C(\Delta)$. In this way, $\sigma(\varphi) \in C(\Delta) = C'(\Delta)$, for $\Delta \in \wp_\omega(\mathrm{Fm}^{\omega}_{\Sigma})$, so, by (2.4), $\sigma(\varphi) \in \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\Delta)$. Moreover, $g[\Delta] \subseteq g[\sigma[\Gamma]] = h[\Gamma] \subseteq D^{\mathcal{A}}$, and so $h(\varphi) = g(\sigma(\varphi)) \in D^{\mathcal{A}}$, as required. □

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, [any submatrix $\mathcal{B}$ of] the Σ-matrix $(\prod_{i \in I} \mathcal{A}_i) \triangleq \langle \prod_{i \in I} \mathfrak{A}_i, \prod_{i \in I} D^{\mathcal{A}_i} \rangle$ is called the [a] *[sub]direct product of* $\overline{\mathcal{A}}$ [whenever, for each $i \in I$, $\pi_i[B] = A_i$]. As usual, when $I = 2$, $\mathcal{A}_0 \times \mathcal{A}_1$ stands for the direct product involved. Likewise, if $(\mathrm{img}\, \overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$ (and $I = 2$), where $\mathcal{A}$ is a Σ-matrix, $\mathcal{A}^I \triangleq (\prod_{i \in I} \mathcal{A}_i)$ [resp., $\mathcal{B}$] is called the [a] *[sub]direct I-power (square) of* $\mathcal{A}$.

Given a class M of Σ-matrices, the class of all (truth-non-empty) [consistent] submatrices/isomorphic copies/strict homomorphic images of members of M is denoted by $(\mathbf{S}^{(*)}_{[*]}/\mathbf{I}/\mathbf{H})(\mathsf{M})$, respectively. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$. Clearly, logic/calculus model classes are closed under both $\mathbf{P}$ and $\mathbf{S}$ and $\mathbf{H}$.

Lemma 2.19 (Subdirect Product Lemma)**.** *Let M be a [finite] class of [finite] Σ-matrices and $\mathcal{A}$ a {truth-non-empty} (simple) $([\omega\cap](\omega+1))$-generated model of the logic of M. Then, $\mathcal{A}/\partial(\mathcal{A})$ (in particular, $\mathcal{A}$ itself) belongs to $\mathbf{H}(\mathbf{P}^{\mathrm{SD}}_{[\omega]}(\mathbf{S}^{\{*\}}_{*}(\mathsf{M})))$.*

Proof. Take any $A' \in \wp_{[\omega\cap](\omega+1)}(A)$ generating $\mathfrak{A}$ and any $a \in A \neq \varnothing$, in which case $A'' \triangleq (A' \cup \{a\}) \in (\wp_{[\omega\cap](\omega+1)}(A) \setminus 1)$ generates $\mathfrak{A}$, and so $\alpha \triangleq |A''| \in (([\omega\cap](\omega+1)) \setminus 1) \subseteq \wp_{\infty\setminus 1}(\omega)$. Next, take any bijection from V_α onto A'' to be extended to a surjective $h \in \hom(\mathrm{Fm}^\alpha_\Sigma, \mathfrak{A})$, in which case it is a surjective strict homomorphism from $\mathcal{B} \triangleq \langle \mathrm{Fm}^\alpha_\Sigma, X\rangle$, where $\{\varnothing \neq\}X \triangleq h^{-1}[D^{\mathcal{A}}]$, onto $\mathcal{A}$, and so, by (2.6), $\mathcal{B}$ is a {truth-non-empty} model of the logic of M. Then, applying (2.4) twice, we get $\mathrm{Cn}^\alpha_{\mathsf{M}}(X) \subseteq \mathrm{Cn}^\alpha_{\mathcal{B}}(X) \subseteq X \subseteq \mathrm{Cn}^\alpha_{\mathsf{M}}(X)$. Furthermore, we have the [finite] set $I \triangleq \{\langle h', \mathcal{D}\rangle \mid h' \in \hom(\mathcal{B}, \mathcal{D}), \mathcal{D} \in \mathsf{M}, (\mathrm{img}\, h') \nsubseteq D^{\mathcal{D}}\}$, in which case, for every $i \in I$, we set $h_i \triangleq \pi_0(i)$, and so $\mathcal{C}_i \triangleq (\pi_1(i){\restriction}(\mathrm{img}\, h_i))$ is a consistent {truth-non-empty} submatrix of $\pi_1(i) \in \mathsf{M}$. Clearly, $X = \mathrm{Cn}^\alpha_{\mathsf{M}}(X) = (\mathrm{Fm}^\alpha_\Sigma \cap \bigcap_{i\in I} h_i^{-1}[D^{\mathcal{C}_i}])$. Therefore, $g \triangleq (\prod_{i\in I} h_i) : \mathrm{Fm}^\alpha_\Sigma \to (\prod_{i\in I} C_i)$ is a strict homomorphism from $\mathcal{B}$ to $\prod_{i\in I} \mathcal{C}_i$ such that, for each $i \in I$, $(\pi_i \circ g) = h_i$, in which case $\pi_i[g[\mathrm{Fm}^\alpha_\Sigma]] = h_i[\mathrm{Fm}^\alpha_\Sigma] = C_i$, and so g is a surjective strict homomorphism from $\mathcal{B}$ onto the subdirect product $\mathcal{E} \triangleq ((\prod_{i\in I} \mathcal{C}_i){\restriction}(\mathrm{img}\, g))$ of $\overline{\mathcal{C}}$. Put $\theta \triangleq \partial(\mathcal{A})(= \Delta_A)$ and $\mathcal{F} \triangleq (\mathcal{A}/\theta)$. Then, $f \triangleq (\nu_\theta \circ h) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{F})$. Therefore, by Corollaries 2.13, 2.17 and Proposition 2.16, we have $(\ker g) = g^{-1}[\Delta_E] \subseteq \partial(\mathcal{B}) = f^{-1}[\Delta_F] = (\ker f)$, in which case, by Proposition 2.15, $e \triangleq (f \circ h^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{F})$ (and so $(\nu_\theta^{-1} \circ e) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{A})$), as required. □

Theorem 2.20. *Let* K *and* M *be classes of* Σ*-matrices,* C *the logic of* M *and* C' *an extension of* C*. Suppose (both* M *and all members of it are finite and)* $[\Re](\mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))) \subseteq \mathsf{K}$ *{in particular,* $[\Re](\mathbf{S}(\mathbf{P}_{(\omega)}(\mathsf{M}))) \subseteq \mathsf{K}$ *⟨in particular,* $\mathsf{K} \supseteq \mathsf{M}$ *is closed under both* $\mathbf{S}$ *and* $\mathbf{P}_{(\omega)}$ *[as well as* $\Re$*] ⌈in particular,* $\mathsf{K} = \mathrm{Mod}(C)$*; cf. (2.6)⌉⟩}. Then,* C' *is (finitely-)defined by* $\mathsf{S} \triangleq (\mathrm{Mod}_{[*]}(C') \cap \mathsf{K})$*, and so by* $\mathrm{Mod}_{[*]}(C')$*.*

Proof. Clearly, $C' \subseteq \mathrm{Cn}^\omega_{\mathsf{S}}$, for $\mathsf{S} \subseteq \mathrm{Mod}(C')$. Conversely, consider any $(\Gamma \cup \{\varphi\}) \in \wp_{(\omega)}(\mathrm{Fm}^\omega_\Sigma)$, in which case (there is some $\alpha' \in (\omega \setminus 1)$ such that $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\alpha'}_\Sigma$, and so) $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^\alpha_\Sigma$, where $\alpha \triangleq ((\alpha' \cap)\omega) \in \wp_{\infty\setminus 1}(\omega)$, such that $\varphi \notin C'(\Gamma)$. Then, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, C'(\Gamma)\rangle$ is a model of C' {in particular, of C}, and so is its $(\alpha+1)$-generated (in particular, finitely-generated) submatrix $\mathcal{A} \triangleq \langle \mathfrak{Fm}^\alpha_\Sigma, C'(\Gamma) \cap \mathrm{Fm}^\alpha_\Sigma\rangle$, in view of (2.6), in which case $\varphi \notin \mathrm{Cn}^\alpha_{\mathcal{A}}(\Gamma)$, by the idempotencity of C', and so $\varphi \notin \mathrm{Cn}^\omega_{\mathcal{A}}(\Gamma)$, in view of (2.4). Therefore, by Lemma 2.19, there are some $\mathcal{B} \in \mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))$, in which case $\mathcal{D} \triangleq [\Re](\mathcal{B}) \in [\Re](\mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))) \subseteq \mathsf{K}$, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{A}/\partial(\mathcal{A}))$. Then, by (2.6), $\mathrm{Cn}^\omega_{\mathcal{D}} = \mathrm{Cn}^\omega_{\mathcal{A}}$, in which case [by Corollary 2.17] $\mathcal{D} \in \mathsf{S}$, and so $\varphi \notin \mathrm{Cn}^\omega_{\mathsf{S}}(\Gamma)$, as required. □

Corollary 2.21. *Let* M *be a class of* Σ*-matrices and* $\mathcal{A}$ *an axiomatic* Σ*-calculus. Then, the axiomatic extension* C' *of the logic* C *of* M *relatively axiomatized by* $\mathcal{A}$ *is defined by* $\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A})$*.*

Proof. Then, $\mathrm{Mod}(C') = (\mathrm{Mod}(C) \cap \mathrm{Mod}(\mathcal{A}))$, and so (2.6), (2.7) and Theorem 2.20 with $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}(\mathbf{S}_*(\mathsf{M})) \subseteq \mathrm{Mod}(C)$, in which case $(\mathrm{Mod}(C') \cap \mathsf{K}) = (\mathrm{Mod}(\mathcal{A}) \cap \mathsf{K}) = \mathbf{P}^{\mathrm{SD}}(\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A}))$, complete the argument. □

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^\alpha_{\Sigma'} \subseteq \mathrm{Fm}^\alpha_\Sigma$ forms a subalgebra of $\mathfrak{Fm}^\alpha_\Sigma{\restriction}\Sigma'$ and $\hom(\mathfrak{Fm}^\alpha_{\Sigma'}, \mathfrak{Fm}^\alpha_{\Sigma'}) = \{h{\restriction}\mathrm{Fm}^\alpha_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{Fm}^\alpha_\Sigma), h[\mathrm{Fm}^\alpha_{\Sigma'}] \subseteq \mathrm{Fm}^\alpha_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^\omega_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^\omega_{\Sigma'}$, called the Σ'*-fragment of* C, in which case C is said to be a *(*Σ*-)expansion of* C', while $\equiv_{C'} = (\equiv_C \cap (\mathrm{Fm}^\omega_{\Sigma'})^2)$, and so C' is self-extensional, whenever C is so. In that case, given also any class M of Σ-matrices defining C, C' is, in its turn, defined by $\mathsf{M}{\restriction}\Sigma'$.

2.3.1. *Classical negations, matrices and logics.* Let $\wr \in \Sigma$ be unary.

A Σ-matrix $\mathcal{A}$ is said to be *[weakly] (classically)* $\wr$*-negative*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$.

A two-valued consistent Σ-matrix $\mathcal{A}$ is said to be $\wr$*-classical*, whenever it is $\wr$-negative, in which case it is truth-non-empty, for it is consistent, and so is both truth- and false-singular but is not $\wr$-paraconsistent.

Remark 2.22. Let $\diamond$ be any (possibly, secondary) binary connective of Σ and $(x_0 \tilde{\diamond} x_1) \triangleq \wr(\wr x_0 \diamond \wr x_1)$. Then, any $\diamond$-disjunctive/conjunctive $\wr$-negative (in particular, $\wr$-classical) Σ-matrix is $\tilde{\diamond}$-conjunctive/disjunctive, respectively. □

A Σ-logic is said to be $\wr$*-[sub]classical*, whenever it is [a sublogic of] the logic of a $\wr$-classical Σ-matrix. Then, a Σ-logic is said to be *inferentially* $\wr$*-classical*, whenever it is either $\wr$-classical or the purely inferential version of a $\wr$-classical Σ-logic.

Next, $\wr$ is called a *subclassical negation for* a Σ-logic C, whenever the $\wr$-fragment of C is $\wr$-subclassical, in which case:

$$\wr^m x_0 \notin C(\wr^n x_0), \tag{2.8}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd.

3. Preliminary key issues

3.1. **Congruence and equality determinants.** A *[binary] relational* Σ*-scheme* is any Σ-calculus $\varepsilon \subseteq (\wp_\omega(\mathrm{Fm}^{[2\cap]\omega}_\Sigma) \times \mathrm{Fm}^{[2\cap]\omega}_\Sigma)$, in which case, given any Σ-matrix $\mathcal{A}$, we set $\theta^{\mathcal{A}}_\varepsilon \triangleq \{\langle a, b\rangle \in A^2 \mid \mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]\} \subseteq A^2$. Note that, given a one more Σ-matrix $\mathcal{B}$ and an $h \in \hom^{(\mathrm{S})}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, we have:

$$h^{-1}[\theta^{\mathcal{B}}_\varepsilon] \subseteq (=)[=]\theta^{\mathcal{A}}_\varepsilon. \tag{3.1}$$

A *[unary] unitary relational* Σ*-scheme* is any $\Upsilon \subseteq \mathrm{Fm}^{[1\cap]\omega}_\Sigma$, in which case we have the [binary] relational Σ-scheme $\varepsilon_\Upsilon \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in \sigma_{1:+1}[\Upsilon]\}$ such that $\theta^{\mathcal{A}}_{\varepsilon_\Upsilon}$, where $\mathcal{A}$ is any Σ-matrix, is an equivalence relation on A.

A *[binary] congruence/equality determinant for* a class of Σ-matrices M is any [binary] relational Σ-scheme ε such that, for each $\mathcal{A} \in \mathsf{M}$, $\theta^{\mathcal{A}}_\varepsilon \in \mathrm{Con}(\mathcal{A})/ = \Delta_A$, respectively.

Then, according to [29]/[28], a *[unary] unitary congruence/equality determinant for* a class of Σ-matrices M is any [unary] unitary relational Σ-scheme Υ such that ε_Υ is a/an congruence/equality determinant for M. (It is unary unitary equality determinants that are equality determinants in the sense of [28].)

Lemma 3.1 (cf., e.g., [29])**.** $\mathrm{Fm}^{\omega}_{\Sigma}$ *is a unitary congruence determinant for every* Σ*-matrix* $\mathcal{A}$*.*

Proof. We start from proving the fact the equivalence relation $\theta^{\mathcal{A}} \triangleq \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}} \in \mathrm{Con}(\mathfrak{A})$. For consider any $\varsigma \in \Sigma$ of arity $n \in \omega$, any $i \in n$, in which case $n \neq 0$, any $\vec{a} \in \theta^{\mathcal{A}}$, any $\bar{b} \in A^{n-1}$, any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and any $\bar{c} \in A^{\omega}$. Put $\psi \triangleq \varsigma(\langle\langle x_{j+1}\rangle_{j\in i}, x_0\rangle * \langle x_k\rangle_{k\in(n\setminus i)})$ and $\varphi \triangleq ((\sigma_{1:+n}\phi)[x_0/\psi]) \in \mathrm{Fm}^{\omega}_{\Sigma}$. Then, we have

$$(\sigma_{1:+1}\phi)^{\mathfrak{A}}[x_{l+1}/c_l; x_0/\varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_0\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})]_{l\in\omega} = (\sigma_{1:+1}\varphi)^{\mathfrak{A}}[x_{l+n+1}/c_l; x_0/a_0; x_{m+1}/b_m]_{l\in\omega; m\in(n-1)} \in D^{\mathcal{A}} \Leftrightarrow$$
$$D^{\mathcal{A}} \ni (\sigma_{1:+1}\varphi)^{\mathfrak{A}}[x_{l+n+1}/c_l; x_0/a_1; x_{m+1}/b_m]_{l\in\omega; m\in(n-1)} = (\sigma_{1:+1}\phi)^{\mathfrak{A}}[x_{l+1}/c_l; x_0/\varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_1\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})]_{l\in\omega},$$

in which case we eventually get $\langle\varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_0\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)}), \varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_1\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})\rangle \in \theta^{\mathcal{A}}$, and so $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$. Finally, as $x_0 \in \mathrm{Fm}^{\omega}_{\Sigma}$, we clearly have $\theta^{\mathcal{A}}[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}$, as required. □

Example 3.2 (cf. Example 1 of [28])**.** $\{x_0\}$ is a unary unitary equality determinant for any consistent truth-non-empty two-valued (in particular, classical) matrix. □

Example 3.3. [cf. Example 2 of [28]] Let $j \in 2$, $\vec{k} \in 2^2$, $\wr$ a (possibly, secondary) unary connective of Σ and $\mathcal{A}$ a Σ-matrix. Suppose $A \subseteq 2^2$, $D^{\mathcal{A}} = (A \cap \pi_j^{-1}[\{k_1\}])$ and $(\wr^{\mathfrak{A}})^{-1}[D^{\mathcal{A}}] = (A \cap \pi_{1-j}^{-1}[\{k_0\}])$. Then, $\Upsilon_{\wr} \triangleq \{x_0, \wr x_0\}$ is a unary unitary equality determinant for $\mathcal{A}$. □

Lemma 3.4. *Let* $\mathcal{A}$ *be a* Σ*-matrix and* ε *a congruence determinant for* $\mathcal{A}$*. Then,* $\eth(\mathcal{A}) = \theta^{\mathcal{A}}_{\varepsilon}$*. In particular,* $\mathcal{A}$ *is simple, whenever* ε *is an equality determinant for it.*

Proof. Consider any $\theta \in \mathrm{Con}(\mathcal{A})$ and any $\langle a, b\rangle \in \theta$. Then, as $\mathrm{Con}(\mathcal{A}) \ni \theta^{\mathcal{A}}_{\varepsilon} \supseteq \Delta_A \ni \langle a, a\rangle$, we have $\mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/a]$, in which case, by the reflexivity of θ, we get $\mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]$, and so $\langle a, b\rangle \in \theta^{\mathcal{A}}_{\varepsilon}$, as required. □

It is remarkable that Proposition 2.16 equally ensues from Lemmas 3.1, 3.4 and (3.1).

Lemma 3.5. *Let* M *be a class of* Σ*-matrices,* C *the logic of* M *and* $\mathcal{B} \in \mathrm{Mod}_*(C)$*. Then,* $\mathfrak{B} \in \mathbf{V}(\pi_0[\mathsf{M}])$*.*

Proof. Consider any $(\phi \approx \psi) \in \mathrm{Eq}^{\omega}_{\Sigma}$ being true in $\pi_0[\mathsf{M}]$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{B})$. Take any $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and any $v : V_{\omega\setminus 2} \to B$. Then, there is some $k \in (\omega \setminus 1)$ such that $(\phi \approx \psi) \in \mathrm{Eq}^{k}_{\Sigma}$. Put $\varphi' \triangleq \sigma_{1:+k}(\varphi)$. Then, for each $\mathcal{A} \in \mathsf{M}$ and every $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, we have $g(\phi) = g(\psi)$, in which case $g(\varphi'[x_0/\phi]) = g(\varphi'[x_0/\psi])$, and so the rules $(\varphi'[x_0/\phi]) \vdash (\varphi'[x_0/\psi])$ and $(\varphi'[x_0/\psi]) \vdash (\varphi'[x_0/\phi])$ are true in M, and so in $\mathcal{B}$. Let $h' \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{B})$ extend $(h \upharpoonright V_k) \cup [x_{i+k}/v(x_{i+1})]_{i\in(\omega\setminus 1)}$. Then, $(\sigma_{1:+1}(\varphi)[x_0/h(\phi); v]) = h'(\varphi'[x_0/\phi]) \in D^{\mathcal{B}}$ iff $D^{\mathcal{B}} \ni h'(\varphi'[x_0/\psi]) = (\sigma_{1:+1}(\varphi)[x_0/h(\psi); v])$. Thus, $\mathcal{B} \models (\forall_{\omega\setminus 2}((\sigma_{1:+1}(\varphi)[x_0/x_1]) \leftrightarrow \sigma_{1:+1}(\varphi)))[x_0/h(\phi), x_1/h(\psi)]$, for all $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$. Hence, by Lemma 3.1, we eventually get $\langle h(\phi), h(\psi)\rangle \in \eth(\mathcal{B}) = \Delta_B$, as required. □

Lemma 3.6. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* Σ*-matrices,* ε *a/an congruence/equality determinant for* $\mathcal{B}$ *and* h *a/an strict homomorphism/embedding from/of* $\mathcal{A}$ *to/into* $\mathcal{B}$*. Suppose either* ε *is binary or* $h[A] = B$*. Then,* ε *is a/an congruence/equality determinant for* $\mathcal{A}$*.*

Proof. In that case, by (3.1), we have $\theta^{\mathcal{A}}_{\varepsilon} = h^{-1}[\theta^{\mathcal{B}}_{\varepsilon}]$. In this way, Corollary 2.13/the injectivity of h completes the argument. □

Lemma 3.7. *Let* $\mathcal{A}$ *be a* Σ*-matrix with unary unitary equality determinant* Υ*,* $\mathcal{B}$ *a submatrix of* $\mathcal{A}$ *and* $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$*. Then,* h *is diagonal.*

Proof. Consider any $a \in B$. Then, for any $\upsilon \in \Upsilon$, we have $(\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (\upsilon^{\mathfrak{B}}(a) \in D^{\mathcal{B}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(h(a)) = h(\upsilon^{\mathfrak{B}}(a)) \in D^{\mathcal{A}})$, so we get $h(a) = a$, as required. □

Lemma 3.8. *Any axiomatic binary equality determinant* ε *for a class* M *of* Σ*-matrices is so for* $\mathbf{P}(\mathsf{M})$*.*

Proof. In that case, members of M are models of the infinitary universal strict Horn theory $\varepsilon[x_1/x_0] \cup \{(\bigwedge \varepsilon) \to (x_0 \approx x_1)\}$ with equality, and so are well-known to be those of $\mathbf{P}(\mathsf{M})$, as required. □

3.1.1. *Self-extensionality versus unary unitary equality determinants.*

Theorem 3.9. *Let* $[\bar{\wedge}(,\veebar) \in \mathrm{Fm}^2_{\Sigma}$*,]* M *a class of* $[\bar{\wedge}$*-conjunctive]* Σ*-matrices with unary unitary equality determinant* Υ *and* C *the logic of* M*. [Suppose the idempotencity and commutativity (as well as distributive lattice) identities for* $\bar{\wedge}$ *(and* $\veebar$*) are true in* $\pi_0[\mathsf{M}]$*.] Then, (i)⇔(ii)⇔(iii)[⇔(iv)⇔(v)⇒(vi)⇒(vii)(⇒(viii)⇒(iv))], where:*

(i) *C is self-extensional;*
(ii) $(\phi \equiv_C \psi) \Leftrightarrow (\pi_0[\mathsf{M}] \models (\phi \approx \psi))$*, for all* $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*;*
(iii) *there is some class* K *of* Σ*-algebras such that* $(\psi \equiv_C \phi) \Leftrightarrow (\mathsf{K} \models (\phi \approx \psi))$*, for all* $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*;*
(iv) $(\psi \in C(\phi)) \Leftrightarrow (\pi_0[\mathsf{M}] \models ((\phi \mathbin{\bar{\wedge}} \psi) \approx \phi))$*, for all* $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*;*
(v) *there is some class* K *of* Σ*-algebras satisfying the idempotencity and commutativity identities for* $\bar{\wedge}$ *such that* $(\psi \in C(\phi)) \Leftrightarrow (\mathsf{K} \models ((\phi \mathbin{\bar{\wedge}} \psi) \approx \phi))$*, for all* $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*;*
(vi) *every truth-non-empty* $\bar{\wedge}$*-conjunctive* Σ*-matrix with underlying algebra in* $\mathbf{V}(\pi_0[\mathsf{M}])$ *is a model of* C*;*
(vii) *every truth-non-empty* $\bar{\wedge}$*-conjunctive* Σ*-matrix with underlying algebra in* $\pi_0[\mathsf{M}]$ *is a model of* C*;*
(viii) *every consistent truth-non-empty* $\bar{\wedge}$*-conjunctive* $\veebar$*-disjunctive* Σ*-matrix with underlying algebra in* $\pi_0[\mathsf{M}]$ *is a model of* C*,*

in which case $\mathrm{IV}(C) = \mathbf{V}(\pi_0[\mathsf{M}])$*.*

Proof. First, (ii)⇒(iii)⇒(i) are immediate.

Next, assume (i) holds. Then, the metaimplication from right to left in (ii) is immediate. Conversely, consider any $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\mathcal{A} \in \mathsf{M}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Assume $\phi \equiv_C \psi$. Then, by (i), for every $\upsilon \in \Upsilon$, we have $\upsilon(\phi) \equiv_C \upsilon(\psi)$, in which case $(\upsilon^{\mathfrak{A}}(h(\phi)) = h(\upsilon(\phi)) \in D^{\mathcal{A}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(h(\psi)) = h(\upsilon(\psi)) \in D^{\mathcal{A}})$, and so $h(\phi) = h(\psi)$. Thus, (ii) holds yielding $\mathrm{IV}(C) = \mathbf{V}(\pi_0[\mathsf{M}])$.

[Further, (ii)⇒(iv) is by the $\bar{\wedge}$-conjunctivity of C, while (v/iv)⇒(iii/ii) is by the idempotencity and commutativity identities for $\bar{\wedge}/$ true in $\pi_0[\mathsf{M}]$, whereas ((viii/)vii/v) is a particular case of ((vii/)vi/iv). Furthermore, (iv)⇒(vi) is by the following claim:

Claim 3.10. *Let C' be an inductive weakly $\bar{\wedge}$-conjunctive Σ-logic. Then, any {truth-non-empty dual-weakly $\wedge$-conjunctive} Σ-matrix $\mathcal{B}$ is a model of C' [if and] only if $C'(\varphi) \subseteq \mathrm{Cn}_{\mathcal{B}}(\varphi)$, for all $\varphi \in \mathrm{Fm}_{\Sigma}$.*

Proof. Immediate {with using Proposition 2.10}. □

(Finally, (viii)⇒(iv) is by the Prime Ideal Theorem for distributive lattices.)] □

3.2. **False-singular consistent weakly conjunctive matrices.** Given any consistent false-singular Σ-matrix $\mathcal{A}$, the unique element of $A \setminus D^{\mathcal{A}}$ is denoted by $\dashv^{\mathcal{A}}$.

Lemma 3.11. *Let $\diamond$ be a (possibly, secondary) binary connective of Σ, $\mathcal{A}$ a consistent false-singular weakly $\diamond$-conjunctive Σ-matrix, I a finite set, $\overline{\mathcal{C}}$ an I-tuple constituted by consistent submatrices of $\mathcal{A}$ and $\mathcal{B}$ a subdirect product of $\overline{\mathcal{C}}$. Then, $(I \times \{\dashv^{\mathcal{A}}\}) \in B$.*

Proof. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $a \in B$ including $(J \times \{\dashv^{\mathcal{A}}\})$. First, when $J = \varnothing$, take any $a \in C \neq \varnothing$, in which case $(J \times \{\dashv^{\mathcal{A}}\}) = \varnothing \subseteq a$. Now, assume $J \neq \varnothing$. Take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, as $\mathcal{C}_i$ is a consistent submatrix of the false-singular matrix $\mathcal{A}$, we have $\dashv^{\mathcal{A}} \in C_j = \pi_j[B]$. Hence, there is some $b \in B$ such that $\pi_j(b) = \dashv^{\mathcal{A}}$, while, by induction hypothesis, there is some $a \in B$ including $(K \times \{\dashv^{\mathcal{A}}\})$. Therefore, since $J = (K \cup \{j\})$, while $\mathcal{A}$ is both weakly $\diamond$-conjunctive and false-singular, we have $B \ni c \triangleq (a \diamond^{\mathfrak{B}} b) \supseteq (J \times \{\dashv^{\mathcal{A}}\})$. Thus, when $J = I$, we eventually get $B \ni (I \times \{\dashv^{\mathcal{A}}\})$, as required. □

3.3. **Disjunctivity.** Fix any set A and any $\delta : A^2 \to A$. Given any $X, Y \subseteq A$, set $\delta(X, Y) \triangleq \delta[X \times Y]$. Then, a $Z \subseteq A$ is said to be *[weakly] δ-disjunctive*, provided, for all $a, b \in A$, it holds that $((\{a, b\} \cap Z) \neq \varnothing) \Leftrightarrow [\Rightarrow](\delta(a, b) \in Z)$, in which case, for all $X, Y \subseteq A$, we have $((X \subseteq Z)|(Y \subseteq Z)) \Leftrightarrow [\Rightarrow](\delta(X, Y) \subseteq Z)$. Next, a closure operator C over A is said to be *[weakly] δ-disjunctive*, provided, for all $a, b \in A$ and every $Z \subseteq A$, it holds that

$$C(Z \cup \delta(a, b))[\subseteq] = (C(Z \cup \{a\}) \cap C(Z \cup \{b\})), \tag{3.2}$$

in which case the following [resp., (3.3) and (3.4) alone, being equivalent to the weak δ-disjunctivity of C] clearly hold, by (3.2) with $Z = \varnothing$:

$$\delta(a, b) \in C(a), \tag{3.3}$$
$$\delta(a, b) \in C(b), \tag{3.4}$$
$$a \in C(\delta(a, a)), \tag{3.5}$$
$$\delta(b, a) \in C(\delta(a, b)), \tag{3.6}$$
$$C(\delta(\delta(a, b), c)) = C(\delta(a, \delta(b, c))), \tag{3.7}$$

for all $a, b, c \in A$.

Lemma 3.12. *Let C be a closure operator over A and $\mathcal{B}$ a closure basis of $\mathrm{img}\, C$. Suppose each element of $\mathcal{B}$ is δ-disjunctive. Then,*

$$(C(Z \cup X) \cap C(Z \cup Y)) = C(Z \cup \delta(X, Y)), \tag{3.8}$$

for all $X, Y, Z \subseteq A$. In particular, C is δ-disjunctive and the following holds:

$$\delta(C(X), a) \subseteq C(\delta(X, a)), \tag{3.9}$$

for all $(X \cup \{a\}) \subseteq A$.

Proof. First, for all $a \in A$, we have:

$$\begin{aligned}
&(a \in C(Z \cup X) \cap C(Z \cup Y)) \\
\Leftrightarrow \forall W \in \mathcal{B} : ((((Z \subseteq W)\&(X \subseteq W)) \Rightarrow (a \in W)) \\
\&(((Z \subseteq W)\&(Y \subseteq W)) \Rightarrow (a \in W))) \\
\Leftrightarrow \forall W \in \mathcal{B} : (((Z \subseteq W)\&(X \subseteq W|Y \subseteq W)) \Rightarrow (a \in W)) \\
\Leftrightarrow \forall W \in \mathcal{B} : (((Z \subseteq W)\&(\delta(X, Y) \subseteq W)) \Rightarrow (a \in W)) \\
\Leftrightarrow (a \in C(Z \cup \delta(X, Y))),
\end{aligned}$$

in which case (3.8) holds, and so immediately does its particular case (3.2). Finally, applying (3.8) with $Z = \varnothing$ twice, we also get $\delta(C(X), a) \subseteq C(\delta(C(X), a)) = (C(C(X)) \cap C(a)) = (C(X) \cap C(a)) = C(\delta(X, a))$, in which case (3.9) holds, as required. □

Lemma 3.13. *Let C be a δ-disjunctive closure operator over A and $X \in (\mathrm{img}\, C)$. Then, X is δ-disjunctive iff it is pair-wise-meet-irreducible in $\mathrm{img}\, C$, and so it is finitely-meet-irreducible in $\mathrm{img}\, C$ iff it is δ-disjunctive and proper.*

Proof. First, assume X is not δ-disjunctive. Then, in view of (3.3) and (3.4), X is weakly δ-disjunctive, so there is some $\vec{a} \in (A \setminus X)^2$, in which case, for each $i \in 2$, it holds that $X \neq C(X \cup \{a_i\}) \in (\mathrm{img}\, C)$, such that $\delta(\vec{a}) \in X$. Therefore, by (3.2), we have $X = (\bigcap_{i \in 2} C(X \cup \{a_i\}))$. Hence, X is not pair-wise-meet-irreducible in $\mathrm{img}\, C$.

Conversely, assume X is not pair-wise-meet-irreducible in $\mathrm{img}\, C$. Then, there is some $\vec{Y} \in ((\mathrm{img}\, C) \setminus \{X\})^2$ such that $X = (\bigcap_{i \in 2} Y_i)$, in which case, for each $i \in 2$, $X \subsetneq Y_i$, so there is some $a_i \in (Y_i \setminus X) \neq \varnothing$. In this way, by (3.2), we have $\delta(\vec{a}) \in C(X \cup \delta(\vec{a})) = (\bigcap_{i \in 2} C(X \cup \{a_i\})) \subseteq (\bigcap_{i \in 2} Y_i) = X$. Thus, X is not δ-disjunctive, as required. □

3.3.1. *Disjunctive logics and matrices.* Fix any (possibly, secondary) binary connective $\veebar$ of Σ. Clearly, a Σ-matrix $\mathcal{A}$ is [weakly] $\veebar$-disjunctive iff $D^{\mathcal{A}}$ is [weakly] $\veebar^{\mathfrak{A}}$-disjunctive.

Remark 3.14. Given any more (possibly, secondary) binary connective $\diamond$ of Σ and any $\veebar$-disjunctive Σ-logic C, in view of (3.2) and the structurality of C, C is $\diamond$-disjunctive iff $(x_0 \diamond x_1) \equiv_C (x_0 \veebar x_1)$. In particular, any extension/model of C is $\veebar$-disjunctive iff it is $\diamond$-disjunctive. □

Remark 3.15. In view of (2.5) and (2.6), given two Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ such that there is a [surjective] strict homomorphism from $\mathcal{A}$ [on]to $\mathcal{B}$, $\mathcal{A}$ is (weakly) $\veebar$-disjunctive if[f] $\mathcal{B}$ is so. □

Corollary 3.16. *Let I be a finite set, $\overline{\mathcal{A}}$ an I-tuple of $\veebar$-disjunctive Σ-matrices and $\mathcal{B}$ a consistent $\veebar$-disjunctive subdirect product of $\overline{\mathcal{A}}$. Then, $(\pi_i \restriction B) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{A}_i)$, for some $i \in I$.*

Proof. Then, by Remark 3.15, $\mathcal{B} \triangleq \{B \cap \pi_i^{-1}[D^{\mathcal{A}_i}] \mid i \in I\}$ is a finite set of $\veebar^{\mathfrak{B}}$-disjunctive subsets of B. Let C be the closure operator over B dual to the closure system with basis $\mathcal{B}$. Then, $D^{\mathcal{B}} = (B \cap \bigcap \mathcal{B}) \in (\mathrm{img}\, C)$ is both $\veebar^{\mathfrak{B}}$-disjunctive and proper. Hence, by Lemmas 3.12 and 3.13, $D^{\mathcal{B}} \in \mathcal{B}$, as required. □

Corollary 3.17. *Let $\alpha \in \wp_{\infty \setminus 1}(\omega)$ and M a class of [non-]weakly $\veebar$-disjunctive Σ-matrices. Then, $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ is [non-]weakly $\veebar$-disjunctive [and satisfies (3.9)].*

Proof. The "weak" case is evident. [Conversely, for each $\mathcal{A} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$, $h^{-1}[D^{\mathcal{A}}]$ is $\veebar$-disjunctive, by Remark 3.15. Then, Lemma 3.12 completes the argument.] □

Corollary 3.18. *Let $\mathcal{A}$ be a false-singular Σ-matrix and C the logic of $\mathcal{A}$. Then, the following are equivalent:*

(i) *C is [non-]weakly $\veebar$-disjunctive;*
(ii) *$\mathcal{A}$ is [non-]weakly $\veebar$-disjunctive;*
(iii) *C satisfies both (3.3) and (3.4) [as well as (3.5)].*

Proof. First, (ii)⇒(i) is by Corollary 3.17. Next, (iii) is a particular case of (i). Finally, assume (iii) holds. Consider any $a, b \in A$. In case $(a/b) \in D^{\mathcal{A}}$, by (3.3)/(3.4), we have $(a \veebar^{\mathfrak{A}} b) \in D^{\mathcal{A}}$. [Now, assume $(\{a, b\} \cap D^{\mathcal{A}}) = \varnothing$. Then, $D^{\mathcal{A}} \not\ni a = b$. Therefore, by (3.5), we get $D^{\mathcal{A}} \not\ni (a \veebar^{\mathfrak{A}} a) = (a \veebar^{\mathfrak{A}} b)$.] Thus, (ii) holds, as required. □

Corollary 3.19. *Let C be an inductive Σ-logic. Then, the following are equivalent:*

(i) *C is $\veebar$-disjunctive;*
(ii) *$\mathrm{img}\, C$ has a basis consisting of $\veebar$-disjunctive sets;*
(iii) *(3.3), (3.5), (3.6) and (3.9) hold;*
(iv) *(3.3), (3.5), (3.6) hold and, for any axiomatization $\mathcal{C}$ of C, every $(\Gamma \vdash \phi) \in \mathrm{SI}_{\Sigma}(\mathcal{C})$ and each $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\phi \veebar \psi) \in C(\Gamma \veebar \psi)$.*

Proof. First, (i)⇒(ii) is by Remark 2.1 and Lemma 3.13. Next, (ii)⇒(iii) is by Lemma 3.12. Further, (iv) is a particular case of (iii). Then, the converse is proved by induction on the length of $\mathcal{C}$-derivations. Finally, assume (iii) holds, in which case (3.4) holds by (3.3) and (3.6), and so does the inclusion from left to right in (3.2), by (3.3) and (3.4). Conversely, consider any $\varphi \in (C(Z \cup \{\phi\}) \cap C(Z \cup \{\psi\}))$. Then, by (3.3), (3.6) and (3.9), we have $(\psi \veebar \varphi) \in C(Z \cup \{\phi \veebar \psi\})$. Likewise, by (3.3), (3.5) and (3.9), we also have $\varphi \in C(Z \cup \{\psi \veebar \varphi\})$. Hence, we eventually get $\varphi \in C(Z \cup \{\phi \veebar \psi\})$, in which case (3.2) holds, and so does (i), as required. □

Finally, by (2.2), we immediately have:

Proposition 3.20. *Any axiomatic extension of a $\veebar$-disjunctive Σ-logic is $\veebar$-disjunctive itself.*

3.3.1.1. Disjunctive extensions of logics defined by finite classes of finite disjunctive matrices. Given a Σ-rule $\Gamma \vdash \phi$ and a Σ-formula ψ, put $((\Gamma \vdash \phi) \veebar \psi) \triangleq ((\Gamma \veebar \psi) \vdash (\phi \veebar \psi))$. (This notation is naturally extended to Σ-calculi member-wise.)

Lemma 3.21. *Let $\Gamma \vdash \phi$ be a Σ-rule and $\mathcal{A}$ a $\veebar$-disjunctive Σ-matrix. Then, $\mathcal{A} \in \mathrm{Mod}(\sigma_{+1}(\Gamma \vdash \phi) \veebar x_0)$ iff $\mathcal{A} \in \mathrm{Mod}(\Gamma \vdash \phi)$.*

Proof. The "if" part is by the strucuturality of $\mathrm{Cn}^{\omega}_{\mathcal{A}}$ and Corollary 3.17(3.9). Conversely, assume $\mathcal{A} \in \mathrm{Mod}(\sigma_{+1}(\Gamma \vdash \phi) \veebar x_0)$. Consider any $h \in \hom(\mathfrak{Fm}^{\omega}, \mathfrak{A})$ such that $h(\phi) \notin D^{\mathcal{A}}$. Let $g \in \hom(\mathfrak{Fm}^{\omega}, \mathfrak{A})$ extend $[x_0/h(\phi); x_{i+1}/h(x_i)]_{i \in \omega}$, in which case $(g \circ \sigma_{+1}) = h$, and so, by the $\veebar$-disjunctivity of $\mathcal{A}$, we have $g(\sigma_{+1}(\phi) \veebar x_0) = (h(\phi) \veebar^{\mathfrak{A}} h(\phi)) \notin D^{\mathcal{A}}$. Hence, there is some $\psi \in \Gamma$ such that $(h(\psi) \veebar^{\mathfrak{A}} h(\phi)) = g(\sigma_{+1}(\psi) \veebar x_0) \notin D^{\mathcal{A}}$, in which case, by the $\veebar$-disjunctivity of $\mathcal{A}$, we eventually get $h(\psi) \notin D^{\mathcal{A}}$, and so $\mathcal{A} \in \mathrm{Mod}(\Gamma \vdash \phi)$, as required. □

Lemma 3.22. *Let C be an inductive $\veebar$-disjunctive logic, $\mathcal{C}$ a Σ-calculus and $\mathcal{A} \subseteq \mathcal{C}$ an axiomatic Σ-calculus. Then, the extension C' of C relatively axiomatized by $\mathcal{C}' \triangleq (\mathcal{A} \cup (\sigma_{+1}[\mathcal{C} \setminus \mathcal{A}] \veebar x_0))$ is $\veebar$-disjunctive.*

Proof. Then, C being inductive, is axiomatized by a Σ-calculus $\mathcal{C}''$, in which case C' is axiomatized by the Σ-calculus $\mathcal{C}''\cup\mathcal{C}'$, and so is inductive. Moreover, C', being an extension of C, inherits (3.3), (3.5), (3.6) and (3.7) held for C. Then, we prove the $\veebar$-disjunctivity of C' with applying Corollary 3.19(i)$\Leftrightarrow$(iv) to both C and C'. For consider any Σ-substitution σ and any $\psi\in\mathrm{Fm}^{\omega}_{\Sigma}$. First, consider any $\phi\in\mathcal{A}$. Then, by the structurality of C' and (3.3), we have $(\sigma(\phi)\veebar\psi)\in C'(\varnothing)$. Now, consider any $(\Gamma\vdash\phi)\in(\mathcal{C}\backslash\mathcal{A})$. Let ς be the Σ-substitution extending $(\sigma{\restriction}(V_\omega\backslash V_1))\cup[x_0/(\sigma(x_0)\veebar\psi)]$, in which case $(\varsigma\circ\sigma_{+1})=(\sigma\circ\sigma_{+1})$, and so, by (3.7) and the structurality of C', we eventually get $(\sigma[\sigma_{+1}[\Gamma]\veebar x_0]\veebar\psi)=((\varsigma[\sigma_{+1}[\Gamma]]\veebar\sigma(x_0))\veebar\psi)\vdash_{C'}(\varsigma[\sigma_{+1}[\Gamma]]\veebar(\sigma(x_0)\veebar\psi))=\varsigma[\sigma_{+1}[\Gamma]\veebar x_0]\vdash_{C'}\varsigma(\sigma_{+1}(\varphi)\veebar x_0)=(\varsigma(\sigma_{+1}(\varphi))\veebar(\sigma(x_0)\veebar\psi))\vdash_{C'}((\varsigma(\sigma_{+1}(\varphi))\veebar\sigma(x_0))\veebar\psi)=(\sigma(\sigma_{+1}(\varphi)\veebar x_0)\veebar\psi)$. □

Lemma 3.23. *Let K be a finite class of consistent $\veebar$-disjunctive Σ-matrices. Then, the set of all relative [positive] universal Horn model subclasses of K is a closure system over K closed under unions, and so forms a finite distributive lattice.*

Proof. Consider any set I of [positive] universal Horn model subclasses of K, in which case it is finite, for K is so, and so there are some bijection $e:n\to I$, where $n\triangleq|I|\in\omega$, some $\overline{\mathcal{C}}:n\to\wp(\wp_{\omega[\cap 1]}(\mathrm{Fm}^{\omega}_{\Sigma})\times\mathrm{Fm}^{\omega}_{\Sigma})$ and some $\bar\alpha:n\to\wp_{\omega\backslash 1}(\omega\setminus 1)$ such that, for every $i\in n$, $e(i)=(\mathsf{K}\cap\mathrm{Mod}(\mathcal{C}_i))$, $\mathcal{C}_i\subseteq(\wp_\omega(\mathrm{Fm}^{\alpha_i}_{\Sigma})\times\mathrm{Fm}^{\alpha_i}_{\Sigma})$ and $(\alpha_i\cap\alpha_j)=\varnothing$, for all $j\in(n\setminus\{i\})$. Then, we clearly have $(\mathsf{K}\cap\mathrm{Mod}(\bigcup_{i\in n}\mathcal{C}_i))=(\mathsf{K}\cap(\bigcap I))$. Moreover, every member of $(\bigcup I)\subseteq\mathsf{K}$ is a model of $\mathcal{C}\triangleq\{(\bigcup\mathrm{img}(\pi_0\circ\bar R))\vdash\veebar\langle\pi_1\circ\bar R,x_0\rangle\mid\bar R\in\prod\overline{\mathcal{C}}\}\in\wp(\wp_{\omega[\cap 1]}(\mathrm{Fm}^{\omega}_{\Sigma})\times\mathrm{Fm}^{\omega}_{\Sigma})$. Conversely, consider any $\mathcal{A}\in(\mathsf{K}\setminus(\bigcup I))$. Then, for every $i\in n$, $\mathcal{A}\notin e(i)$, in which case there are some $R_i\in\mathcal{C}_i$ and some $h_i:\alpha_i\to A$ such that $\mathcal{A}\not\models R_i[h_i]$, and so $((\bigcup_{i\in n}\pi_0[R_i])\vdash\veebar\langle\langle\pi_1(R_i)\rangle_{i\in n},x_0\rangle)\in\mathcal{C}$ is not true in $\mathcal{A}$ under $[x_0/a]\cup\bigcup_{i\in n}h_i$, where $a\in(A\setminus D^{\mathcal{A}})\neq\varnothing$, for $\mathcal{A}$ is consistent. Thus, $(\bigcup I)=(\mathsf{K}\cap\mathrm{Mod}(\mathcal{C}))$, as required. □

Theorem 3.24. *Let M be a finite class of finite $\veebar$-disjunctive matrices, C the logic of M and $\mathsf{K}^{[*]}\triangleq\mathbf{S}^{[*]}_{*}(\mathsf{M})$. Then:*

(i) *the mappings $C'\mapsto(\mathrm{Mod}(C')\cap\mathsf{K}^{[*]})$ and $\mathsf{S}\mapsto\mathrm{Cn}^{\omega}_{\mathsf{S}}$ are inverse to one another dual isomorphisms between the poset of all $\veebar$-disjunctive [non-pseudo-axiomatic] extensions of C and that of all relative universal Horn model subclasses of $\mathsf{K}^{[*]}$, the latter poset forming a finite distributive lattice, and so doing the former one;*

(ii) *for any Σ-calculus $\mathcal{C}$, the following hold:*
   - **a)** *the extension of C relatively axiomatized by $\mathcal{C}$, being $\veebar$-disjunctive [and non-pseudo-axiomatic], corresponds to the relative universal Horn model subclass of $\mathsf{K}^{[*]}$ relatively axiomatized by $\mathcal{C}$;*
   - **b)** *[providing $(\mathcal{C}\cap\mathrm{Fm}^{\omega}_{\Sigma})\neq\varnothing$] the relative universal Horn model subclass of $\mathsf{K}^{[*]}$ relatively axiomatized by $\mathcal{C}$ corresponds to the $\veebar$-disjunctive [non-pseudo-axiomatic] extension of C relatively axiomatized by $(\mathcal{C}\cap\mathrm{Fm}^{\omega}_{\Sigma})\cup(\sigma_{+1}[\mathcal{C}\backslash\mathrm{Fm}^{\omega}_{\Sigma}]\veebar x_0)$;*

(iii) *[providing every member of M is truth-non-empty] relative positive universal Horn model subclasses of $\mathsf{K}^{[*]}$ correspond exactly to [non-pseudo-axiomatic] axiomatic extensions of C, corresponding objects having same axiomatic relative axiomatizations and forming dual isomorphic finite distributive lattices;*

(iv) *for any $\mathsf{C}\subseteq\mathsf{K}^{[*]}$, $\mathbf{S}^{[*]}_{*}(\mathsf{C})$, being a relative universal Horn model subclass of $\mathsf{K}^{[*]}$, corresponds to the logic of C.*

In particular, all $\veebar$-disjunctive extensions of C are inductive and relatively finitely-axiomatizable.

Proof. (i) First, the fact that $(\mathrm{Mod}(\mathrm{Cn}^{\omega}_{\mathsf{S}})\cap\mathsf{K}^{[*]})=\mathsf{S}$, where S is a relative universal Horn model subclass of $\mathsf{K}^{[*]}$, is immediate, while the fact that $\mathrm{Cn}^{\omega}_{\mathsf{S}}$ is a $\veebar$-disjunctive [and non-pseudo-axiomatic] extension of C is by (2.6), Corollary 3.17 and Remark 3.15 [as well as Proposition 2.10]. Now, consider any $\veebar$-disjunctive [non-pseudo-axiomatic] extension C' of C. Then, we have the inductive $\veebar$-disjunctive [non-pseudo-axiomatic] extension C'' of C (for C is inductive [and non-pseudo-axiomatic]) defined as follows: for every $Z\subseteq\mathrm{Fm}^{\omega}_{\Sigma}$, put $C''(Z)\triangleq(\bigcup C'[\wp_\omega(Z)])$. Consider any Σ-rule $\Gamma\vdash\varphi$ such that $\varphi\notin C''(\Gamma)$ [and $\Gamma\neq\varnothing$]. Then, by Corollary 3.19(i)$\Rightarrow$(ii), there is some $\veebar$-disjunctive $X\in(\mathrm{img}\,C'')\subseteq(\mathrm{img}\,C)$ such that $\Gamma\subseteq X\not\ni\varphi$. Moreover, as Γ is finite, there is some $\alpha\in(\omega\setminus 1)\subseteq\wp_{\infty\backslash 1}(\omega)$ such that $(\Gamma\cup\{\varphi\})\subseteq\mathrm{Fm}^{\alpha}_{\Sigma}$, in which case, in view of (2.4), $\Gamma\subseteq Y\triangleq(X\cap\mathrm{Fm}^{\alpha}_{\Sigma})\in(\mathrm{img}\,\mathrm{Cn}^{\alpha}_{\mathsf{M}})$ is [both] $\veebar$-disjunctive [and non-empty] as well as proper, for $\varphi\in(\mathrm{Fm}^{\alpha}_{\Sigma}\backslash Y)$. Furthermore, by the structurality of C'', $\langle\mathfrak{Fm}^{\omega}_{\Sigma},X\rangle$ is a model of C'', and so is its consistent [truth-non-empty] submatrix $\mathcal{D}\triangleq\langle\mathfrak{Fm}^{\alpha}_{\Sigma},Y\rangle$, in view of (2.6). On the other hand, by Corollary 3.17, $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ is $\veebar$-disjunctive. Hence, by Lemma 3.13, Y is finitely-meet-irreducible in $\mathrm{img}\,\mathrm{Cn}^{\alpha}_{\mathsf{M}}$. And what is more, since both α, M and all members of M are finite, $\mathcal{B}\triangleq\{h^{-1}[D^{\mathcal{A}}]\mid\mathcal{A}\in\mathsf{M},h\in\hom(\mathfrak{Fm}^{\alpha}_{\Sigma},\mathfrak{A})\}$ is a finite basis of $\mathrm{img}\,\mathrm{Cn}^{\alpha}_{\mathsf{M}}$. Therefore, $Y\in\mathcal{B}$, in which case there are some $\mathcal{A}\in\mathsf{M}$ and some $h\in\hom(\mathfrak{Fm}^{\alpha}_{\Sigma},\mathfrak{A})$ such that $Y=h^{-1}[D^{\mathcal{A}}]$, and so h is a surjective strict homomorphism from $\mathcal{D}$ onto $\mathcal{B}\triangleq(\mathcal{A}{\restriction}(\mathrm{img}\,h))$. In this way, by (2.6), $\mathcal{B}$ is a consistent [truth-non-empty] model of C''. Finally, as $\Gamma\subseteq Y=h^{-1}[D^{\mathcal{B}}]\not\ni\varphi$, we conclude that $\Gamma\vdash\varphi$ is not true in $\mathcal{B}\in\mathsf{S}\triangleq(\mathrm{Mod}(C'')\cap\mathsf{K}^{[*]})$ under h. Thus, since both S and all members of it are finite, in which case $C'''\triangleq\mathrm{Cn}^{\omega}_{\mathsf{S}}$ is inductive [and non-pseudo-axiomatic, by Proposition 2.10], and so $C''=C'''$, by Proposition 2.18, we eventually get $C'=C'''=C''$, as required, for, in that case, C', being inductive, is axiomatized by a Σ-calculus. In this way, Lemma 3.23 completes the argument.

(ii) Consider any Σ-calculus $\mathcal{C}$. Then:
   - **a)** is immediate, in view of (2.6), due to which $\mathsf{K}\subseteq\mathrm{Mod}(C)$.
   - **b)** Let C' be the extension of C relatively axiomatized by $(\mathcal{C}\cap\mathrm{Fm}^{\omega}_{\Sigma})\cup(\sigma_{+1}[\mathcal{C}\setminus\mathrm{Fm}^{\omega}_{\Sigma}]\veebar x_0)$. Then, by Lemma 3.22 with $\mathcal{A}=(\mathcal{C}\cap\mathrm{Fm}^{\omega}_{\Sigma})$, C' is $\veebar$-disjunctive. [And what is more, since $\mathcal{A}\neq\varnothing$, C' is not theorem-less, and so is non-pseudo-axiomatic.] Then, **a)** and Lemma 3.21 complete the argument.

(iii) is by (i), (ii), Proposition 3.20, Lemma 3.23 and Remark 3.15 [as well as Proposition 2.10, due to which C, being the axiomatic extension of C relatively axiomatized by the axiomatic Σ-calculus $\varnothing$, is non-pseudo-axiomatic].

(iv) is by (2.6). □

As it is demonstrated by Theorem 5.53 below, $(\mathcal{C}\cap\mathrm{Fm}^{\omega}_{\Sigma})\cup(\sigma_{+1}[\mathcal{C}\setminus\mathrm{Fm}^{\omega}_{\Sigma}]\veebar x_0)$ cannot be replaced by $\mathcal{C}$ in the item (ii)**b)** of Theorem 3.24.

3.3.1.2. Axiomatic extensions of logics defined by finite classes of finite implicative matrices. Let $\rhd$ be any (possibly, secondary) binary connective of Σ. For any $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, by induction on the length $l = (\mathrm{dom}\,\bar{\phi}) \in \omega$ of any $\bar{\phi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^{*}$, put:

$$(\bar{\phi} \rhd \psi) \triangleq \begin{cases} \psi & \text{if } l = 0, \\ \phi_0 \rhd (((\bar{\phi}\restriction(l \setminus 1)) \circ ((+1)\restriction(l-1))) \rhd \psi) & \text{otherwise.} \end{cases}$$

A Σ-matrix $\mathcal{A}$ is said to be $\rhd$-*implicative*, provided, for all $a, b \in A$, it holds that $((a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \rhd^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, in which case it is $\vee_{\rhd}$-disjunctive, where $(x_0 \vee_{\rhd} x_1) \triangleq ((x_0 \rhd x_1) \rhd x_1)$, while every submatrix of $\mathcal{A}$ is $\rhd$-implicative,

Remark 3.25. Let M be a finite class of finite $\rhd$-implicative as well as $\veebar$-disjunctive (in particular, $\veebar = \veebar_{\rhd}$) Σ-matrices, in which case the axiom $x_0 \rhd x_0$ is true in it, and so every member of $\mathsf{K}_{[*]} \triangleq \mathbf{S}_{[*]}(\mathsf{M})$, satisfying the axiom involved, in view of (2.6), is truth-non-empty. Then, any Σ-rule $\Gamma \vdash \psi$ is true in any member of K iff $\bar{\phi} \rhd \psi$ is so, where $\bar{\phi} : |\Gamma| \to \Gamma$ is any bijection, in which case any universal Horn model subclass of K_{*} is positive, and so $\veebar$-disjunctive extensions of the logic of M are exactly axiomatic ones, in view of Theorem 3.24(i,iii). ☐

A Σ-logic C is said to *have Deduction-Detachment Theorem (DDT) with respect to* $\rhd$, provided $(\phi \in C(\Gamma \cup \{\psi\})) \Leftrightarrow ((\psi \rhd \phi) \in C(\Gamma))$, for all $(\Gamma \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$.

Proposition 3.26. *Let $\mathcal{A}$ be a false-singular Σ-matrix and C the logic of $\mathcal{A}$. Then, the following are equivalent:*

(i) *C has DDT with respect to $\veebar$;*
(ii) *C satisfies the following rule and axioms:*

$$\{x_0, x_0 \rhd x_1\} \vdash x_1, \tag{3.10}$$

$$x_0 \rhd (x_1 \rhd x_0), \tag{3.11}$$

$$x_0 \rhd x_0; \tag{3.12}$$

(iii) (3.10)*,* (3.11) *and* (3.12) *are true in $\mathcal{A}$;*
(iv) *$\mathcal{A}$ is $\rhd$-implicative.*

Proof. First, (iv)⇒(i)⇒(ii)⇒(iii) are immediate. Finally, assume (iii) holds. Consider any $a, b \in A$. Then, the fact that $((a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftarrow ((a \rhd^{\mathfrak{A}} b) \in D^{\mathcal{A}})$ is by (3.10). Conversely, assume $(a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})$. Then, in case $b \in D^{\mathcal{A}}$, by (3.10) and (3.11), we get $(a \rhd^{\mathfrak{A}} b) \in D^{\mathcal{A}}$. Otherwise, we have $a \notin D^{\mathcal{A}}$, in which case $a = b$, by the false-singularity of $\mathcal{A}$, and so, by (3.12), we eventually get $(a \rhd^{\mathfrak{A}} b) = (b \rhd^{\mathfrak{A}} b) \in D^{\mathcal{A}}$, as required. ☐

3.3.1.3. Disjunctive extensions of the logics of single finite disjunctive matrices with unary initary equality determinant.

Lemma 3.27. *Let $\mathcal{A}$ be a finite $\veebar$-disjunctive Σ-matrix with unary unitary equality determinant Υ, $\mathsf{S} \subseteq \mathbf{S}(\mathcal{A})$ and $\mathcal{B} \in \mathbf{S}_{*}(\mathcal{A})$. Suppose $\mathcal{B} \notin \mathbf{S}(\mathsf{S})$. Then, there is some Σ-rule satisfied in S but is not satisfied in $\mathcal{B}$.*

Proof. In case $\mathsf{S} = \varnothing$, the axiom $\vdash x_0$ is satisfied in it but is not satisfied in any consistent Σ-matrix (in particular, in $\mathcal{B}$). Now, assume $\mathsf{S} \neq \varnothing$, in which case $n \triangleq |\mathsf{S}| \in (\omega \setminus 1)$, and so there is a bijection $\overline{\mathcal{C}} : n \to \mathsf{S}$. Consider any $i \in n$, in which case $B \nsubseteq C_i$, and so there is some $a_i \in (B \setminus C_i) \neq \varnothing$. Define a $\Delta_i \in \wp_{\omega}(\mathrm{Fm}^{\omega}_{\Sigma})$ and a $\bar{\psi}^i \in (\mathrm{Fm}^{\omega}_{\Sigma})^{*}$ as follows. Let $m \triangleq |C_i| \in (\omega \setminus 1)$. Take any bijection $\vec{c} : m \to C_i$. By induction on any $j \in (m+1)$, define a $\Gamma_j \in \wp_{\omega}(\mathrm{Fm}^{1}_{\Sigma})$ and a $\bar{\phi}^j \in (\mathrm{Fm}^{1}_{\Sigma})^{*}$ such that, for all $b \in (A \setminus D^{\mathcal{A}})$, it holds that $\mathcal{A} \not\models (\Gamma_j \vdash (\veebar\langle \bar{\phi}^j, x_n\rangle)[x_0/a_i, x_n/b]$, while, for all $k \in j$ and all $a \in A$, it holds that $\mathcal{A} \models (\Gamma_j \vdash (\veebar\langle \bar{\phi}^j, x_n\rangle)[x_0/c_k, x_n/a]$, as follows. First, put $\Gamma_j \triangleq \varnothing$ and $\bar{\phi}^j \triangleq \varnothing$, in case $j = 0$. Next, assume $j > 0$, in which case $(j-1) \in m$, and so $c_{j-1} \neq a_i$. Therefore, there is some $\upsilon \in \Upsilon$ such that $\upsilon^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(c_{j-1}) \notin D^{\mathcal{A}}$. Then, set:

$$\langle \Gamma_j, \bar{\phi}^j \rangle \triangleq \begin{cases} \langle \Gamma_{j-1}, \langle \bar{\phi}^{j-1}, \upsilon\rangle\rangle & \text{if } \upsilon^{\mathfrak{A}}(a_i) \notin D^{\mathcal{A}}, \upsilon \notin (\mathrm{img}\,\bar{\phi}^{j-1}), \\ \langle \Gamma_{j-1}, \bar{\phi}^{j-1}\rangle & \text{if } \upsilon^{\mathfrak{A}}(a_i) \notin D^{\mathcal{A}}, \upsilon \in (\mathrm{img}\,\bar{\phi}^{j-1}), \\ \langle \Gamma_{j-1} \cup \{\upsilon\}, \bar{\phi}^{j-1}\rangle & \text{otherwise.} \end{cases}$$

Finally, put $\Delta_i \triangleq (\Gamma_m[x_0/x_i])$ and $\bar{\psi}^i \triangleq (\bar{\phi}^m[x_0/x_i])$. Let $\Xi \triangleq (\bigcup_{i \in n} \Delta_i)$, $\bar{\xi} \triangleq (*\langle \bar{\psi}^i \rangle_{i \in n})$ and

$$\varphi \triangleq \begin{cases} x_n & \text{if } \bar{\xi} = \varnothing, \\ \veebar\bar{\xi} & \text{otherwise.} \end{cases}$$

In this way, the Σ-rule $\Xi \vdash \varphi$ is true in S but is not true in $\mathcal{B}$ under $[x_i/a_i; x_n/b]_{i \in n}$, where $b \in (B \setminus D^{\mathcal{A}}) \neq \varnothing$, for $\mathcal{B}$ is consistent, as required. ☐

As an immediate consequence of (2.6) and Lemma 3.27, we get:

Theorem 3.28. *Let M, C and $\mathsf{K}^{[*]}$ be as in Theorem 3.24. Suppose $\mathsf{M} = \{\mathcal{A}\}$, where $\mathcal{A}$ is a Σ-matrix with equality determinant. Then, relative universal Horn model subclasses of $\mathsf{K}^{[*]}$ are exactly lower cones of it, under identification of its members with the carriers of their underlying algebras.*

In this way, Lemma 3.27 collectively with Theorems 3.24 and 3.28 provide an effective procedure of finding the lattice of disjunctive extensions of the logic of a finite disjunctive matrix with equality determinant collectively with their finite relative axiomatizations and finite anti-chain matrix semantics. Concluding this discussion, we should like to highlight that the effective procedure of finding relative axiomatizations of disjunctive extensions to be extracted from the constructive proof of Lemma 3.27 (properly adapting the ideas underlying that of Lemma 5.7 of [27]) is definitely and obviously much less computationally complex than the straightforward one of direct search among all finite sets of rules.

3.3.1.3.1. Implicative matrices with unary unitary equality determinant. By (2.2), Theorem 3.24, Remark 3.25 and Lemma 3.27, we immediately get:

Corollary 3.29. *Let $\mathcal{A}$ be a finite $\diamond$-implicative Σ-matrix with unary unitary equality determinant and $\mathsf{S} \triangleq \mathbf{S}_*(\mathcal{A})$. Then, the mappings:*

$$\begin{array}{rcl}\mathcal{E} & \mapsto & (\mathrm{Mod}(\mathcal{E}) \cap \mathsf{S}) = (\mathrm{Mod}(\mathcal{E} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cap \mathsf{S}),\\ \mathsf{C} & \mapsto & \mathrm{Cn}^{\omega}_{\mathsf{C}}\end{array}$$

are inverse to one another dual isomorphisms between the lattices of all axiomatic extensions of $\mathrm{Cn}^{\omega}_{\mathcal{A}}$ and of all lower cones of S (under identification of submatrices of $\mathcal{A}$ with the carriers of their underlying algebras), corresponding axiomatic extensions of $\mathrm{Cn}^{\omega}_{\mathcal{A}}$ and lower cones of S having same relative axiomatizations, both lattices being finite and distributive.

This elaboration, being exemplified by applying it to implicative expansions of Belnap's logic (cf. Subsubsection 7.1.3 collectively with Remark 3.25 and Theorem 7.10), is equally applicable to another interesting examples including Łukasiewicz' finitely-valued logics (cf. [14]), for their defining matrices both are implicative (cf. Example 7 of [29]) and have unary unitary equality determinant, in view of Example 3 of [28] (cf. Proposition 6.10 of [30] for a constructive proof of this result), being however beyond the scopes of the present work.

3.4. **Distributive and De Morgan lattices.** Let $\Sigma^{+}_{[01]} \triangleq (\{\wedge,\vee\}[\cup\{\bot,\top\}])$ be the [bounded] lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)].

Lemma 3.30. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be lattices, a a unit/zero of $\mathfrak{A}$, b a unit/zero of $\mathfrak{B}$ and $h \in \hom(\mathfrak{A},\mathfrak{B})$. Suppose $h[A] = B$. Then, $h(a) = b$.*

Proof. Then, there is some $c \in A$ such that $h(c) = b$, in which case $(a(\vee/\wedge)^{\mathfrak{A}}c) = a$, and so $h(a) = (h(a)(\vee/\wedge)^{\mathfrak{B}}b) = b$. □

Given any $\Sigma \supseteq \Sigma^{+}$, $\phi \lessapprox \psi$ is used as an abbreviation for $(\phi \wedge \psi) \approx \phi$, where $\phi,\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$. Then, any Σ-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\Sigma^{+}$ is a lattice is well-known to be congruence-distributive (cf., e.g., Example 2 on p. 12 of [17]), the partial ordering of $\mathfrak{A}{\restriction}\Sigma^{+}$ being denoted by $\leqslant^{\mathfrak{A}}$.

Given any $n \in (\omega \setminus 1)$, by $\mathfrak{D}_{n[01]}$ we denote the [bounded] distributive lattice given by the chain n, viz., the $\Sigma^{+}_{[01]}$-algebra with carrier n such that $(\wedge/\vee)^{\mathfrak{D}_n} \triangleq ((\min/\max){\restriction}n^2)$ [and $(\bot/\top)^{\mathfrak{D}_n} \triangleq (0/(n-1))$].

Here, we deal with the signature $\Sigma_{0[1]} \triangleq (\Sigma^{+}_{[01]} \cup \{\sim\})$ with unary $\sim$ (weak negation).

A *[bounded] De Morgan lattice* (cf. [3], [23], [24]) is any $\Sigma_{0[1]}$-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\Sigma^{+}_{[01]}$ is a [bounded] distributive lattice (cf. [3]) and the following Σ_0-identities are true in $\mathfrak{A}$:

$$\sim\sim x_0 \approx x_0, \tag{3.13}$$

$$\sim(x_0 \vee x_1) \approx \sim x_0 \wedge \sim x_1, \tag{3.14}$$

$$\sim(x_0 \wedge x_1) \approx \sim x_0 \vee \sim x_1, \tag{3.15}$$

the variety of all them being denoted by $[\mathsf{B}]\mathsf{DML}$. Then, a *[bounded] Kleene lattice* is any [bounded] De Morgan lattice satisfying the Σ_0-identity:

$$(x_0 \wedge \sim x_0) \lessapprox (x_1 \vee \sim x_1), \tag{3.16}$$

the variety of all them being denoted by $[\mathsf{B}]\mathsf{KL}$. Next, a *[bounded] Boolean lattice* is any [bounded] De Morgan lattice satisfying the Σ_0-identity:

$$x_0 \lessapprox (x_1 \vee \sim x_1), \tag{3.17}$$

the variety of all them being denoted by $[\mathsf{B}]\mathsf{BL} \subseteq [\mathsf{B}]\mathsf{KL}$.[3]

By $\mathfrak{DM}_{4[01]}$ we denote the [bounded] De Morgan lattice such that $(\mathfrak{DM}_{4[01]}{\restriction}\Sigma^{+}_{[01]}) \triangleq \mathfrak{D}^2_{2[01]}$ and $\sim^{\mathfrak{DM}_{4[01]}}\vec{a} \triangleq \langle 1 - a_{1-i}\rangle_{i\in 2}$, for all $\vec{a} \in 2^2$.

Next, a Σ-algebra $\mathfrak{A}$ is said to be *b-idempotent*, where $b \in B$, provided its primary (and so secondary) operations are so, that is, $\{b\}$ forms a subalgebra of it. Likewise, in case $A \subseteq 2^2$, $\mathfrak{A}$ is said to be *regular*, whenever its primary (and so secondary) operations are so. Finally, in case $A = 2^2$, a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be *specular*, provided $(\mu{\restriction}B) \in \hom(\mathfrak{B},\mathfrak{A})$. (Clearly, every subalgebra of $\mathfrak{DM}_{4[01]}$ is both regular and specular.)

Remark 3.31. Since any non-empty proper prime filter of $\mathfrak{D}^2_{2[01]}$ contains t but not f, and so contains b iff it does not contain n, $F_j \triangleq (2^2 \cap \pi_j^{-1}[\{1\}])$, where $j \in 2$, are exactly all non-empty proper prime filters of $\mathfrak{D}^2_{2[01]}$, in which case $\langle \mathfrak{DM}_{4[01]}, F_j\rangle$ is both $\wedge$-conjunctive and $\vee$-disjunctive, while, by Example 3.3 with $\vec{k} = \Delta_2$ and $\wr = \sim$, we see that $\Upsilon_{\sim}$ is a unary unitary equality determinant for it. □

Recall also the following rather well-known (within Universal Algebra) fact:

Lemma 3.32. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{DM}_4$. Then, $\mathrm{Con}(\mathfrak{B}) \subseteq \{\Delta_B, B^2\}$. In particular, $\mathfrak{B}$ is simple iff $|B| > 1$.*

Proof. Consider any $\theta \in (\mathrm{Con}(\mathfrak{B}) \setminus \{\Delta_B\})$. Take any $\vec{a} \in (\theta \setminus \Delta_B) \neq \varnothing$. Consider the following exhaustive cases:

(1) $\mathrm{img}\,\vec{a} \subseteq \{\mathsf{f},\mathsf{t}\}$.
Then, $\mathrm{img}\,\vec{a} = \{\mathsf{f},\mathsf{t}\}$, for $a_0 \neq a_1$, and so $\mathsf{f}\;\theta\;\mathsf{t}$.

[3] According to [3], "Boolean/Kleene/De Morgan algebra" traditionally stands for "bounded Boolean/Kleene/De Morgan lattice".

(2) $\mathrm{img}\,\vec{a} \subseteq \{\mathsf{n}, \mathsf{b}\}$.
Then, $\mathrm{img}\,\vec{a} = \{\mathsf{n}, \mathsf{b}\}$, for $a_0 \neq a_1$, in which case $\mathsf{n}\ \theta\ \mathsf{b}$, and so $\mathsf{f} = (\mathsf{n} \wedge^{\mathfrak{B}} \mathsf{b})\ \theta\ (\mathsf{n} \wedge^{\mathfrak{B}} \mathsf{n}) = \mathsf{n} = (\mathsf{n} \vee^{\mathfrak{B}} \mathsf{n})\ \theta\ (\mathsf{n} \vee^{\mathfrak{B}} \mathsf{b}) = \mathsf{t}$.

(3) $a_i \in \{\mathsf{f}, \mathsf{t}\}$, while $a_{1-i} \in \{\mathsf{b}, \mathsf{n}\}$, for some $i \in 2$.
Then, $a_i\ \theta\ a_{1-i} = {\sim}^{\mathfrak{B}} a_{1-i}\ \theta\ {\sim}^{\mathfrak{B}} a_i$, and so $\mathsf{f}\ \theta\ \mathsf{t}$, because ${\sim}^{\mathfrak{B}}\langle j, j\rangle = \langle 1-j, 1-j\rangle$, for all $j \in 2$.

Thus, in any case, we have $\mathsf{f}\ \theta\ \mathsf{t}$. Therefore, for every $c \in B$, we get $c = (\mathsf{f} \vee^{\mathfrak{B}} c)\ \theta\ (\mathsf{t} \vee^{\mathfrak{B}} c) = \mathsf{t}$. Hence, $\theta = B^2$, as required. ☐

Given any $n \in (\omega \setminus 1)$, by $\mathfrak{K}_{n[01]}$ we denote the chain [bounded] Kleene lattice such that $(\mathfrak{K}_{n[01]} {\upharpoonright} \Sigma^{+}_{[01]}) \triangleq \mathfrak{D}_{n[01]}$ and ${\sim}^{\mathfrak{K}_{n[01]}} i \triangleq (n-1-i)$, for all $i \in n$, $\mathfrak{K}_{2[01]}$ being a [bounded] Boolean lattice. Then, $e_n \triangleq \{\langle 0, 0\rangle, \langle 1, n-1\rangle\} \in \hom(\mathfrak{K}_{2[01]}, \mathfrak{K}_{n[01]})$ is injective. Moreover, for any $n \in (\omega \setminus 3)$, $\hbar_n \triangleq (\{\langle 0, 0\rangle, \langle n-1, 2\rangle\} \cup (((n-1) \setminus 1) \times \{1\})) \in \hom(\mathfrak{K}_{n[01]}, \mathfrak{K}_{3[01]})$ is surjective. Finally, for any $i \in 2$, $e_{3,i} \triangleq \{\langle 0, \mathsf{f}\rangle, \langle 2, \mathsf{t}\rangle, \langle 1, \langle i, 1-i\rangle\rangle\} \in \hom(\mathfrak{K}_{3[01]}, \mathfrak{DM}_{4[01]})$ is injective.

4. Three-valued paraconsistent logics with subclassical negation

The present section collectively with Subsection 7.2 exemplifying the former incorporates the material prepared by and announced in 1995 (cf. the paragraph after Theorem 2.1 in [21] and the reference [Pyn 95b] therein).

Fix any unary connective $\wr$ of Σ.

A Σ-matrix $\mathcal{A}$ is said to be *$\wr$-superclassical*, provided $A = \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, $D^{\mathcal{A}} = \{\mathsf{b}, \mathsf{t}\}$, $\wr^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$, $\wr^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$ and $\wr^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}}$, in which case it is three-valued, both consistent and false-singular with $\dashv^{\mathcal{A}} = \mathsf{f}$ as well as $\wr$-paraconsistent, while $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}{\upharpoonright}\{\wr\}$, whereas $(\mathfrak{A}{\upharpoonright}\{\wr\}){\upharpoonright}2^2_{\not\mathsf{b}}$ is $\wr$-classical, in which case $\wr$ is a subclassical negation for the logic of $\mathcal{A}$, in view of (2.6), and so we have argued the routine part (viz., (ii)⇒(iii)⇒(i)) of the following preliminary marking the framework of the present subsection:

Proposition 4.1. *Let C be a Σ-logic. Then, the following are equivalent:*

(i) *C is three-valued and $\wr$-paraconsistent, while $\wr$ is a subclassical negation for C;*
(ii) *C is three-valued, while any three-valued Σ-matrix defining C is isomorphic to a $\wr$-superclassical one;*
(iii) *C is defined by a $\wr$-superclassical Σ-matrix.*

Proof. Assume (i) holds. Let $\mathcal{B}$ be any three-valued Σ-matrix defining C. Define an $e : \{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \to B$ as follows. In that case, $\mathcal{B}$ is $\wr$-paraconsistent, so there are some $e(\mathsf{b}) \in D^{\mathcal{B}}$ such that $\wr^{\mathfrak{B}} e(\mathsf{b}) \in D^{\mathcal{B}}$ and some $e(\mathsf{f}) \in (B \setminus D^{\mathcal{B}})$, in which case $e(\mathsf{f}) \neq e(\mathsf{b})$. Next, by (2.8) with $m = 1$ and $n = 0$, there is some $e(\mathsf{t}) \in D^{\mathcal{B}}$ such that $\wr^{\mathfrak{B}} e(\mathsf{t}) \notin D^{\mathcal{B}}$, in which case $e(\mathsf{f}) \neq e(\mathsf{t}) \neq e(\mathsf{b})$. In this way, $e : \{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \to B$ is injective, and so bijective, for $|B| = 3$. Hence, it is an isomorphism from $\mathcal{A} \triangleq \langle e^{-1}[\mathfrak{B}], \{\mathsf{b}, \mathsf{t}\}\rangle$ onto $\mathcal{B}$. Therefore, by (2.6), C is defined by $\mathcal{A}$. Furthermore, $\wr^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}}$, while $\wr^{\mathfrak{A}}\mathsf{t} \notin D^{\mathcal{A}}$, in which case $\wr^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$, and so, for proving that $\mathcal{A}$ is $\wr$-superclassical, in which case (ii) holds, it only remains to show that $\wr^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$. We do it by contradiction. For suppose $\wr^{\mathfrak{A}}\mathsf{f} \neq \mathsf{t}$, in which case we have the following two exhaustive cases:

(1) $\wr^{\mathfrak{A}}\mathsf{f} = \mathsf{f}$.
This contradicts to (2.8) with $m = 0$ and $n = 1$.

(2) $\wr^{\mathfrak{A}}\mathsf{f} = \mathsf{b}$.
As $\wr^{\mathfrak{A}}\mathsf{b} \in \{\mathsf{b}, \mathsf{t}\}$, we then have the following two exhaustive subcases:
 (a) $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$.
 Then, $\wr^{\mathfrak{A}} \wr^{\mathfrak{A}} \wr^{\mathfrak{A}} a = \mathsf{b} \in D^{\mathcal{A}}$, for each $a \in D^{\mathcal{A}}$. This contradicts to (2.8) with $m = 3$ and $n = 0$.
 (b) $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$.
 Then, $\wr^{\mathfrak{A}} \wr^{\mathfrak{A}} \wr^{\mathfrak{A}}\mathsf{f} = \mathsf{f}$. This contradicts to (2.8) with $m = 0$ and $n = 3$.

Thus, in any case, we come to a contradiction, as required. ☐

Proposition 4.2. *Any three-valued $\wr$-paraconsistent Σ-logic C with subclassical negation $\wr$ is minimally three-valued.*

Proof. By contradiction. For supposed C is defined by a Σ-matrix $\mathcal{A}$ such that $|A| < 3$, in which case it is $\wr$-paraconsistent, and so both consistent and truth-non-empty. Therefore, there is some $a \in A$ such that $D^{\mathcal{A}} = \{a\}$. Hence, ${\sim}^{\mathfrak{A}} a = a$. This contradicts to (2.8) with $m = 1$ and $n = 0$, as required. ☐

Remark 4.3. By Example 3.3 with $j = 0$ and $\vec{k} = \Delta_2$, $\Upsilon_{\wr}$ is a unary unitary equality determinant for any $\wr$-superclassical Σ-matrix $\mathcal{A}$. ☐

Fix any $\wr$-superclassical Σ-matrix $\mathcal{A}$. Let C be the logic of $\mathcal{A}$ and C^{NP} the least non-$\wr$-paraconsistent extension of C, viz., that which is relatively axiomatized by (2.3).

Proposition 4.4. *Suppose $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, C^{NP} is not an axiomatic extension of C.*

Proof. In that case, by (2.6), $\mathcal{B} \triangleq (\mathcal{A} \times (\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}))$ is a model of C. Moreover, it is not $\wr$-paraconsistent, for $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$, being $\wr$-classical, is $\wr$-negative, and so is a model of C^{NP}. Then, (2.7) and the fact that $(\pi_0{\upharpoonright}B) \in \hom^{\mathrm{S}}(\mathcal{B}, \mathcal{A})$ complete the argument. ☐

Lemma 4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\wr$-superclassical Σ-matrices and $e \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Then, e is diagonal. In particular, $\mathcal{A} = \mathcal{B}$.*

Proof. In that case, $(\mathcal{A}{\upharpoonright}\{\wr\}) = (\mathcal{B}{\upharpoonright}\{\wr\})$ is $\wr$-superclassical and $e \in \hom_{\mathrm{S}}(\mathcal{A}{\upharpoonright}\{\wr\}, \mathcal{B}{\upharpoonright}\{\wr\})$. Therefore, by Lemma 3.7 and Remark 4.3, e is diagonal, and so $\mathcal{A} = \mathcal{B}$, for $A = B$, as required. ☐

Theorem 4.6. *Let $\mathcal{B}$ be a $\wr$-superclassical Σ-matrix. Suppose $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$). Then, $\mathcal{B} = \mathcal{A}$.*

Proof. In that case, $\mathcal{B}$ is a finite (and so finitely-generated) $\wr$-paraconsistent model of C. Then, by Lemmas 2.19, 3.4 and Remark 4.3, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D},\mathcal{B})$, in which case $\mathcal{D}$ is both weakly $\wr$-negative and, by (2.6), is $\wr$-paraconsistent, for $\mathcal{B}$ is so, and so there are some $a \in D^{\mathcal{D}}$ such that $\sim^{\mathfrak{D}} a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$, in which case $c \triangleq \wr^{\mathfrak{D}} b \in D^{\mathcal{D}} \subseteq \{\mathsf{b},\mathsf{t}\}^I$, for $\mathcal{D}$ is weakly $\wr$-negative. Then, $D \ni a = (I \times \{\mathsf{b}\})$. Consider the following complementary cases:

(1) $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$,
in which case $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, and so $\wr^{\mathfrak{D}} c = b \notin D^{\mathcal{B}}$. Hence, $J \triangleq \{i \in I \mid \pi_i(c) = \mathsf{t}\} \neq \varnothing$. Given any $\vec{a} \in A^2$, set $(a_0|a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. In this way, $D \ni a = (\mathsf{b}|\mathsf{b})$, $D \ni c = (\mathsf{t}|\mathsf{b})$ and $D \ni b = (\mathsf{f}|\mathsf{b})$. Then, as $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, while $J \neq \varnothing$, $\{\langle d,(d|\mathsf{b})\rangle \mid d \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

(2) $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\{\mathsf{b},\varphi^{\mathfrak{A}}(\mathsf{b}),\wr^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\mathsf{b})\} = A$, and so $D \supseteq \{a,\varphi^{\mathfrak{D}}(a),\wr^{\mathfrak{D}}\varphi^{\mathfrak{D}}(a)\} = \{I \times \{d\} \mid d \in A\}$. Therefore, as $I \neq \varnothing$, for $b \notin D^{\mathcal{D}}$, $\{\langle d, I \times \{d\}\rangle \mid d \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Thus, anyway, there is some $f \in \hom_{\mathrm{S}}(\mathcal{A},\mathcal{D})$, in which case $(g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A},\mathcal{E})$, and so Lemma 4.5 completes the argument. ☐

Corollary 4.7. *Let $\Sigma' \supseteq \Sigma$ be a signature and C' a three-valued Σ'-expansion of C. Then, C' is defined by a unique Σ'-expansion of $\mathcal{A}$.*

Proof. In that case, C' is $\wr$-paraconsistent, while $\wr$ is a subclassical negation for C'. Hence, by Proposition 4.1, C' is defined by a $\wr$-superclassical Σ'-matrix $\mathcal{A}'$, in which case C is defined by the $\wr$-superclassical Σ-matrix $\mathcal{A}'{\upharpoonright}\Sigma$, and so $(\mathcal{A}'{\upharpoonright}\Sigma) = \mathcal{A}$, in view of Theorem 4.6. Finally, Theorem 4.6 completes the argument. ☐

4.1. **Classical extensions.**

Lemma 4.8. *Let I be a set and $\mathcal{B}$ a consistent $\wr$-negative submatrix of $\mathcal{A}^I$. Suppose $a \triangleq (I \times \{\mathsf{f}\}) \in B$. Then,*

(i) $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}$*;*
(ii) $\mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}$ *is embeddable into* $\mathcal{B}$*.*

Proof. In that case, $B \ni b \triangleq \wr^{\mathfrak{B}} a = (I \times \{\mathsf{t}\})$.

(i) By contradiction. For suppose $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{b}$, in which case $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (I \times \{\mathsf{b}\})$, and so $\{c,\wr^{\mathfrak{B}} c\} \subseteq D^{\mathcal{B}}$. This contradicts to the fact that $\mathcal{B}$ is $\wr$-negative, as required.
(ii) In this way, as $I \neq \varnothing$, for $\mathcal{B}$ is consistent, taking (i) into account, we eventually conclude that $\{\langle d, I \times \{d\}\rangle \mid d \in \{\mathsf{f},\mathsf{t}\}\}$ is an embedding of $\mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}$ into $\mathcal{B}$, as required. ☐

Corollary 4.9. *Let I be a set, $\diamond$ a (possibly, secondary) binary connective of Σ and $\mathcal{B}$ a consistent $\wr$-negative submatrix of $\mathcal{A}^I$. Suppose either $\mathcal{B}$ is $\diamond$-conjunctive or $\mathcal{A}$ is weakly $\diamond$-conjunctive. Then, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}$ is embeddable into $\mathcal{B}$.*

Proof. Take any $a \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$, in which case, as $\mathcal{B}$ is $\wr$-negative, $b \triangleq \wr^{\mathfrak{B}} a \in D^{\mathcal{B}} \subseteq \{\mathsf{b},\mathsf{t}\}^I$, and so $B \ni c \triangleq \wr^{\mathfrak{B}} b \notin D^{\mathcal{B}}$ and $d \triangleq \wr^{\mathfrak{B}} c \in D^{\mathcal{B}}$. Hence, $J \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\} \neq \varnothing$. Given any $\vec{d} \in A^2$, set $(d_0|d_1) \triangleq ((J \times \{d_0\}) \cup ((I \setminus J) \times \{d_1\})) \in A^I$. In this way, $b = (\mathsf{t}|\mathsf{b})$. Consider the following exhaustive cases:

(1) $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$.
Then, $c = (\mathsf{f}|\mathsf{t})$ and $d = (\mathsf{t}|\mathsf{f}) \in D^{\mathcal{B}}$, in which case $J = I$, and so $B \ni c = (I \times \{\mathsf{f}\})$, as required, in view of Lemma 4.8.

(2) $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$.
Then, $c = (\mathsf{f}|\mathsf{b}) \notin D^{\mathcal{B}}$, in which case $J \neq \varnothing$, and $d = b$. Let us prove, by contradiction, that $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. For suppose $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{b}$, in which case $\phi \triangleq \varphi(x_0,\wr x_0) \in \mathrm{Fm}^1_\Sigma$ and $\phi^{\mathfrak{A}}(\mathsf{f}) = \mathsf{b}$, and so $\phi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, for, otherwise, we would have $e \triangleq (\mathsf{b}|\mathsf{b}) = \phi^{\mathfrak{B}}(c) \in B$, in which case we would get $\{e,\wr^{\mathfrak{B}} e\} \subseteq D^{\mathcal{B}}$, contrary to the $\wr$-negativity of $\mathcal{B}$. In this way, $f \triangleq (\mathsf{b}|\mathsf{f}) \in \{\phi^{\mathfrak{B}}(c),\wr^{\mathfrak{B}}\phi^{\mathfrak{B}}(c)\} \subseteq B$, in which case $\wr^{\mathfrak{B}} f = (\mathsf{b}|\mathsf{t}) \in D^{\mathcal{B}}$. Consider the following exhaustive subcases:

- $\mathcal{B}$ is $\diamond$-conjunctive.
Then, we have $\{d \diamond^{\mathfrak{B}} \wr^{\mathfrak{B}} f, \wr^{\mathfrak{B}} f \diamond^{\mathfrak{B}} d\} \subseteq D^{\mathcal{B}}$. Therefore, as $J \neq \varnothing$, we get $\{\mathsf{t} \diamond^{\mathfrak{A}} \mathsf{b}, \mathsf{b} \diamond^{\mathfrak{A}} \mathsf{t}\} \subseteq D^{\mathcal{A}}$. Then, by the $\diamond$-conjunctivity of $\mathcal{B}$, we also have $(\{c \diamond^{\mathfrak{B}} \wr^{\mathfrak{B}} f, \wr^{\mathfrak{B}} f \diamond^{\mathfrak{B}} c\} \cap D^{\mathcal{B}}) = \varnothing$, in which case we get $(\mathsf{f} \diamond^{\mathfrak{A}} \mathsf{b}) = \mathsf{f} = (\mathsf{b} \diamond^{\mathfrak{A}} \mathsf{f})$.
- $\mathcal{A}$ is weakly $\diamond$-conjunctive.
Then, $(\mathsf{f} \diamond^{\mathfrak{A}} \mathsf{b}) = \mathsf{f} = (\mathsf{b} \diamond^{\mathfrak{A}} \mathsf{f})$.

Thus, in any case, we have $(\mathsf{f} \diamond^{\mathfrak{A}} \mathsf{b}) = \mathsf{f} = (\mathsf{b} \diamond^{\mathfrak{A}} \mathsf{f})$, and so we eventually get $B \ni (c \diamond^{\mathfrak{B}} f) = (\mathsf{f}|\mathsf{f}) = (I \times \{\mathsf{f}\})$, contrary to Lemma 4.8. Thus, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Consider the following complementary subcases:

(a) $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, as $J \neq \varnothing$, $\{\langle e,(e|\mathsf{b})\rangle \mid e \in \{\mathsf{f},\mathsf{t}\}\}$ is an embedding of $\mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}$ into $\mathcal{B}$, as required.

(b) $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}(\mathsf{b}) \in \{\mathsf{f},\mathsf{t}\}$, in which case $\psi^{\mathfrak{A}}(\mathsf{f}) \in \{\mathsf{f},\mathsf{t}\}$, for $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Consider the following complementary subsubcases:

 (i) $\psi^{\mathfrak{A}}(\mathsf{b}) = \psi^{\mathfrak{A}}(\mathsf{f})$,
 in which case $(I \times \{\mathsf{f}\}) = (\mathsf{f}|\mathsf{f}) \in \{\psi^{\mathfrak{B}}(c),\wr^{\mathfrak{B}}\psi^{\mathfrak{B}}(c)\} \subseteq B$, as required, in view of Lemma 4.8.

 (ii) $\psi^{\mathfrak{A}}(\mathsf{b}) \neq \psi^{\mathfrak{A}}(\mathsf{f})$.
 Then, as $d \in D^{\mathcal{B}}$, by the $\diamond$-disjunctivity of $\mathcal{B}$, we have $\{d \diamond^{\mathfrak{B}} c, c \diamond^{\mathfrak{B}} d\} \subseteq D^{\mathcal{B}}$, in which case we get $\{\mathsf{t} \diamond^{\mathfrak{A}} \mathsf{f}, \mathsf{f} \diamond^{\mathfrak{A}} \mathsf{t}\} \subseteq D^{\mathcal{A}}$, and so we eventually get $(\mathsf{t} \diamond^{\mathfrak{A}} \mathsf{f}) = \mathsf{t} = (\mathsf{f} \diamond^{\mathfrak{A}} \mathsf{t})$, for $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. In this way, $(I \times \{\mathsf{f}\}) = (\mathsf{f}|\mathsf{f}) = \wr^{\mathfrak{B}}(\psi^{\mathfrak{B}}(c) \diamond^{\mathfrak{B}} \wr^{\mathfrak{B}}\psi^{\mathfrak{B}}(c)) \in B$, as required, in view of Lemma 4.8. ☐

Theorem 4.10. *C is $\wr$-subclassical, if $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$ is a $\wr$-classical model of C, and so the logic of it is a $\wr$-classical extension of C. Conversely, given a (possibly, secondary) binary connective $\diamond$ of Σ, if C [is weakly $\diamond$-conjunctive and] has a $\wr$-classical [weakly] $\diamond$-conjunctive extension, then $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$ is a $\wr$-classical [weakly] $\diamond$-conjunctive model of C isomorphic to any $\wr$-classical [weakly] $\diamond$-conjunctive model of C, and so the logic of it is a unique [weakly] $\diamond$-conjunctive $\wr$-classical extension of C.*

Proof. The "if" part is by (2.6).

Conversely, [assume C is weakly $\diamond$-conjunctive and] consider any $\wr$-classical [weakly] $\diamond$-conjunctive model $\mathcal{B}$ of C, in which case it is finite (in particular, finitely-generated) as well as both consistent and $\wr$-negative. Hence, by Lemmas 2.19, 3.4 and Example 3.2, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, in which case this is a submatrix of $\mathcal{A}^I$, and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D},\mathcal{B})$, in which case $\mathcal{D}$ is both consistent, [weakly] $\diamond$-conjunctive and $\wr$-negative, for $\mathcal{B}$ is so. Therefore, by Corollary 4.9, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, while there is some $e \in \hom_{\mathrm{S}}(\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\},\mathcal{D})$, in which case $(g\circ e) \in \hom_{\mathrm{S}}(\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\},\mathcal{B})$, and so $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$, being embeddable into $\mathcal{B}$, in view of Corollary 2.14, Example 3.2 and Lemma 3.4, is isomorphic to $\mathcal{B}$, for $|B| = 2 = |\{\mathsf{f},\mathsf{t}\}|$. In particular, it is [weakly] $\diamond$-conjunctive, for $\mathcal{B}$ is so. Finally, (2.6) completes the argument. □

It is remarkable that the $\overline{\wedge}$-conjunctivity of C is not required in the formulation of Theorem 4.10, making it the right algebraic criterion of C's being "genuinely subclassical" in the sense of having a *genuinely* (viz., functionally-complete) classical extension.

In case $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, the $\wr$-classical extension of C defined by $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$ (cf. Theorem 4.10), being uniquely determined by C (cf. Theorem 4.6), is denoted by C^{PC}.

4.2. Maximal paraconsistency of three-valued paraconsistent logics with subclassical negation.

Lemma 4.11. *Let Σ' be an algebraic signature, $\wr$ a (possibly, secondary) unary connective of Σ', $\mathcal{A}'$ a Σ'-matrix, I a set, $\overline{\mathcal{D}}$ an I-tuple constituted by submatrices of $\mathcal{A}'$, $\mathcal{E}$ a submatrix of $\prod_{i\in I}\mathcal{D}_i$ and $a \in D^{\mathcal{E}}$. Suppose $\wr^{\mathfrak{E}}a \in D^{\mathcal{E}}$. Then, $a \in (D^{\mathcal{A}'} \cap (\wr^{\mathfrak{A}'})^{-1}[D^{\mathcal{A}'}])^I$.*

Proof. Then, for each $i \in I$, both $\pi_i(a) \in D^{\mathcal{A}'}$ and $\wr^{\mathfrak{A}'}\pi_i(a) = \pi_i(\wr^{\mathfrak{E}}a) \in D^{\mathcal{A}'}$, as required. □

A *ternary* b*-relative (weak classical) conjunction for* $\mathfrak{A}$ is any $\varphi \in \mathrm{Fm}^3_\Sigma$ such that both $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f},\mathsf{t}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{t},\mathsf{f}) \neq \mathsf{t}$.

Lemma 4.12. *Let $\mathcal{B}$ be a (simple) finitely-generated $\wr$-paraconsistent model of C. Suppose either $\mathfrak{A}$ has a ternary b-relative conjunction or $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A}$ is embeddable into $\mathcal{B}/\partial(\mathcal{B})$ (resp., into $\mathcal{B}$).*

Proof. Put $\mathcal{E} \triangleq (\mathcal{B}/\partial(\mathcal{B}))$ (resp., $\mathcal{E} \triangleq \mathcal{B}$). Then, by Lemma 2.19 with $\mathsf{M} = \{\mathcal{A}\}$, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D},\mathcal{E})$, in which case, by (2.6), $\mathcal{D}$ is $\wr$-paraconsistent, and so there are some $a \in D^{\mathcal{D}}$ such that $\sim^{\mathfrak{D}}a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$. Then, $D \ni a = (I \times \{\mathsf{b}\})$. Consider the following complementary cases:

1. $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$.
   Then, $\mathfrak{A}$ has a ternary b-relative conjunction $\varphi \in \mathrm{Fm}^3_\Sigma$. Put $c \triangleq \varphi^{\mathfrak{D}}(a,b,\wr^{\mathfrak{D}}b) \in D$, $d \triangleq \wr^{\mathfrak{D}}c \in D$, $J \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$ and $K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{f}\} \neq \varnothing$, for $b \notin D^{\mathcal{D}}$. Given any $\vec{a} \in A^3$, set $(a_0|a_1|a_2) \triangleq ((J \times \{a_0\}) \cup (K \times \{a_1\}) \cup ((I \setminus (J \cup K)) \times \{a_2\})) \in A^I$. Then, $a = (\mathsf{b}|\mathsf{b}|\mathsf{b})$ and $b = (\mathsf{t}|\mathsf{f}|\mathsf{b})$. Consider the following complementary subcases:
   (a) $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{t},\mathsf{f}) = \mathsf{f}$.
   In that case, we have $c = (\mathsf{f}|\mathsf{f}|\mathsf{b})$ and $d = (\mathsf{t}|\mathsf{t}|b)$. Then, since $K \neq \varnothing$, while $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle e,(e|e|\mathsf{b})\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.
   (b) $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{t},\mathsf{f}) \neq \mathsf{f}$,
   in which case we have $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{t},\mathsf{f}) = \mathsf{b}$, and so we get $c = (\mathsf{b}|\mathsf{f}|\mathsf{b})$ and $d = (\mathsf{b}|\mathsf{t}|\mathsf{b})$. Then, since $K \neq \varnothing$, while $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle e,(\mathsf{b}|e|\mathsf{b})\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.
2. $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$.
   Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\{\mathsf{b},\varphi^{\mathfrak{A}}(\mathsf{b}),\wr^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\mathsf{b})\} = A$, and so $D \supseteq \{a,\varphi^{\mathfrak{D}}(a), \wr^{\mathfrak{D}}\varphi^{\mathfrak{D}}(a)\} = \{I \times \{e\} \mid e \in A\}$. Therefore, as $I \neq \varnothing$, for $b \notin D^{\mathcal{D}}$, $\{\langle e, I \times \{e\}\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Thus, anyway, there is some $f \in \hom_{\mathrm{S}}(\mathcal{A},\mathcal{D})$, in which case $(g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A},\mathcal{E})$, and so Corollary 2.14, Lemma 3.4 and Remark 4.3 complete the argument. □

Theorem 4.13. *The following are equivalent [provided C is $\wr$-subclassical]:*

(i) *C has no proper $\wr$-paraconsistent [$\wr$-subclassical] extension;*
(ii) *either $\mathfrak{A}$ has a ternary b-relative conjunction or $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$ (in particular, $\wr^{\mathfrak{A}}\mathsf{b} \neq \mathsf{b}$, that is, $\wr\wr x_0 \notin C(x_0)$);*
(iii) *$D_3 \triangleq \{\langle\mathsf{b},\mathsf{b}\rangle,\langle\mathsf{f},\mathsf{t}\rangle,\langle\mathsf{t},\mathsf{f}\rangle\}$ does not form a subalgebra of $\mathfrak{A}^2$;*
(iv) *$\mathcal{A}$ has no truth-singular $\wr$-paraconsistent subdirect square;*
(v) *$\mathcal{A}^2$ has no truth-singular $\wr$-paraconsistent submatrix;*
(vi) *C has no truth-singular $\wr$-paraconsistent model.*

Proof. First, assume (ii) holds. Consider any $\wr$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0,\wr x_0\}) \supseteq \{x_0,\wr x_0\}$, while, by the structurality of C', $\langle\mathfrak{Fm}^{\omega}_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated $\wr$-paraconsistent submatrix $\mathcal{B} \triangleq \langle\mathfrak{Fm}^2_\Sigma, T \cap \mathrm{Fm}^2_\Sigma\rangle$, in view of (2.6). Then, by Lemma 4.12, $\mathcal{A}$ is embeddable into $\mathcal{B}/\partial(\mathcal{B})$, in which case, by (2.6), it is a model of C', and so $C' = C$. Thus, (i) holds.

Next, (iv)⇒(iii) is by the fact $\wr^{\mathfrak{A}}\mathsf{b} \in \{\mathsf{b},\mathsf{t}\}$, $(D_3 \cap \{\mathsf{b},\mathsf{t}\}^2) = \{\langle\mathsf{b},\mathsf{b}\rangle\} \neq D_3$ and $\pi_{0[+1]}[D_3] = A$, while (iv) is a particular case of (v), whereas (vi)⇒(v) is by (2.6).

Now, let $\mathcal{B} \in \mathrm{Mod}(C)$ be both $\wr$-paraconsistent and truth-singular, in which case the rule $x_0 \vdash \wr x_0$ is true in $\mathcal{B}$, and so is its logical consequence $\{x_0, x_1, \wr x_1\} \vdash \wr x_0$, not being true in $\mathcal{A}$ under $[x_0/\mathsf{t}, x_1/\mathsf{b}]$ [but true in any $\wr$-classical model $\mathcal{C}'$ of C, for $\mathcal{C}'$ is $\wr$-negative]. In this way, the logic of $\{\mathcal{B}[, \mathcal{C}']\}$ is a proper $\wr$-paraconsistent [$\wr$-subclassical] extension of C, Thus, (i)⇒(vi) holds.

Finally, assume $\mathfrak{A}$ has no ternary b-relative conjunction and $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$. In that case, $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by D_3. If $\langle \mathsf{f}, \mathsf{f}\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{f}, \mathsf{t}) = \mathsf{f} = \varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{t}, \mathsf{f})$, in which case it would be a ternary b-relative conjunction for $\mathfrak{A}$. Likewise, if either $\langle \mathsf{b}, \mathsf{f}\rangle$ or $\langle \mathsf{f}, \mathsf{b}\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{f}, \mathsf{t}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{t}, \mathsf{f}) = \mathsf{b}$, in which case it would be a ternary b-relative conjunction for $\mathfrak{A}$. Therefore, as $\wr^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$ and $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, we conclude that $(\{\langle \mathsf{f}, \mathsf{b}\rangle, \langle \mathsf{t}, \mathsf{b}\rangle, \langle \mathsf{b}, \mathsf{t}\rangle, \langle \mathsf{b}, \mathsf{f}\rangle, \langle \mathsf{f}, \mathsf{f}\rangle, \langle \mathsf{t}, \mathsf{t}\rangle\} \cap B) = \varnothing$. Thus, $B = D_3$ forms a subalgebra of $\mathfrak{A}^2$. In this way, (iii)⇒(ii) holds, as required. □

Theorem 4.13(i)⇔(ii[i]) is especially useful for [effective dis]proving the maximal $\wr$-paraconsistency of C. In this connection, we have the following negative instance:

Example 4.14. Suppose $\Sigma = \{\wr\}$ and $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, in which case D_3 forms a subalgebra of $\mathfrak{A}^2$, for $\wr^{\mathfrak{A}}(\mathsf{f}/\mathsf{t}) = (\mathsf{t}/\mathsf{f})$, and so, by Theorem 4.13, C is not maximally $\wr$-paraconsistent. □

On the other hand, Subsubsections 7.1.1 and 7.1.2 definitely show that the maximal paraconsistency is not at all a prerogative of merely three-valued logics. And what is more, as it is shown in the next subsection, there is no limit of the number of truth values, for which minimally many-valued maximally paraconsistent logics exist.

4.3. **Weakly conjunctive three-valued paraconsistent logics with subclassical negation.** Fix (in addition to $\wr$) any (possibly, secondary) binary connective $\barwedge$ of Σ.

Remark 4.15. Suppose either $\mathcal{A}$ is weakly $\barwedge$-conjunctive or both $\{0, 2\}$ forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}{\restriction}\{0, 2\}$ is weakly $\barwedge$-conjunctive. Then, $(x_1 \barwedge x_2)$ is a ternary b-relative conjunction for $\mathfrak{A}$. □

By Proposition 4.1, Theorems 4.13, 4.6 and Remark 4.15, we immediately get:

Corollary 4.16. *Any three-valued $\wr$-paraconsistent weakly $\barwedge$-conjunctive Σ-logic C with subclassical negation $\wr$ is maximally $\wr$-paraconsistent.*

4.3.1. *Subclassical three-valued paraconsistent weakly conjunctive logics.* By Theorem 4.10, we immediately have:

Corollary 4.17. *[Providing C is weakly $\barwedge$-conjunctive] C is $\wr$-subclassical if[f] $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ [in which case $\mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}$ is isomorphic to any $\wr$-classical model of C, and so defines a unique $\wr$-classical extension of C].*

Theorem 4.18. *Let C' be a consistent extension of C. Suppose C (viz., $\mathcal{A}$) is weakly $\barwedge$-conjunctive and $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\wr$-subclassical; cf. Corollary 4.17). Then, $\mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}$ is a model of C', that is, C^{PC} is an extension of C'.*

Proof. Then, $x_0 \notin C'(\varnothing)$, while, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, C'(\varnothing)\rangle$ is a model of C' (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_\Sigma, \mathrm{Fm}^1_\Sigma \cap C'(\varnothing)\rangle$, in view of (2.6). Hence, by Lemma 2.19, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D}$ of it such that $\mathcal{B}$ is a strict surjective homomorphic counter-image of a strict surjective homomorphic image of $\mathcal{D}$, in which case $\mathcal{D}$ is a consistent model of C', in view of (2.6), and so, in particular, $I \neq \varnothing$. Then, by Lemma 3.11, $a \triangleq (I \times \{\mathsf{f}\}) \in D$, in which case $D \ni \wr^{\mathfrak{D}} a = (I \times \{\mathsf{t}\})$, and so, as $I \neq \varnothing$, $\{\langle b, I \times \{b\}\rangle \mid b \in \{\mathsf{f}, \mathsf{t}\}\}$ is an embedding of $\mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}$ into $\mathcal{D}$. In this way, (2.6) completes the argument. □

4.4. **Disjunctive three-valued paraconsistent logics with subclassical negation.** Fix (in addition to $\wr$ and $\barwedge$) any (possibly, secondary) binary connective $\veebar$ of Σ. Then, by Corollary 3.18, we first have:

Corollary 4.19. *C is [weakly] $\veebar$-disjunctive iff $\mathcal{A}$ is so.*

Corollary 4.20. *Any $\wr$-classical extension of C is [weakly] $\veebar$-disjunctive, whenever C is so.*

By C^{MP} we denote the extension of C relatively axiomatized by the Modus Ponens rule for the *material* implication $\wr x_0 \veebar x_1$:

$$\{x_0, \wr x_0 \veebar x_1\} \vdash x_1. \tag{4.1}$$

Clearly, providing C is $\veebar$-disjunctive, we have $C^{\mathrm{NP}} \subseteq C^{\mathrm{MP}}$, by (3.3) held in C and inherited by its extensions.

Lemma 4.21. *Any $\veebar$-disjunctive extension C' of C^{NP} is an extension of C^{MP}.*

Proof. In that case, we have $x_1 \in (C'(\{x_0, \wr x_0\}) \cap C'(\{x_0, x_1\})) = C'(\{x_0, \wr x_0 \veebar x_1\})$, as required. □

4.4.1. *Subclassical disjunctive three-valued paraconsistent logics.*

Lemma 4.22. *Let $\wr$ and $\veebar$ be (possibly, secondary) unary and binary connectives of Σ, C' a $\veebar$-disjunctive Σ-logic and C'' an extension of C'. Then,*

$$\{x_1 \veebar x_0, \wr x_1 \veebar x_0\} \vdash (x_2 \veebar x_0) \tag{4.2}$$

is satisfied in C'' iff

$$\{x_1 \veebar x_0, \wr x_1 \veebar x_0\} \vdash x_0 \tag{4.3}$$

is so.

Proof. In that case, (3.4) and (3.5), being valid for C', remain so for C''. First, assume (4.2) is satisfied in C'', in which case (4.2)$[x_2/x_0]$ is so, in view of the structurality of C'', and so is (4.3), in view of (3.5) and the transitivity of C''. Conversely, the fact that (4.3) and (3.4) are satisfied in C'' implies the fact that (4.2) is so, in view of the transitivity of C'', as required. □

Note that $\mathbf{S}_*(\mathcal{A}) \setminus \{\mathcal{A}\}$ is either the singleton $\{\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}\}$, if $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, or empty, otherwise. In this way, the fact that $\wr$-[super]classical matrices are not [resp., are] $\wr$-paraconsistent, by Corollary 4.19, Lemma 4.22 and Theorem 3.24, we then get:

Theorem 4.23. *Suppose C is $\veebar$-disjunctive and $\{\mathsf{f},\mathsf{t}\}$ does not form [resp., forms] a subalgebra of $\mathfrak{A}$. Then, there is no [resp., a unique] proper consistent $\veebar$-disjunctive extension of C [in which case it is defined by $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$ and relatively axiomatized by (4.3)].*

Recall that (4.3) is nothing but the *Resolution* rule. Since any $\wr$-classical Σ-logic is consistent but not $\wr$-paraconsistent, as opposed to C, by (2.6), Corollary 4.20 and Theorem 4.23, we eventually get the following "disjunctive" analogue of Corollary 4.17:

Corollary 4.24. *[Providing C is $\veebar$-disjunctive] C is $\wr$-subclassical if[f] $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case the logic of $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$ is a [unique] $\wr$-classical extension of C.*

Corollary 4.25. *Suppose $\mathcal{A}$ is $\rhd$-implicative, where $\rhd$ is a (possibly, secondary) binary connective of Σ, and C is $\wr$-subclassical. Then, C^{PC} is a unique proper consistent axiomatic extension of C and is relatively axiomatized by:*

$$\wr x_0 \rhd (x_0 \rhd x_1). \tag{4.4}$$

Proof. In that case, by Corollary 4.24, $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$ is the only consistent submatrix of $\mathcal{A}$. Moreover, it, being both $\wr$-negative and $\rhd$-implicative, is a model of (4.4) not being true in $\mathcal{A}$ under $[x_0/\mathsf{b}, x_1/\mathsf{f}]$, for it is $\rhd$-implicative. In this way, Corollary 2.21 completes the argument. □

Combining Corollaries 4.19, 4.24, Remarks 3.15, 4.15, 2.22, Propositions 3.26, 4.1 and Theorem 4.13, we eventually get:

Theorem 4.26. *Any disjunctive (in particular, having DDT) $\wr$-subclassical three-valued $\wr$-paraconsistent Σ-logic is maximally $\wr$-paraconsistent.*

The following counterexample shows that the condition of being $\wr$-subclassical in the formulation of Theorem 4.26 is essential:

Example 4.27. Let $\Sigma = \{\wr, \vee\}$, where $\vee$ is binary, while $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, whereas:

$$(a \vee^{\mathfrak{A}} b) = \begin{cases} a & \text{if } a = b, \\ \mathsf{b} & \text{otherwise,} \end{cases}$$

for all $a, b \in A$, in which case $\mathcal{A}$ is $\vee$-disjunctive, and so is C, in view of Corollary 3.17. On the other hand, D_3 forms a subalgebra of $\mathfrak{A}^2$, in which case, by Theorem 4.13, C is not maximally $\wr$-paraconsistent, and so is not $\wr$-subclassical, in view of Theorem 4.26. □

4.4.2. *Three-valued paraconsistent logics with subclassical negation and lattice conjunction and disjunction.* Fix (in addition to $\wr$) binary (possibly, secondary) connectives $\barwedge$ and $\veebar$ of Σ.

A Σ-algebra $\mathfrak{B}$ is said to be a *[distributive] $(\barwedge,\veebar)$-lattice*, provided it satisfies [distributive] lattice identities for $\barwedge$ and $\veebar$, that is, $\langle B, \barwedge^{\mathfrak{B}}, \veebar^{\mathfrak{B}}\rangle$ is a [distributive] lattice (in the standard algebraic sense; cf. [3]), whose partial ordering is denoted by $\leq^{\mathfrak{B}}$.

Throughout this subsubsection, it is supposed that:

- $\mathfrak{A}$ is a $(\barwedge,\veebar)$-lattice, in which case $\langle A, \leq^{\mathfrak{A}}\rangle$ is a chain poset for $|A| = 3$, and so $\mathfrak{A}$ is a distributive $(\barwedge,\veebar)$-lattice;
- f is the least element of the poset involved or, equivalently, $\mathcal{A}$ is $\barwedge$-conjunctive/$\veebar$-disjunctive, that is, C is so/, in view of Corollary 4.19.

4.4.2.1. Extensions.

Lemma 4.28. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{B}$ a consistent non-$\wr$-paraconsistent subdirect product of $\overline{\mathcal{C}}$. Then, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\hom(\mathcal{B}, \mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}) \neq \varnothing$.*

Proof. Then, as $\langle A, \leq^{\mathfrak{A}}\rangle$ is a chain, we have $\mathsf{b}(\leq/\geq)^{\mathfrak{A}}\mathsf{t}$. Moreover, $\wr^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}} = \{\mathsf{b},\mathsf{t}\}$. Therefore, $\mathsf{b}(\leq/\geq)^{\mathfrak{A}} \wr^{\mathfrak{A}} \mathsf{b}$. Let us prove, by contradiction, that there is some $i \in I$ such that $\mathsf{b} \notin C_i$. For suppose, for each $i \in I$, $\mathsf{b} \in C_i$. By induction on the cardinality on any $J \subseteq I$, let us prove that there is some $a \in (B \cap \{\mathsf{f}/\mathsf{t},\mathsf{b}\}^I)$ including $J \times \{\mathsf{b}\}$. First, in case $J = \varnothing$, by Lemma 3.11, we have $d \triangleq (I \times \{\mathsf{f}\}) \in B$, and so $(J \times \{\mathsf{b}\}) = \varnothing \subseteq a \triangleq (d/\wr^{\mathfrak{B}} d) = (I \times \{\mathsf{f}/\mathsf{t}\}) \in (B \cap \{\mathsf{f}/\mathsf{t},\mathsf{b}\}^I)$. Now, assume $J \neq \varnothing$, in which case there is some $j \in J \subseteq I$, and so $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$. Hence, by induction hypothesis, there is some $a \in (B \cap \{\mathsf{f}/\mathsf{t},\mathsf{b}\}^I)$ including $K \times \{\mathsf{b}\}$. Moreover, as $j \in I$, we have $\mathsf{b} \in C_j = \pi_j[B]$, in which case there is some $b \in B$ such that $\pi_j(b) = \mathsf{b}$, and so $c \triangleq (b(\barwedge/\veebar)^{\mathfrak{B}} \wr^{\mathfrak{B}} b) \in B$, while, for every $i \in I$, $\pi_i(c) = \mathsf{b}$, if $\pi_i(b) = \mathsf{b}$, and $\pi_i(c) = (\mathsf{f}/\mathsf{t})$, otherwise, in which case $c \in \{\mathsf{f}/\mathsf{t},\mathsf{b}\}^I$, while $\pi_j(c) = \mathsf{b}$, and so, as $J = (K \cup \{j\})$, we eventually get $(J \times \{\mathsf{b}\}) \subseteq (a \veebar^{\mathfrak{B}} c) \in (B \cap \{\mathsf{f}/\mathsf{t},\mathsf{b}\}^I)$, as required. In particular, when $J = I$, we have $a \triangleq (I \times \{\mathsf{b}\}) \in B$, in which case we get $\{a, \wr^{\mathfrak{B}} a\} \subseteq D^{\mathcal{B}}$, and so $\mathcal{B}$, being consistent, is $\wr$-paraconsistent. This contradiction shows that there is some $i \in I$ such that $\mathsf{b} \notin C_i$, in which case $h \triangleq (\pi_i \upharpoonright B) \in \hom(\mathcal{B}, \mathcal{C}_i)$, while C_i forms a subalgebra of $\mathfrak{A}$, whereas $\mathcal{C}_i = (\mathcal{A}\upharpoonright C_i)$. Finally, as $\mathcal{C}_i$ is consistent, in which case $\mathsf{f} \in C_i$, and so $\mathsf{t} = \wr^{\mathfrak{A}}\mathsf{f} \in C_i$, we eventually conclude that $C_i = \{\mathsf{f},\mathsf{t}\}$, for $\mathsf{b} \notin C_i$, as required. □

Theorem 4.29. *C^{NP} is consistent iff C is $\sim$-subclassical, in which case $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and C^{NP} is defined by $\mathcal{A} \times (\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\})$.*

Proof. First, assume C is $\sim$-subclassical.

Then, any $\sim$-classical extension of C is a both consistent and non-$\sim$-paraconsistent extension of C, and so a consistent extension of C^{NP}, in which case this is consistent too.

Moreover, by Corollary 4.17, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case we have the Σ-matrix $\mathcal{B} \triangleq (\mathcal{A} \times (\mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}))$. Consider any finite set I, any $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and any subdirect product $\mathcal{D} \in \mathrm{Mod}(C^{\mathrm{NP}})$ of $\overline{\mathcal{C}}$, in which case $\mathcal{D}$ is not $\sim$-paraconsistent. Put $J \triangleq \hom(\mathcal{D},\mathcal{B})$. Consider any $a \in (D \setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent, and so, by Lemma 4.28, there is some $g \in \hom(\mathcal{D},\mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}) \neq \varnothing$. Moreover, there is some $i \in I$, in which case $f \triangleq (\pi_i{\upharpoonright}D) \in \hom(\mathcal{D},\mathcal{A})$, such that $f(a) \notin D^{\mathcal{A}}$. Then, $h \triangleq (f \times g) \in J$ and $h(a) \notin D^{\mathcal{B}}$. In this way, $(\prod \Delta_J) \in \hom_{\mathrm{S}}(\mathcal{D},\mathcal{B}^J)$. Thus, by (2.6) and Theorem 2.20, C^{NP} is finitely-defined by the six-valued $\mathcal{B}$, and so, being finitary, for the three-valued C is so, is defined by $\mathcal{B}$.

Conversely, assume C^{NP} is consistent, in which case $x_0 \notin T \triangleq C^{\mathrm{NP}}(\varnothing)$, while, by the structurality of C^{NP}, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C^{NP} (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B}' \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, T \cap \mathrm{Fm}^{1}_{\Sigma}\rangle$, in view of (2.6). Hence, by Lemma 2.19, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{B}'/\eth(\mathcal{B}'))$, in which case, by (2.6), $\mathcal{D}$ is a consistent model of C^{NP}, so it is not $\sim$-paraconsistent. Thus, by Lemma 4.28 and Corollary 4.17, C is $\sim$-subclassical, as required. □

Remark 4.30. Let C' be any extension of C. Since $\mathcal{A}$ is $\veebar$-disjunctive, while f is the least element of the poset $\langle A, \leq^{\mathfrak{A}}\rangle$, we have $\{\wr x_0 \veebar x_1\} \vdash_C (\wr(x_0 \veebar x_1) \veebar x_1)$. Therefore, in view of (3.3), C' satisfies (4.3) iff it satisfies (4.1). □

Corollary 4.31. *C^{MP} is consistent iff C is $\sim$-subclassical, in which case $C^{\mathrm{NP}} \subsetneq C^{\mathrm{MP}} = C^{\mathrm{PC}}$, and so C^{NP} is not $\veebar$-disjunctive.*

Proof. First, if C is $\sim$-subclassical, then, by Corollary 4.17, Theorem 4.23 and Remark 4.30, $C^{\mathrm{MP}} = C^{\mathrm{PC}}$ is consistent. Conversely, if C^{MP} is consistent, then so is its sublogic C^{NP}, in which case case, by Theorem 4.29, C is $\sim$-subclassical, while $C^{\mathrm{NP}} \subsetneq C^{\mathrm{MP}}$, for (4.1) is not true in the Σ-matrix $\mathcal{A} \times (\mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\})$, defining C^{NP}, under $[x_0/\langle\mathsf{b},\mathsf{t}\rangle, x_1/\langle\mathsf{f},\mathsf{t}\rangle]$. In this way, (3.3), (3.5) and Corollary 3.19(i)⇒(iii)(3.9) complete the argument. □

Lemma 4.32. *Let $a \in \{\mathsf{b},\mathsf{n}\}$ and $\mathcal{B}$ a Σ-matrix. Suppose both of $\{\mathsf{f},[a,]\mathsf{t}\} \subseteq B$ form subalgebras of $\mathfrak{B}$, $\wr^{\mathfrak{B}}(\mathsf{f}|\mathsf{t}) = (\mathsf{t}|\mathsf{f})$ (while $\mathsf{f} \notin D^{\mathcal{B}} \ni \mathsf{t}$, whereas $a \in / \notin D^{\mathcal{A}}$). Then, (i)⇒(ii)(⇒(iii)⇒(iv)), where:*

(i) *$\mathfrak{B}{\upharpoonright}\{\mathsf{f},a,\mathsf{t}\}$ is regular;*
(ii) *$K^a_4 \triangleq \{\langle\mathsf{f},\mathsf{f}\rangle,\langle a,\mathsf{f}\rangle,\langle a,\mathsf{t}\rangle,\langle\mathsf{t},\mathsf{t}\rangle\}$ forms a subalgebra of $\mathfrak{B}^2$;*
(iii) *$(\pi_{0[+1]/0}{\upharpoonright}K^a_4) \in \hom^{\mathrm{S}}_{[\mathrm{S}]/\mathrm{S}}(\mathcal{B}^2{\upharpoonright}K^a_4, \mathcal{B}{\upharpoonright}((\{\mathsf{f},a,\mathsf{t}\}[\setminus\{a\}])/\{\mathsf{f},a,\mathsf{t}\}))$;*
(iv) *$(\mathrm{Cn}_{\mathcal{B}{\upharpoonright}\{\mathsf{f},a,\mathsf{t}\}}(\varnothing) = \mathrm{Cn}_{\mathcal{B}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}}(\varnothing))/(\mathrm{Cn}_{\mathcal{B}{\upharpoonright}\{\mathsf{f},a,\mathsf{t}\}} = \mathrm{Cn}_{\mathcal{B}^2{\upharpoonright}K^a_4})$.*

Proof. First, assume (i) holds. Let $\mathfrak{D}$ be the subalgebra of $\mathfrak{B}^2$ generated by K^a_4, in which case it is a subalgebra of $(\mathfrak{B}{\upharpoonright}\{\mathsf{f},a,\mathsf{t}\}) \times (\mathfrak{B}{\upharpoonright}\{\mathsf{f},\mathsf{t}\})$, for both $\{\mathsf{f},[a,]\mathsf{t}\} = \pi_{0[+1]}[K^a_4]$ form subalgebras of $\mathfrak{B}$. Then, if $\langle\mathsf{t},\mathsf{f}\rangle$ was in D, there would be some $\varphi \in \mathrm{Fm}^4_{\Sigma}$ such that both $\varphi^{\mathfrak{B}}(\mathsf{f},a,a,\mathsf{t}) = \mathsf{t}$ and $\varphi^{\mathfrak{B}}(\mathsf{f},\mathsf{f},\mathsf{t},\mathsf{t}) = \mathsf{f}$, in which case, since $(\mathsf{n}/\mathsf{b}) \sqsubseteq / \sqsupseteq b$, for every $b \in \{\mathsf{f},\mathsf{t}\}$, by the regularity of $\mathfrak{B}{\upharpoonright}\{\mathsf{f},a,\mathsf{t}\}$, we would get $\mathsf{t} \sqsubseteq / \sqsupseteq \mathsf{f}$. Therefore, as $\wr^{\mathfrak{B}}(\mathsf{f}|\mathsf{t}) = (\mathsf{t}|\mathsf{f})$, we conclude that $D = K^a_4$, and so (ii) holds.

(Finally, (ii)⇒(iii) is immediate, while (iii)⇒(iv) is by (2.6) and (2.7) as well as the fact that $\mathcal{B}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}$ is a submatrix of $\mathcal{B}{\upharpoonright}\{\mathsf{f},a,\mathsf{t}\}$, as required.) □

Lemma 4.33. *Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\wr$-subclassical; cf. Corollary 4.17). Then, (i)⇔(ii)⇐(iii)⇒(iv), where:*

(i) *$\wr(x_0 \mathbin{\overline{\wedge}} \wr x_0) \notin C(\varnothing)$;*
(ii) *neither $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$ (that is, $C(x_0) = C(\wr\wr x_0)$) nor $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$;*
(iii) *$L_5 \triangleq ((A \times \{\mathsf{f},\mathsf{t}\}) \setminus \{\langle\mathsf{b},\mathsf{f}\rangle\})$ forms a subalgebra of $\mathfrak{A}^2$;*
(iv) *C^{NP} has a proper non-axiomatic extension being both that of C and a proper sublogic of C^{MP}, being, in its turn, an axiomatic extension of C, and so of C^{NP}.*

Proof. First, (i)⇔(ii) is immediate.

Next, if $(\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b})/(\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t})$, then we have $(\wr^{\mathfrak{A}^2}\langle\mathsf{b},\mathsf{t}\rangle/(\langle\mathsf{b},\mathsf{t}\rangle \mathbin{\overline{\wedge}}^{\mathfrak{A}^2} \langle\mathsf{t},\mathsf{f}\rangle)) = \langle\mathsf{b},\mathsf{f}\rangle \notin L_5$, in which case $L_5 \supseteq \{\langle\mathsf{b},\mathsf{t}\rangle,\langle\mathsf{t},\mathsf{f}\rangle\}$ does not form a subalgebra of $\mathfrak{A}^2$, and so (iii)⇒(ii) holds.

Further, assume (iii) holds, in which case (ii) holds too, as it has been proved above. Then, by (2.6) and Theorem 4.29, the consistent Σ-logic C' of the consistent submatrix $\mathcal{D} \triangleq (\mathcal{A}^2{\upharpoonright}L_5)$ of $\mathcal{B} \triangleq (\mathcal{A}^2{\upharpoonright}(A \times \{\mathsf{f},\mathsf{t}\}))$, defining C^{NP}, is an extension of $C^{[\mathrm{NP}]}$, and so a sublogic of $C^{\mathrm{PC}} = C^{\mathrm{MP}}$ (cf. Theorem 4.18 and Corollary 4.31). Moreover, (4.1) is not true in $\mathcal{D}$ under $[x_0/\langle\mathsf{b},\mathsf{t}\rangle, x_1/\langle\mathsf{f},\mathsf{t}\rangle]$, and so C' is a proper sublogic of C^{MP}. And what is more, since, for all $a \in D = L_5$, it holds that $(\wr^{\mathfrak{D}}a \in D^{\mathcal{D}}) \Rightarrow (a = \langle\mathsf{f},\mathsf{f}\rangle)$, while $\mathcal{A}$ is $\veebar$-disjunctive, whereas $\mathsf{f} \notin D^{\mathcal{A}}$, we conclude that

$$\{\wr x_0, x_0 \veebar x_1\} \vdash x_1 \tag{4.5}$$

is true in $\mathcal{D}$. However, (4.5) is not true in $\mathcal{B}$ under $[x_0/\langle\mathsf{b},\mathsf{f}\rangle, x_1/\langle\mathsf{f},\mathsf{t}\rangle]$, and so C' is a proper extension of $C^{[\mathrm{NP}]}$. In addition, $(\pi_0{\upharpoonright}D) \in \hom^{\mathrm{S}}(\mathcal{D},\mathcal{A})$, in which case, by (2.7), we have $C(\varnothing) \subseteq C^{\mathrm{NP}}(\varnothing) \subseteq C'(\varnothing) \subseteq C(\varnothing)$, and so C' is not an axiomatic extension of $C^{[\mathrm{NP}]}$. Finally, by (ii), $\mathcal{A}$ is $\neg$-negative, where $\neg x_0 \triangleq \wr(x_0 \mathbin{\overline{\wedge}} (\wr\wr x_0 \veebar \wr x_0))$, in which case it, being $\veebar$-disjunctive, is $\rhd$-implicative, where $(x_0 \rhd x_1) \triangleq (\neg x_0 \veebar x_1)$, and so Corollary 4.25 completes the argument of (iv), as required. □

Lemma 4.34. *Let C' be an extension of C. Suppose (4.1) is not satisfied in C' and L_5 does not form a subalgebra of $\mathfrak{A}^2$ (in particular, $\wr(x_0 \mathbin{\overline{\wedge}} \wr x_0) \in C(\varnothing)$, i.e., either $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$ — that is, $C(x_0) = C(\wr\wr x_0)$ — or $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$; cf. Lemma 4.33(iii)⇒(ii)⇔(i)). Then, C' is a sublogic of C^{NP}.*

Proof. The case, when C^{NP} is inconsistent, is evident. Otherwise, by Theorem 4.29, C is $\wr$-subclassical, in which case $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, C^{NP} being defined by the submatrix $\mathcal{B} \triangleq (\mathcal{A} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}))$ of $\mathcal{A}^2$, and so it suffices to prove that $\mathcal{B} \in \mathrm{Mod}(C')$. On the other hand, as C' does not satisfy (4.1), by Theorem 2.20, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it not being a model of (4.1), in which case there are some $a \in D^{\mathcal{D}} \subseteq \{\mathsf{b},\mathsf{t}\}^I$ and some $b \in (D \setminus D^{\mathcal{D}})$ such that $(\wr^{\mathfrak{D}} a \veebar^{\mathfrak{D}} b) \in D^{\mathcal{D}}$, and so $J \triangleq \{i \in I \mid \pi_i(a) = \mathsf{b}\} \supseteq K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{f}\} \neq \varnothing$. Put $L \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$. Then, given any $\vec{a} \in A^5$, set $(a_0|a_1|a_2|a_3|a_4) \triangleq ((((I \setminus (L \cup K)) \cap J) \times \{a_0\}) \cup ((I \setminus (L \cup J)) \times \{a_1\}) \cup ((L \setminus J) \times \{a_2\}) \cup ((L \cap J) \times \{a_3\}) \cup (K \times \{a_4\})) \in A^I$. In this way:

$$D \ni a = (\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{b}|\mathsf{b}), \tag{4.6}$$

$$D \ni b = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{f}). \tag{4.7}$$

Moreover, by Lemma 3.11, we also have:

$$D \ni f \triangleq (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}), \tag{4.8}$$

$$D \ni \wr^{\mathfrak{D}} f = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}). \tag{4.9}$$

Consider the following exhaustive (as $\wr^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}} = \{\mathsf{b},\mathsf{t}\}$) cases:

(1) $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$.
Then, in case $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$, by (4.6) and (4.7), we have:

$$D \ni e \triangleq (a \mathbin{\bar{\wedge}}^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{f}), \tag{4.10}$$

$$D \ni \wr^{\mathfrak{D}} e = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{t}), \tag{4.11}$$

$$D \ni c \triangleq (e \veebar^{\mathfrak{D}} \wr^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{t}), \tag{4.12}$$

$$D \ni \wr^{\mathfrak{D}} c = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{f}). \tag{4.13}$$

Likewise, in case $\mathsf{b}(\leq / \geq)^{\mathfrak{A}}\mathsf{t}$, by (4.6) and (4.10)/(4.7), we have:

$$D \ni d \triangleq ((e/b) \veebar^{\mathfrak{D}} \wr^{\mathfrak{D}} a) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{b}), \tag{4.14}$$

$$D \ni \wr^{\mathfrak{D}} d = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{b}). \tag{4.15}$$

Consider the following complementary subcases:

(a) $L \subseteq J$.
Then, since $I \supseteq K \neq \varnothing = (L \setminus J)$, by (4.8), (4.9) and (4.14), $\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.6), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{B}$, for $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

(b) $L \nsubseteq J$.
Then, consider the following complementary subsubcases:

(i) there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{f}) = \mathsf{t}$,
in which case, by (4.8) and (4.15), we have:

$$D \ni \varphi^{\mathfrak{D}}(\wr^{\mathfrak{D}} d, f) = (\mathsf{f}|\mathsf{f}|\mathsf{t}|\mathsf{f}|\mathsf{f}), \tag{4.16}$$

$$D \ni \wr^{\mathfrak{D}}\varphi^{\mathfrak{D}}(\wr^{\mathfrak{D}} d, f) = (\mathsf{t}|\mathsf{t}|\mathsf{f}|\mathsf{t}|\mathsf{t}). \tag{4.17}$$

Then, since $(L \setminus J) \neq \varnothing \neq K$, taking (4.8), (4.9), (4.14), (4.15), (4.16) and (4.17) into account, we conclude that $\{\langle\langle g, h\rangle, (g|g|h|g|g)\rangle \mid \langle g, h\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$, and so, by (2.6), $\mathcal{B}$ is a model of C', for $\mathcal{D}$ is so.

(ii) there is no $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{f}) = \mathsf{t}$,
Then, $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$, for, otherwise, we would have $\mathsf{t} \leq^{\mathfrak{A}} \mathsf{b}$, in which case we would get $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{f}) = \mathsf{t}$, where $\varphi \triangleq \wr(x_0 \mathbin{\bar{\wedge}} \wr x_1) \in \mathrm{Fm}^2_\Sigma$. Consider the following complementary subsubsubcases:

(A) $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) = \varnothing$.
Then, taking (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15) into account, as $K \neq \varnothing \neq (L \setminus J)$, we see that $\{\langle\langle g, h\rangle, (\mathsf{b}|\mathsf{b}|h|\mathsf{b}|g)\rangle \mid \langle g, h\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$, and so, by (2.6), $\mathcal{B}$ is a model of C', for $\mathcal{D}$ is so.

(B) $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing$.
Let $\mathfrak{G}$ be the subalgebra of $\mathfrak{B} \times \mathfrak{A}$ generated by $(B \dotplus 2) \triangleq ((B \times \{\mathsf{b}\}) \cup \{\langle\langle i, i\rangle, i\rangle \mid i \in \{\mathsf{f},\mathsf{t}\}\})$. Then, as $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing \notin \{K, L \setminus J\}$, by (4.8), (4.9), (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15), we see that $\{\langle\langle\langle g, h\rangle, j\rangle, (j|j|h|j|g)\rangle \mid \langle\langle g, h\rangle, j\rangle \in G\}$ is an embedding of $\mathcal{G} \triangleq ((\mathcal{B} \times \mathcal{A}){\restriction}G$ into $\mathcal{D}$, in which case, by (2.6), $\mathcal{G}$ is a model of C', for $\mathcal{D}$ is so. Let us prove, by contradiction, that $((D^{\mathcal{B}} \times \{\mathsf{f}\}) \cap G) = \varnothing$. For suppose $((D^{\mathcal{B}} \times \{\mathsf{f}\}) \cap G) \neq \varnothing$. Then, there is some $\psi \in \mathrm{Fm}^8_\Sigma$ such that $\psi^{\mathfrak{A}}(\mathsf{t},\mathsf{b},\mathsf{b},\mathsf{b},\mathsf{b},\mathsf{b},\mathsf{b},\mathsf{f}) = \mathsf{f}$ and $\psi^{\mathfrak{A}}(\mathsf{t},\mathsf{t},\mathsf{t},\mathsf{t},\mathsf{f},\mathsf{f},\mathsf{f},\mathsf{f}) = \mathsf{t}$, for $\pi_1[D^{\mathcal{B}}] = \{\mathsf{t}\}$. Let $\varphi \triangleq (\wr x_1, \wr x_0, \wr x_0, \wr x_0, x_0, x_0, x_0, x_1) \in \mathrm{Fm}^2_\Sigma$. Then, $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{f}) = \mathsf{t}$. This contradiction shows that $((D^{\mathcal{B}} \times \{\mathsf{f}\}) \cap G) = \varnothing$, in which case $(\pi_0{\restriction}G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G},\mathcal{B})$, and so, by (2.6), $\mathcal{B}$ is a model of C', for $\mathcal{G}$ is so.

(2) $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$,
Consider the following exhaustive (as $\langle A, \leq^{\mathfrak{A}}\rangle$ is a chain poset) subcases:

(a) $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$.
Then, by (4.6) and (4.7), we get:

$$D \ni c' \triangleq (a \veebar^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{b}), \tag{4.18}$$

(4.19) $$D \ni d' \triangleq \wr^{\mathfrak{D}} c' = (\mathsf{t}|\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{t}),$$

(4.20) $$D \ni e' \triangleq \wr^{\mathfrak{D}} d' = (\mathsf{f}|\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{f}),$$

(4.21) $$D \ni f' \triangleq (c' \mathbin{\overline{\wedge}}^{\mathfrak{D}} d') = (\mathsf{b}|\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{b}).$$

Consider the following complementary subsubcases:

(i) $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) = \varnothing$.
Then, since $I \supseteq K \neq \varnothing$, by (4.8), (4.9) and (4.18), we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.6), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{B}$, for $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

(ii) $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) \neq \varnothing$.
Then, as $K \neq \varnothing$, by (4.8), (4.9), (4.18), (4.19), (4.20) and (4.21), we conclude that $\{\langle\langle g, h\rangle, (g|h|h|h|g)\rangle \mid \langle g, h\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$, in which case, by (2.6), $\mathcal{B}$ is a model of C', for $\mathcal{D}$ is so.

(b) $\mathsf{t} \leq^{\mathfrak{A}} \mathsf{b}$.
Then, by (4.6) and (4.7), we get:

(4.22) $$D \ni c'' \triangleq (a \mathbin{\underline{\vee}}^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{b}),$$

(4.23) $$D \ni d'' \triangleq \wr^{\mathfrak{D}} c'' = (\mathsf{t}|\mathsf{t}|\mathsf{f}|\mathsf{t}|\mathsf{t}),$$

(4.24) $$D \ni e'' \triangleq \wr^{\mathfrak{D}} d'' = (\mathsf{f}|\mathsf{f}|\mathsf{t}|\mathsf{f}|\mathsf{f}).$$

Consider the following complementary subsubcases:

(i) $L \subseteq J$.
Then, as $K \neq \varnothing = (L \setminus J)$, taking (4.8), (4.9) and (4.22) into account, we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.6), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{B}$, for $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

(ii) $L \nsubseteq J$.
Then, since L_5 does not form a subalgebra of $\mathfrak{A}^2$, and so of its subalgebra $\mathfrak{B}$, there is some $\varphi \in \mathrm{Fm}^5_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{t}) = \mathsf{b}$ and $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{f}, \mathsf{t}, \mathsf{t}, \mathsf{t}) = \mathsf{f}$, in which case, by (4.8), (4.9), (4.22), (4.23) and (4.24), we get:

(4.25) $$D \ni f'' \triangleq \varphi^{\mathfrak{D}}(f, d'', e'', c'', \wr^{\mathfrak{D}} f) = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{b}),$$

and so, as $K \neq \varnothing \neq (L \setminus J)$, taking (4.8), (4.9), (4.22), (4.23), (4.24) and (4.25) into account, we see that $\{\langle\langle g, h\rangle, (g|g|h|g|g)\rangle \mid \langle g, h\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$, in which case, by (2.6), $\mathcal{B}$ is a model of C', for $\mathcal{D}$ is so. ☐

Theorem 4.35. *Suppose C is [not] non-$\wr$-subclassical. Then, extensions of C form the $(2[+2])$-element chain $C \subsetneq C^{\mathrm{NP}} = [\mathrm{Cn}^\omega_{\mathcal{A}\times(\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\})} \subsetneq] C^{\mathrm{MP}} = [C^{\mathrm{PC}} \subsetneq] \mathrm{Cn}^\omega_\varnothing$, C^{NP} [not] being axiomatic$|\underline{\vee}$-disjunctive, [iff L_5 does not form a subalgebra of $\mathfrak{A}^2$ (in particular, $\wr(x_0 \mathbin{\overline{\wedge}} \wr x_0) \in C(\varnothing)$, i.e., either $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$ — that is, $C(x_0) = C(\wr\wr x_0)$ — or $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$), in which case C^{PC} is $\underline{\vee}$-disjunctive, while, providing $\mathcal{A}$ is $\rhd$-implicative, where $\rhd \in \mathrm{Fm}^2_\Sigma$,/ $K^{\mathsf{b}}_{3(+1)}$ forms a subalgebra of $\mathfrak{A}^2$ (in particular, $\mathfrak{A}$ is regular), C^{PC} is relatively axiomatized by (4.4)/ $C^{\mathrm{PC}}(\varnothing) = C(\varnothing)$, in which case C^{PC} is an axiomatic extension of C/ both proper consistent extensions of C are not axiomatic, and so C has a unique/no proper consistent axiomatic extension].*

Proof. By Theorems 4.18, 4.23, 4.29, Lemmas 4.32, 4.33, 4.34 and Corollaries 3.17, 4.16, 4.17, 4.25, 4.31 and Proposition 3.20. ☐

The following sample counterexample collectively with Lemma 4.33(iii)⇒(iv) show that the condition of L_5's not forming a subalgebra of $\mathfrak{A}^2$ cannot be omitted in the formulations of Lemma 4.34 and Theorem 4.35:

Example 4.36. Let $\Sigma = (\Sigma_+ \cup \{\wr\})$, $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$, $\overline{\wedge} = \wedge$, $\underline{\vee} = \vee$ and $\mathsf{f} \leq^{\mathfrak{A}} \mathsf{t} \leq^{\mathfrak{A}} \mathsf{b}$, in which case $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\wr$-subclassical; cf. Corollary 4.17), while L_5 forms a subalgebra of $\mathfrak{A}^2$. ☐

4.4.2.2. Self-extensionality. Let $h_{\mathsf{b}\to\mathsf{f}} \triangleq ((\{\mathsf{b}, \mathsf{f}\} \times \{\mathsf{f}\}) \cup \{\langle \mathsf{t}, \mathsf{t}\rangle\}) : A \to \{\mathsf{f}, \mathsf{t}\}$.

Lemma 4.37. *Suppose $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\wr$-subclassical; cf. Corollary 4.17). Let $h \in \hom(\mathfrak{A}, \mathfrak{A}\restriction\{\mathsf{f}, \mathsf{t}\})$. Then, $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{t} \geq^{\mathfrak{A}} \mathsf{b}$, in which case the* De Morgan *identities:*

(4.26) $$\wr(x_0 \mathbin{\overline{\wedge}} x_1) \approx (\wr x_0 \mathbin{\underline{\vee}} \wr x_1),$$

(4.27) $$\wr(x_0 \mathbin{\underline{\vee}} x_1) \approx (\wr x_0 \mathbin{\overline{\wedge}} \wr x_1)$$

are true in $\mathfrak{A}$, while $h = h_{\mathsf{b}\to\mathsf{f}}$, in which case $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\langle\mathfrak{A}, \{\mathsf{t}\}\rangle, \mathcal{A}\restriction\{\mathsf{f}, \mathsf{t}\})$, and so C^{PC} is defined by $\langle\mathfrak{A}, \{\mathsf{t}\}\rangle$, in view of (2.6).

Proof. If $\wr^{\mathfrak{A}}\mathsf{b}$ was not equal to t, then it would be equal to b, in which case we would get $\wr^{\mathfrak{A}}h(\mathsf{b}) = h(\wr^{\mathfrak{A}}\mathsf{b}) = h(\mathsf{b}) \in \{\mathsf{f}, \mathsf{t}\}$. Therefore, we first get $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$. Likewise, if it did hold that $\mathsf{t} \not\geq^{\mathfrak{A}} \mathsf{b}$, we would have $\mathsf{b} \geq^{\mathfrak{A}} \mathsf{t}$, in which case, by Lemma 3.30 and the fact that $\mathfrak{A}\restriction\{\mathsf{f}, \mathsf{t}\}$ has no proper subalgebra, we would get $h(\mathsf{b}) = \mathsf{t}$ and $h(\mathsf{f}) = \mathsf{f}$, and so we would eventually get $\mathsf{f} = \wr^{\mathfrak{A}}\mathsf{t} = \wr^{\mathfrak{A}}h(\mathsf{b}) = h(\wr^{\mathfrak{A}}\mathsf{b}) = h(\mathsf{t}) = h(\wr^{\mathfrak{A}}\mathsf{f}) = \wr^{\mathfrak{A}}h(\mathsf{f}) = \wr^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$. Thus, $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$, in which case, by Lemma 3.30 and the fact that $\mathfrak{A}\restriction\{\mathsf{f}, \mathsf{t}\}$ has no proper subalgebra, we get $h(\mathsf{t}) = \mathsf{t}$ and $h(\mathsf{f}) = \mathsf{f}$, and so, as $h(\mathsf{b}) \in \{\mathsf{f}, \mathsf{t}\}$, we also get $h(\mathsf{b}) = \wr^{\mathfrak{A}}\wr^{\mathfrak{A}}h(\mathsf{b}) = \wr^{\mathfrak{A}}h(\wr^{\mathfrak{A}}\mathsf{b}) = \wr^{\mathfrak{A}}h(\mathsf{t}) = \wr^{\mathfrak{A}}\mathsf{t} = \mathsf{f} = h(\mathsf{f})$, as required. ☐

Theorem 4.38. *The following are equivalent:*

(i) *C is self-extensional;*

(ii) *$h_{\mathsf{b}\to\mathsf{f}}$ is an endomorphism of $\mathfrak{A}$;*

(iii) $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}$ *and* $\hom(\mathfrak{A},\mathfrak{A}\upharpoonright\{\mathsf{f},\mathsf{t}\})\neq\varnothing$*;*
(iv) $\langle\mathfrak{A},\{\mathsf{t}\}\rangle\in\mathrm{Mod}(C)$*;*
(v) C *has a truth-singular* $\veebar$*-disjunctive model with underlying algebra* $\mathfrak{A}$*;*
(vi) $(\psi\in C(\phi))\Leftrightarrow(\mathfrak{A}\models((\phi\barwedge\psi)\approx\phi))$*, for all* $\phi,\psi\in\mathrm{Fm}^{\omega}_{\Sigma}$*;*
(vii) *there is some class* K *of* Σ*-algebras satisfying the idempotencity and commutativity identities for* $\barwedge$ *such that* $(\psi\in C(\phi))\Leftrightarrow(\mathsf{K}\models((\phi\barwedge\psi)\approx\phi))$*, for all* $\phi,\psi\in\mathrm{Fm}^{\omega}_{\Sigma}$*;*
(viii) $(\psi\equiv_C\phi)\Leftrightarrow(\mathfrak{A}\models(\phi\approx\psi))$*, for all* $\phi,\psi\in\mathrm{Fm}^{\omega}_{\Sigma}$*;*
(ix) *there is some class* K *of* Σ*-algebras such that* $(\psi\equiv_C\phi)\Leftrightarrow(\mathsf{K}\models(\phi\approx\psi))$*, for all* $\phi,\psi\in\mathrm{Fm}^{\omega}_{\Sigma}$*;*
(x) *any truth-non-empty* $\barwedge$*-conjunctive* Σ*-matrix with underlying algebra in* $\mathbf{V}(\mathfrak{A})$ *is a model of* C*;*
(xi) *any truth-non-empty* $\barwedge$*-conjunctive* Σ*-matrix with underlying algebra* $\mathfrak{A}$ *is a model of* C*;*
(xii) C *has the property of Weak Contraposition with respect to* $\wr$*;*

in which case $\mathrm{IV}(C)=\mathbf{V}(\mathfrak{A})$.

Proof. First, the equivalence of (i,vi–xi) as well as the final conclusion are due to Theorem 3.9, Remark 4.3.

Next, (ii)⇒(iii) is by the fact that $(\mathrm{img}\,h_{\mathsf{b}\to\mathsf{f}})=\{\mathsf{f},\mathsf{t}\}$, while the converse is by Lemma 4.37, whereas (iii)⇒(iv) is by Corollary 4.17 and Lemma 4.37.

Further, assume (iv) holds. Then, $\langle\mathfrak{A},\{\mathsf{t}\}\rangle$ is $\barwedge$-conjunctive, for $\mathcal{A}$ is so, in which case t is the greatest element of the chain poset $\langle A,\leq^{\mathfrak{A}}\rangle$, and so $\langle\mathfrak{A},\{\mathsf{t}\}\rangle$ is $\veebar$-disjunctive. Thus, (v) holds.

Furthermore, assume (v) holds. Take any truth-singular $\veebar$-disjunctive $\mathcal{B}\in\mathrm{Mod}(C)$ such that $\mathfrak{B}=\mathfrak{A}$, in which case it is consistent, for $|B|=|A|=3>1=|D^{\mathcal{B}}|$, while $\mathcal{E}\triangleq(\mathcal{B}/\theta)$, where $\theta\triangleq\eth(\mathcal{B})$, is truth-singular and, by Corollary 2.17, is simple. Then, by Lemma 2.19, there are some finite set I, some $\overline{\mathcal{C}}\in\mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it and some $g\in\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{E})$, in which case, by (2.6) and Remark 3.15, $\mathcal{D}$ is both consistent and $\veebar$-disjunctive, for $\mathcal{B}$ is so. Hence, by Remark 3.15 and Corollary 3.16, there is some $i\in I$ such that $h\triangleq(\pi_i\upharpoonright D)\in\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{C}_i)$. Moreover, by Lemmas 3.4, 3.6 and Remark 4.3, $\mathcal{C}_i$ is simple. Therefore, by Proposition 2.16, we have $(\ker h)=\eth(\mathcal{D})=(\ker g)$. Hence, by Proposition 2.15, $e\triangleq(h\circ g^{-1})$ is an isomorphism from $\mathcal{E}$ onto $\mathcal{C}_i$, in which case $\mathcal{C}_i$ is truth-singular, for $\mathcal{E}$ is so, and so $\mathcal{C}_i\neq\mathcal{A}$, for $\mathcal{A}$ is not truth-singular. On the other hand, $\mathcal{A}$ may have no proper consistent submatrix other than that with carrier $\{\mathsf{f},\mathsf{t}\}$. In this way, $C_i=\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, while $(e\circ\nu_\theta)\in\hom(\mathfrak{A},\mathfrak{A}\upharpoonright\{\mathsf{f},\mathsf{t}\})$. Thus, (iii) holds.

In this way, we have proved the equivalence of (ii–v).

Now, assume (xi) holds. Let a be the greatest element of the finite chain poset $\langle A,\leq^{\mathfrak{A}}\rangle$. Then, $\langle\mathfrak{A},\{a\}\rangle$ is both truth-non-empty, truth-singular, $\barwedge$-conjunctive and $\veebar$-disjunctive, in which case, by (xi), it is a model of C, and so (v) holds.

Conversely, assume (v) holds, in which case (iii,iv) hold as well, and so, by Lemma 4.37, $\mathsf{b}\leq^{\mathfrak{A}}\mathsf{t}$. In that case, $\mathcal{A}$ and $\langle\mathfrak{A},\{\mathsf{t}\}\rangle$ are exactly all consistent truth-non-empty $\barwedge$-conjunctive Σ-matrices with underlying algebra $\mathfrak{A}$. Thus, (xi) holds.

In this way, we have proved the equivalence of (i–xi).

Finally, assume (i) holds, in which case (iii) holds as well, and so, by Lemma 4.37, $(4.26)\in\equiv_C$. Then, (xii) is by Corollary 3.17 and the following claim:

Claim 4.39. *Any self-extensional extension* C' *of a* $\barwedge$*-conjunctive weakly* $\veebar$*-disjunctive* Σ*-logic such that* $(4.26)\in\equiv_{C'}$ *has the property of Weak Contraposition with respect to* $\wr$*.*

Proof. In that case, C' is $\barwedge$-conjunctive and satisfies (3.4). Consider any $\phi,\psi\in\mathrm{Fm}^{\omega}_{\Sigma}$ such that $\psi\in C'(\phi)$. Then, we have $C'(\phi\wedge\psi)=C'(\{\phi,\psi\})=C'(\phi)$, in which case, by the self-extensionality of C', we get $C'(\sim\psi)\supseteq C'(\sim\phi\vee\sim\psi)=C'(\sim(\phi\wedge\psi))=C'(\sim\phi)\ni\sim\phi$, as required. □

Conversely, assume (xii) holds. Clearly, $\langle\mathfrak{A},\{\mathsf{t}\}\rangle$ is truth-non-empty and, being truth-singular, is dual-weakly $\barwedge$-conjunctive, in view of the idempotencity identity for $\barwedge$ true in $\mathfrak{A}$. Then, (iv) is by the following claim:

Claim 4.40. *Let* C' *be an inductive weakly* $\barwedge$*-conjunctive* Σ*-logic,* $\mathcal{B}\in\mathrm{Mod}(C')$ *and* $\mathcal{D}\triangleq\langle\mathfrak{B},(\wr^{\mathfrak{B}})^{-1}[B\setminus D^{\mathcal{B}}]\rangle$*. Suppose* C' *has the Property of Weak Contraposition with respect to* $\wr$*, while* $\mathcal{D}$ *is both truth-non-empty and dual-weakly* $\barwedge$*-conjunctive. Then,* $\mathcal{D}\in\mathrm{Mod}(C')$*.*

Proof. Consider any $\phi\in\mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi\in C'(\phi)$, in which case $\wr\phi\in C'(\wr\psi)$, and so $\wr\phi\in\mathrm{Cn}^{\omega}_{\mathcal{B}}(\wr\psi)$, and any $h\in\hom(\mathfrak{Fm}^{\omega}_{\Sigma},\mathfrak{B})$. Suppose $h(\phi)\in D^{\mathcal{D}}$, in which case $h(\wr\phi)=\wr^{\mathfrak{B}}h(\phi)\notin D^{\mathcal{B}}$, and so $\wr^{\mathfrak{B}}h(\psi)=h(\wr\psi)\notin D^{\mathcal{B}}$, in which case $h(\psi)\in D^{\mathcal{D}}$, and so $\psi\in\mathrm{Cn}^{\omega}_{\mathcal{D}}(\phi)$, as required, in view of Claim 3.10. □

Theorem 4.38(i)⇔(ii) yields an effective (in case Σ is finite) — equally useful heuristically — purely algebraic criterion of the self-extensionality of C. By Corollary 4.17, Theorem 4.38(i)⇒(iii) and Lemma 4.37, we immediately have:

Corollary 4.41. C *is* $\wr$*-subclassical, whenever it is self-extensional.*

Corollary 4.42. *Suppose* $\wr^{\mathfrak{A}}\mathsf{b}=\mathsf{b}$ *(i.e.,* $C(x_0)=C(\wr\wr x_0)$*). Then,* C *is not self-extensional.*

This covers arbitrary three-valued expansions of both LP and HZ (cf. Subsubsections 7.2.1 and 7.2.3).

Corollary 4.43. C^{NP} *is self-extensional iff it is inconsistent.*

Proof. The "if" part is immediate. Conversely, assume C^{NP} is consistent. Let us prove, by contradiction, that it is not self-extensional. For suppose it is self-extensional. Then, by Theorems 4.29, 4.38(i)⇒(ii) and Lemma 4.37, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, while C^{NP} is defined by $\mathcal{B}\triangleq(\mathcal{A}\times(\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}))$, whereas $\wr^{\mathfrak{A}}\mathsf{b}=\mathsf{t}\geq^{\mathfrak{A}}\mathsf{b}$. Moreover, by (2.3) and the $\barwedge$-conjunctivity of $\mathcal{A}$, we have $((x_0\barwedge\wr x_0)\barwedge x_1)\equiv_{C^{\mathrm{NP}}}(x_0\barwedge\wr x_0)$. Hence, by the self-extensionality of C^{NP}, we get $(((x_0\barwedge\wr x_0)\barwedge x_1)\veebar x_2)\equiv_{C^{\mathrm{NP}}}((x_0\barwedge\wr x_0)\veebar x_2)$. However, this contradicts to the fact that the rule $((x_0\barwedge\wr x_0)\veebar x_2)\vdash(((x_0\barwedge\wr x_0)\barwedge x_1)\veebar x_2)$ is not true in $\mathcal{B}$ under $[x_0/\langle\mathsf{b},\mathsf{f}\rangle,x_1/\langle\mathsf{f},\mathsf{f}\rangle,x_2/\langle\mathsf{f},\mathsf{t}\rangle]$. □

Corollary 4.44. *Suppose C is self-extensional. Then, extensions of C form the four-element chain $C \subsetneq C^{\mathrm{NP}} \subsetneq C^{\mathrm{MP}} \subsetneq \mathrm{Cn}^{\omega}_{\varnothing}$, C^{NP} being the only non-self-extensional one. In particular, any extension of C is self-extensional iff it is $\veebar$-disjunctive.*

Proof. By Theorems 4.23, 4.29, 4.35, 4.38, Corollaries 4.17, 4.31, 4.41, 4.43 and Remark 2.12. □

4.4.3. *Disjunctive three-valued paraconsistent logics with subclassical negation and classically-valued operations.* An n-ary, where $n \in \omega$, operation f on A is said to be *classically-valued*, if $(\mathrm{img}\, f) \subseteq \{\mathsf{f}, \mathsf{t}\}$.

Throughout this subsubsection, it is supposed that C is $\veebar$-disjunctive (that is, $\mathcal{A}$ is so; cf. Corollary 4.19) and all primary operations of $\mathfrak{A}$ are classically-valued, in which case:

- $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so C is both $\wr$-subclassical (cf. Corollary 4.24) and maximally $\wr$-paraconsistent (cf. Theorem 4.26);
- $\mathcal{A}$ is both $\neg$-negative, $\bar{\wedge}$-conjunctive and $\rhd$-negative, where:

$$\neg x_0 \triangleq \sim(x_0 \veebar x_0),$$
$$(x_0 \bar{\wedge} x_1) \triangleq \neg(\neg x_0 \veebar \neg x_1),$$
$$(x_0 \rhd x_1) \triangleq (\neg x_0 \veebar x_1),$$

  and so C^{PC} is an extension of any consistent extension of C (cf. Theorem 4.18), while $\varepsilon^{\rhd}_{\wr} \triangleq \{\wr^i x_j \rhd \wr^i x_{1-j} \mid i, j \in 2\}$ is an axiomatic binary equality determinant for $\mathcal{A}$ (cf. Remark 4.3).

Theorem 4.45. *There is an increasing countable chain of finitary extensions of C, and so such finitary extension of C that is not (relatively) finitely-axiomatizable, in which case this is consistent.*

Proof. We use Theorem 2.20 with $\mathsf{K} \triangleq \mathrm{Mod}(C)$ tacitly.

Let $n \in (\omega \setminus 1)$ and C_n the finitary (for C, being three-valued, is so) extension of C relatively axiomatized by the rule $R_n \triangleq ((\{\wr x_i \mid i \in n\} \cup \{\veebar\langle x_i\rangle_{i\in n}\}) \vdash x_n)$. Then, as C, being $\veebar$-disjunctive, satisfies (3.4), and so does any $\mathcal{B} \in \mathsf{K}$, providing R_n is not true in $\mathcal{B}$ under any $v : V_{n+1} \to B$, for every $m \in (\omega \setminus n)$, R_m is not true in $\mathcal{B}$ under $v \cup [x_j/v(x_0); x_m/v(x_n)]_{j\in(m\setminus n)}$. In this way, $\langle C_n\rangle_{i\in n}$ is an increasing denumerable chain of finitary extensions of C.

Claim 4.46. *For every $n \in (\omega \setminus 1)$, there is a subdirect n-power $\mathcal{A}_n \in \mathsf{K}$ of $\mathcal{A}$ such that R_n is [not] true in $\mathcal{A}_{n+1[-1]}$.*

Proof. Since all primary operations of $\mathfrak{A}$ are classically-valued, the set $A_n \triangleq (\{\mathsf{f}, \mathsf{t}\}^n \cup \{\{\langle i, \mathsf{b}\rangle\} \cup ((n \setminus \{i\}) \times \{\mathsf{f}\}) \mid i \in n\})$ forms a subalgebra of $\mathfrak{A}^n$, so we have the subdirect n-power $\mathcal{A}_n \triangleq (\mathcal{A}^n \restriction A_n) \in \mathsf{K}$ of $\mathcal{A}$ (cf. (2.6)). Then, as $\mathcal{A}$ is $\veebar$-disjunctive, R_n is not true in $\mathcal{A}_n$ under $[x_i/(\{\langle i, \mathsf{b}\rangle\} \cup ((n \setminus \{i\}) \times \{\mathsf{f}\})); x_n/(n \times \{\mathsf{f}\})]_{i\in n}$. However, R_n is true in $\mathcal{A}_{n+1}$, as required. □

Then, by Claim 4.46, the increasing chain $\langle C_n\rangle_{n\in(\omega\setminus 1)}$ is injective, and so countable, in which case the finitary (for C, being three-valued, is so) extension C_ω of C relatively axiomatized by $\{R_n \mid n \in (\omega \setminus 1)\}$ is a proper extension of C_n, for any $n \in (\omega \setminus 1)$, and so, by the Compactness Theorem for classes of algebraic systems closed under ultra-products (cf. [15]) — in particular, finitary logic model classes, being universal Horn model classes axiomatized by calculi of all rules satisfied in finitary logics, C_ω is not (relatively) finitely axiomatizable, as required. □

As it is demonstrated by Subsubsections 7.2.1, 7.2.3 and 7.2.4, the condition of $\mathfrak{A}$'s primary operations' being classically-valued cannot be omitted in the formulation of Theorem 4.45. It is remarkable that $R_1 = (2.3)$, in which case $C_1 = C^{\mathrm{NP}}$, while C_ω, being a consistent extension of C, is a sublogic of C^{PC}, and so the infinite chain involved appears intermediate between C^{NP} and C^{PC}, in contrast to Theorem 4.35. And what is more, we have:

Proposition 4.47. *There is no $\varphi \in \mathrm{Fm}^2_\Sigma$ such that the identities:*

$$\varphi(x_0, x_0) \approx x_0, \tag{4.28}$$
$$\varphi(x_0, x_1) \approx \varphi(x_1, x_0) \tag{4.29}$$

are true in $\mathfrak{A}$.

Proof. By contradiction. For suppose there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that (4.28) and (4.29) are true in $\mathfrak{A}$. Then, $\varphi \in V_2$, for, otherwise, (4.28) would not be true in $\mathfrak{A}$ under $[x_0/\mathsf{b}]$, because all its primary operations are classically-valued. However, in that case, (4.29) is not true in $\mathfrak{A}$ under $[x_0/\mathsf{f}, x_1/\mathsf{t}]$. This contradiction completes the argument. □

This makes the present subsubsection essentially disjoint with Subsubsection 4.4.2. In addition, in contrast to Lemma 4.28, we have:

Lemma 4.48. *$\mathcal{A}_2$ (cf. Claim 4.46) is a consistent non-$\wr$-paraconsistent subdirect square of $\mathcal{A}$ such that* $\mathrm{hom}(\mathcal{A}_2, \mathcal{A}\restriction\{\mathsf{f}, \mathsf{t}\}) = \varnothing$, *while* (4.1) *is true in it.*

Proof. Then, $\mathcal{B} \triangleq \mathcal{A}_2$ is consistent and, being a model of $R_1 = (2.3)$ (cf. Claim 4.46), is not $\wr$-paraconsistent. Moreover, $D^{\mathcal{B}} = \{\langle \mathsf{t}, \mathsf{t}\rangle\}$, while, for every $b \in B$, it holds that $(\wr^{\mathfrak{B}}\langle \mathsf{t}, \mathsf{t}\rangle \veebar^{\mathfrak{B}} b) = (\langle \mathsf{f}, \mathsf{f}\rangle \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$ implies $b \in D^{\mathcal{B}}$, in view of the $\veebar$-disjunctivity of $\mathcal{A}$ and the fact that $\mathsf{f} \notin D^{\mathcal{A}}$. Hence, (4.1) is true in $\mathcal{B}$. Finally, let us prove, by contradiction, that $\mathrm{hom}(\mathcal{B}, \mathcal{A}\restriction\{\mathsf{f}, \mathsf{t}\}) = \varnothing$. For suppose $\mathrm{hom}(\mathcal{B}, \mathcal{A}\restriction\{\mathsf{f}, \mathsf{t}\}) \neq \varnothing$. Take any $h \in \mathrm{hom}(\mathcal{B}, \mathcal{A}\restriction\{\mathsf{f}, \mathsf{t}\})$, in which case $h(\langle \mathsf{t}, \mathsf{t}\rangle) = \mathsf{t}$, for $\langle \mathsf{t}, \mathsf{t}\rangle \in D^{\mathcal{B}}$. Therefore, if, for any $a \in \{\langle \mathsf{b}, \mathsf{f}\rangle, \langle \mathsf{f}, \mathsf{b}\rangle\} \subseteq B$, it did hold that $h(a) = \mathsf{t}$, we would have $\mathsf{f} = \wr^{\mathfrak{A}}\mathsf{t} = h(\wr^{\mathfrak{B}} a) = h(\langle \mathsf{t}, \mathsf{t}\rangle) = \mathsf{t}$. Hence, $h(\langle \mathsf{b}, \mathsf{f}\rangle) = \mathsf{f} = h(\langle \mathsf{f}, \mathsf{b}\rangle)$. Then, we get $\mathsf{f} = (\mathsf{f} \veebar^{\mathfrak{A}} \mathsf{f}) = h(\langle \mathsf{b}, \mathsf{f}\rangle \veebar^{\mathfrak{B}} \langle \mathsf{f}, \mathsf{b}\rangle) = h(\langle \mathsf{t}, \mathsf{t}\rangle) = \mathsf{t}$. This contradiction completes the argument. □

As a consequence, in contrast to Theorem/Corollary 4.29/4.31, we get:

Corollary 4.49. *$C^{\mathrm{NP/MP}}$ is not defined by* $\mathcal{B} \triangleq ((\mathcal{A} \times (\mathcal{A}\restriction\{\mathsf{f}, \mathsf{t}\}))/(\mathcal{A}\restriction\{\mathsf{f}, \mathsf{t}\}))$.

Proof. By contradiction. For suppose $C^{\mathrm{NP/MP}}$ is defined by $\mathcal{B}$. Then, by Lemma 4.48, $\mathcal{D} \triangleq \mathcal{A}_2$ is a consistent non-≀-paraconsistent subdirect square of $\mathcal{A}$ such that $\hom(\mathcal{D}, \mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}) = \varnothing$, while (4.1) is true in it, in which case it is a finite, for A is so, and so finitely-generated, and simple (cf. Lemmas 3.4, 3.6 and 3.8) consistent model of $C^{\mathrm{NP/MP}}$ (cf. (2.6)). Therefore, by Lemma 2.19, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{B})^I$, some subdirect product $\mathcal{E}$ of it and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{D})$, in which case $\mathcal{E}$ is consistent, for $\mathcal{D}$ is so (cf. (2.6)), and so $I \neq \varnothing$. On the other hand, by Lemmas 3.4, 3.6 and 3.8, $\mathcal{E}$ is simple, in which case, by Corollary 2.14, g is injective, and so $((\pi_1/\Delta_{\{\mathsf{f},\mathsf{t}\}}) \circ \pi_i \circ g^{-1}) \in \hom(\mathcal{D}, \mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}) = \varnothing$, where $i \in I \neq \varnothing$. This contradiction completes the argument. ☐

Finally, we have the following universal negative result:

Proposition 4.50. *C is not self-extensional, and so is any expansion of it.*

Proof. For just notice that $x_0 \equiv_C ((x_0 \rhd x_0) \rhd x_0)$, while ${\sim}x_0 \not\equiv_C {\sim}((x_0 \rhd x_0) \rhd x_0)$, because the rule ${\sim}x_0 \vdash {\sim}((x_0 \rhd x_0) \rhd x_0)$ is not true in $\mathcal{A}$ under $[x_0/\mathsf{b}]$. ☐

5. Four-valued expansions of Belnap's logic

Fix any language $\Sigma \supseteq \Sigma_{0[1]}$ such that either $\Sigma \supseteq \Sigma_{01}$ or $(\Sigma \cap \Sigma_{01}) = \Sigma_0$ and any Σ-algebra $\mathfrak{A}$ such that $(\mathfrak{A}\restriction\Sigma_{0[1]}) = \mathfrak{DM}_{4[01]}$. Given any Σ-matrix $\mathcal{B}$, set $\overleftarrow{\mathcal{B}} \triangleq \langle \mathfrak{B}, ({\sim}^{\mathfrak{B}})^{-1}[B \setminus D^{\mathcal{B}}]\rangle$. Put $\mathcal{A} \triangleq \langle \mathfrak{A}, 2^2 \cap \pi_0^{-1}[\{1\}]\rangle$, in which case $\overleftarrow{\mathcal{A}} = \langle \mathfrak{A}, 2^2 \cap \pi_1^{-1}[\{1\}]\rangle$, and $\overrightarrow{\mathcal{A}} \triangleq \langle \mathfrak{A}, \{\mathsf{t}\}\rangle$. Since [bounded] Belnap's four-valued logic (cf. [4]), denoted by $C_{[\mathrm{B}]\mathrm{B}}$ from now on, is defined by $\mathcal{DM}_{4[01]} \triangleq (\mathcal{A}\restriction\Sigma_{0[1]})$ (cf. [20]),[4] the logic C of $\mathcal{A}$ is a four-valued expansion of $C_{[\mathrm{B}]\mathrm{B}}$. We start our study from marking its framework.

5.1. Characteristic matrix expansions.

Lemma 5.1. *Let $\mathcal{B}$ and $\mathcal{D}$ be Σ-matrices, I a set, $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{B})^I$ and e an embedding of $\mathcal{D}$ into $\prod_{i\in I} \mathcal{C}_i$. Suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\} \subseteq B \subseteq 2^2$ forms a subalgebra of $\mathfrak{B}$, $\{I \times \{d\} \mid d \in \{\mathsf{f},\mathsf{t}\}\} \subseteq e[D]$ and, for each $i \in I$, both $\{\mathsf{f},\mathsf{b},\mathsf{t}\} \cup C_i$ forms a regular subalgebra of $\mathfrak{B}$ and, providing $\mathsf{n} \in C_i$, $\mathfrak{B}\restriction\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is specular. Then, $(D \dotplus 2) \triangleq ((D \times \{\mathsf{b}\}) \cup \{\langle e^{-1}(I \times \{d\}), d\rangle \mid d \in \{\mathsf{f},\mathsf{t}\}\})$ forms a subalgebra of $\mathfrak{D} \times (\mathfrak{B}\restriction\{\mathsf{f},\mathsf{b},\mathsf{t}\})$, in which case $\pi_0\restriction(D \dotplus 2)$ is a surjective strict homomorphism from $(\mathcal{D} \dotplus 2) \triangleq ((\mathcal{D} \times (\mathcal{B}\restriction\{\mathsf{f},\mathsf{b},\mathsf{t}\}))\restriction(D \dotplus 2))$ onto $\mathcal{D}$.*

Proof. Consider any $\varsigma \in \Sigma$ of arity $n \in \omega$ and any $\bar{b} \in (D \dotplus 2)^n$. In case $\varsigma^{\mathfrak{B}}(\bar{a}) = \mathsf{b}$, where $\bar{a} \triangleq (\pi_1 \circ \bar{b}) \in \{\mathsf{f},\mathsf{b},\mathsf{t}\}^n$, we clearly have $\varsigma^{\mathfrak{D}\times\mathfrak{B}}(\bar{b}) \in (D \times \{\mathsf{b}\}) \subseteq (D \dotplus 2)$. Otherwise, since $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{B}$, we have $\varsigma^{\mathfrak{B}}(\bar{a}) = (\mathsf{f}/\mathsf{t})$. Put $N \triangleq \{k \in n \mid a_k = \mathsf{b}\}$. Consider any $i \in I$. Put $\bar{c} \triangleq (\pi_i \circ e \circ \pi_0 \circ \bar{b})$. Then, for every $j \in (n \setminus N)$, it holds that $C_i \ni c_j = a_j \in \{\mathsf{f},\mathsf{t}\}$. Hence, $c_j \sqsubseteq a_j$, for all $j \in n$. Therefore, by the regularity of $\mathfrak{B}\restriction(\{\mathsf{f},\mathsf{b},\mathsf{t}\} \cup C_i)$, we have $\varsigma^{\mathfrak{B}}(\bar{c}) \sqsubseteq \varsigma^{\mathfrak{B}}(\bar{a})$. Consider the following complementary cases:

1. $\mathsf{n} \in C_i$.
   Then, $C_i \ni \mu(a_j) \sqsubseteq c_j$, for all $j \in n$. Therefore, as, in that case, $\mathfrak{B}\restriction\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is specular, by the regularity of $\mathfrak{B}\restriction(\{\mathsf{f},\mathsf{b},\mathsf{t}\} \cup C_i)$, we have $\varsigma^{\mathfrak{B}}(\bar{a}) = \mu(\varsigma^{\mathfrak{B}}(\bar{a})) = \varsigma^{\mathfrak{B}}(\mu \circ \bar{a}) \sqsubseteq \varsigma^{\mathfrak{B}}(\bar{c})$, and so we get $\varsigma^{\mathfrak{B}}(\bar{c}) = \varsigma^{\mathfrak{B}}(\bar{a})$.
2. $\mathsf{n} \notin C_i$.
   Then, $\varsigma^{\mathfrak{B}}(\bar{c}) \in C_i \subseteq \{\mathsf{f},\mathsf{b},\mathsf{t}\}$. Therefore, since both f and t are minimal elements of the poset $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ ordered by $\sqsubseteq$, we get $\varsigma^{\mathfrak{B}}(\bar{c}) = \varsigma^{\mathfrak{B}}(\bar{a})$.

Thus, in any case, we have $\varsigma^{\mathfrak{B}}(\bar{c}) = \varsigma^{\mathfrak{B}}(\bar{a})$, and so, by the injectivity of e, we get $\varsigma^{\mathfrak{D}\times\mathfrak{B}}(\bar{b}) = \langle e^{-1}(I \times \{\mathsf{f}/\mathsf{t}\}), \mathsf{f}/\mathsf{t}\rangle \in \{\langle e^{-1}(I \times \{d\}), d\rangle \mid d \in \{\mathsf{f},\mathsf{t}\}\} \subseteq (D \dotplus 2)$, as required. ☐

Lemma 5.2. *Let $\mathcal{B}$ be a model of C. Suppose either $\mathfrak{A}$ is b-idempotent or both $\mathfrak{A}$ is regular and $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{0[1]}$), while $\mathcal{B}$ is not a model of the rule:*

$$\{x_0, {\sim}x_0\} \vdash (x_1 \vee {\sim}x_1). \tag{5.1}$$

Then, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to $\Re(\mathcal{D})$.

Proof. In that case, there are some $a, b \in B$ such that (5.1) is not true in $\mathcal{B}$ under $[x_0/a, x_1/b]$. Then, in view of (2.6), the submatrix $\mathcal{E}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a finitely-generated model of C, in which (5.1) is not true under $[x_0/a, x_1/b]$. Hence, by Lemma 2.19 with $\mathsf{M} = \{\mathcal{A}\}$, there are some set J, some J-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{F}$ of $\overline{\mathcal{C}}$, in which case $(\mathfrak{F}\restriction\Sigma_0) \in \mathsf{DML}$, for $\mathsf{DML} \ni \mathfrak{DM}_4$ is a variety, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}, \Re(\mathcal{E}))$, in which case, by (2.6), $\mathcal{F}$ is a model of C, in which case it is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.31 with $j = 0$), but is not a model of (5.1), in which case there are some $c, d \in F$ such that $\{c, {\sim}^{\mathfrak{F}}c\} \subseteq D^{\mathcal{F}} \not\ni d \geqslant^{\mathfrak{F}} {\sim}^{\mathfrak{F}}d$. Then, by Lemma 4.11, $c = (I \times \{\mathsf{b}\})$, in which case ${\sim}^{\mathfrak{F}}c = c$, and so $(F \setminus D^{\mathcal{F}}) \ni e \triangleq ((c \wedge^{\mathfrak{F}} d) \vee^{\mathfrak{F}} {\sim}^{\mathfrak{F}}d) = {\sim}^{\mathfrak{F}}e \leqslant^{\mathfrak{F}} d$. Hence, $e \in \{\mathsf{b},\mathsf{n}\}^J$, while $K \triangleq \{i \in J \mid \pi_i(e) = \mathsf{n}\} \neq \varnothing$. Given any $\bar{a} \in A^2$, set $(a_0|a_1) \triangleq ((K \times \{a_0\}) \cup ((J \setminus K) \times \{a_1\}))$. In this way, we have:

$$F \ni c = (\mathsf{b}|\mathsf{b}), \tag{5.2}$$

$$F \ni e = (\mathsf{n}|\mathsf{b}), \tag{5.3}$$

$$F \ni (c \wedge^{\mathfrak{F}} e) = (\mathsf{f}|\mathsf{b}), \tag{5.4}$$

$$F \ni (c \vee^{\mathfrak{F}} e) = (\mathsf{t}|\mathsf{b}). \tag{5.5}$$

Consider the following complementary cases:

[4] This equally ensues from Theorem 5.61(x)⇒(v) below, (2.6), the $\wedge$-conjuctivity (cf. Remark 3.31 with $j = 0$) and the finiteness (and so the inductivity of the logic) of $\mathcal{DM}_{4[01]}$ as well as the fact that $\mathcal{DM}_4\restriction\{n\}$ is truth-empty, while $\mu \in \hom(\mathfrak{DM}_{4[01]}, \mathfrak{DM}_{4[01]})$.

(1) either $\mathfrak{A}$ is b-idempotent or $K = J$.
Then, $f \triangleq \{\langle x, (x|\mathsf{b})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{F}$, in which case $g' \triangleq (g \circ f) \in \mathrm{hom_S}(\mathcal{A}, \Re(\mathcal{E}))$, and so, by Corollary 2.14, Lemma 3.4 and Remark 3.31 with $j = 0$, g' is injective. In this way, g' is an isomorphism from $\mathcal{A}$ onto the submatrix $\mathcal{G} \triangleq (\Re(\mathcal{E}) \upharpoonright (\mathrm{img}\, g'))$ of $\Re(\mathcal{E})$, and so $h \triangleq g'^{-1} \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$.

(2) $\mathfrak{A}$ is not b-idempotent and $K \neq J$.
Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\phi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{f}$ and $\psi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{t}$, where $\phi \triangleq (x_0 \wedge (\varphi \wedge \sim\varphi))$ and $\psi \triangleq (x_0 \vee (\varphi \vee \sim\varphi))$, and so, by (5.2), we get:

$$F \ni \phi^{\mathfrak{F}}(c) = (\mathsf{f}|\mathsf{f}), \tag{5.6}$$

$$F \ni \psi^{\mathfrak{F}}(c) = (\mathsf{t}|\mathsf{t}). \tag{5.7}$$

Moreover, in that case, both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$. And what is more, $e' \triangleq \{\langle a', \langle a'\rangle\rangle$ is an embedding of $\mathcal{A}$ into $\mathcal{A}^1$ such that $\{1 \times \{x\} \mid x \in \{\mathsf{f}, \mathsf{t}\}\} = e'[\{\mathsf{f}, \mathsf{t}\}] \subseteq e'[A]$. In this way, Lemma 5.1 with $I = 1$ and $\mathcal{A}$ instead of both $\mathcal{B}$ and $\mathcal{D}$ as well as e' instead of e, used tacitly throughout the rest of the proof, is well-applicable to $\mathcal{A}$. Then, since $K \neq \varnothing \neq (J \setminus K)$, by (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7), we see that $f \triangleq \{\langle\langle x, y\rangle, (x|y)\rangle \mid \langle x, y\rangle \in (A \dotplus 2)\}$ is an embedding of $\mathcal{H} \triangleq (\mathcal{A} \dotplus 2)$ into $\mathcal{F}$, while $h' \triangleq (\pi_0 \upharpoonright (A \dotplus 2)) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H}, \mathcal{A})$. Then, $g' \triangleq (g \circ f) \in \mathrm{hom_S}(\mathcal{H}, \Re(\mathcal{E}))$, and so g' is a surjective strict homomorphism from $\mathcal{H}$ onto the submatrix $\mathcal{G} \triangleq (\Re(\mathcal{E}) \upharpoonright (\mathrm{img}\, g'))$ of $\Re(\mathcal{E})$. And what is more, by Lemma 3.4 and Remark 3.31 with $j = 0$, $\mathcal{A}$ is simple. Hence, by Corollary 2.13 and Proposition 2.16, we get $(\ker g') \subseteq \partial(\mathcal{H}) = (\ker h')$. Therefore, by Proposition 2.15, $h \triangleq (h' \circ g'^{-1}) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$.

Thus, in any case, there are some submatrix $\mathcal{G}$ of $\mathcal{E}/\theta$, where $\theta \triangleq \partial(\mathcal{E})$, and some $h \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. Then, $\mathcal{D} \triangleq (\mathcal{E} \upharpoonright \nu_\theta^{-1}[G])$, being a submatrix of $\mathcal{E}$, is so of $\mathcal{B}$, in which case $h'' \triangleq (\nu_\theta \upharpoonright D) \in \mathrm{hom_S}(\mathcal{D}, \mathcal{G})$ is surjective, and so is $h''' \triangleq (h \circ h'') \in \mathrm{hom_S}(\mathcal{D}, \mathcal{A})$. On the other hand, by Lemma 3.4 and Remark 3.31 with $j = 0$, $\mathcal{A}$ is simple. Hence, by Proposition 2.16, $\vartheta \triangleq \partial(\mathcal{D}) = (\ker h''')$. Therefore, by Proposition 2.15, $\nu_\vartheta \circ h'''^{-1}$ is an isomorphism from $\mathcal{A}$ onto $\Re(\mathcal{D})$, as required. □

Corollary 5.3. *Let C' be an extension of C. Suppose either $\mathfrak{A}$ is b-idempotent or both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{0[1]}$), while the rule* (5.1) *is not satisfied in C'. Then, $C' = C$.*

Proof. In that case, $\sim(x_1 \vee \sim x_1) \notin T \triangleq C'(\{x_0, \sim x_0\})$, so, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C) not being a model of (5.1). In this way, (2.6) and Lemma 5.2 complete the argument. □

Proposition 5.4. *Let M be a class of Σ-matrices. Suppose either $\mathfrak{A}$ is b-idempotent or both $\mathfrak{A}$ is regular and $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{0[1]}$), while C is defined by M. Then, there are some $\mathcal{B} \in \mathsf{M}$ and some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to $\Re(\mathcal{D})$.*

Proof. Note that the rule (5.1) is not satisfied in C, because it is not true in $\mathcal{A}$ under $[x_0/\mathsf{b}, x_1/\mathsf{n}]$. Therefore, as C is defined by M, there is some $\mathcal{B} \in \mathsf{M} \subseteq \mathrm{Mod}(C)$ not being a model of (5.1), in which case Lemma 5.2 completes the argument. □

Now, we are in a position to argue several interesting corollaries of Proposition 5.4:

Corollary 5.5. *Let M be a class of Σ-matrices. Suppose the logic of M is an expansion of C_{B} (in particular, $\Sigma = \Sigma_0$ and the logic of M is C_{B} itself). Then, some $\mathcal{B} \in \mathsf{M}$ is not truth-/false-singular. In particular, any four-valued expansion of C_{B} (including C_{B} itself) is defined by no truth-/false-singular matrix.*

Proof. By contradiction. For suppose every member of M is truth-/false-singular. Then, $\mathsf{M} \upharpoonright \Sigma_0$ is a class of truth-/false-singular Σ_0-matrices defining C_{B}. Then, by Proposition 5.4, there are some $\mathcal{B} \in (\mathsf{M} \upharpoonright \Sigma_0)$ and some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{E} \triangleq (\mathcal{D}/\theta)$, where $\theta \triangleq \partial(\mathcal{D})$, in which case $\mathcal{E}$ is truth-/false-singular, for $\mathcal{D}$ is so, because $\mathcal{B}$ is so/, while $((D/\theta) \setminus (D^{\mathcal{D}}/\theta)) \subseteq ((D \setminus D^{\mathcal{D}})/\theta)$, and so is $\mathcal{DM}_4$. This contradiction completes the argument. □

Corollary 5.6. *Any four-valued $\Sigma_{0[1]}$-matrix $\mathcal{B}$ defining $C_{[\mathrm{B}]\mathrm{B}}$ is isomorphic to $\mathcal{DM}_{4[01]}$.*

Proof. By Proposition 5.4, there are then some submatrix $\mathcal{D}$ of $\mathcal{B}$ and some isomorphism e from $\mathcal{DM}_{4[01]}$ onto $\mathcal{D}/\theta$, where $\theta \triangleq \partial(\mathcal{D})$, in which case $4 = |DM_{4[01]}| = |D/\theta| \leqslant |D| \leqslant |B| = 4$, in which case $4 = |D/\theta| = |D| = |B|$, and so ν_θ is injective, whereas $D = B$. In this way, $e^{-1} \circ \nu_\theta$ is an isomorphism from $\mathcal{B}$ onto $\mathcal{DM}_{4[01]}$, as required. □

This, in its turn, enables us to prove:

Theorem 5.7. *Any four-valued expansion of $C_{[\mathrm{B}]\mathrm{B}}$ is defined by an expansion of $\mathcal{DM}_{4[01]}$.*

Proof. Let $\mathcal{B}$ be a four-valued Σ-matrix defining an expansion of $C_{[\mathrm{B}]\mathrm{B}}$. Then, $\mathcal{B} \upharpoonright \Sigma_{0[1]}$ is a four-valued $\Sigma_{0[1]}$-matrix defining $C_{[\mathrm{B}]\mathrm{B}}$ itself. Hence, by Corollary 5.6, there is an isomorphism e from $\mathcal{B} \upharpoonright \Sigma_{0[1]}$ onto $\mathcal{DM}_{4[01]}$. In that case, e is an isomorphism from $\mathcal{B}$ onto the expansion $\langle e[\mathfrak{B}], e[D^{\mathcal{B}}]\rangle$ of $\mathcal{DM}_{4[01]}$. In this way, (2.6) completes the argument. □

Thus, the way of construction of four-valued expansions chosen in the beginning of this section does exhaust *all* of them. And what is more, any of them is defined by a unique expansion of $\mathcal{DM}_4$, as it follows from the theorem immediately ensuing from the following key lemma "killing several (more precisely, $|\mathbf{S}_*(\mathcal{DM}_4)| = 5$; cf. Subsubsection 7.1.4) birds with one stone":

Lemma 5.8 (Four-Valued Key Lemma)**.** *Let $\mathcal{B}$ be a Σ-matrix. Suppose $(\mathcal{B} \upharpoonright \Sigma_0) \in \mathbf{S}_*(\mathcal{DM}_4)$ and $\mathcal{B}$ is a model of C. Then, $\mathcal{B}$ is a submatrix of $\mathcal{A}$.*

Proof. In that case, $\mathcal{B}$ is consistent and, being finite, is finitely-generated. In addition, by Lemmas 3.4, 3.6 and Remark 3.31 with $j = 0$, it is simple. And what is more, by Remarks 3.15 and 3.31 with $j = 0$, $\mathcal{B}$ is $\vee$-disjunctive. Therefore, as $\mathcal{A}$ is finite, by Lemma 2.19 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$, in which case, by Remark 3.15 and (2.6), $\mathcal{D}$ is consistent and $\vee$-disjunctive, and so, by Corollary 3.16, there is some $i \in I$ such that $h \triangleq (\pi_i \restriction D) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$. Moreover, by Lemmas 3.4, 3.6 and Remark 3.31 with $j = 0$, $\mathcal{C}_i$ is simple. Therefore, by Proposition 2.16, $(\ker h) = \partial(\mathcal{D}) = (\ker g)$. Hence, by Proposition 2.15, $e \triangleq (h \circ g^{-1}) \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{C}_i) \subseteq \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A}) \subseteq \hom_{\mathrm{S}}(\mathcal{B}\restriction\Sigma_0, \mathcal{DM}_4)$, in which case, by Lemma 3.7 and Remark 3.31 with $j = 0$, e is diagonal, as required. □

By (2.6) and Lemma 5.8, we immediately get the following universal characterization:

Corollary 5.9. *Let $\mathcal{B} \in \mathbf{S}_*(\mathcal{DM}_4)$. Then, the logic of a Σ-expansion of $\mathcal{B}$ is an extension of C iff $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$.*

Theorem 5.10. *Let $\mathcal{B}$ be a Σ-matrix. Suppose $(\mathcal{B}\restriction\Sigma_0) = \mathcal{DM}_4$ and $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$). Then, $\mathcal{B} = \mathcal{A}$.*

Proof. Then, by Lemma 5.8, $\mathcal{B}$ is a submatrix of $\mathcal{A}$, in which case $\mathcal{B} = \mathcal{A}$, for $B = A$, as required. □

In view of Theorem 5.10, $\mathcal{A}$ is said to be *characteristic for C*. Subsections 5.2, 5.3, 5.4, 5.5 and 5.6 provide characterizations of certain properties of four-valued expansions of C_{B} via respective properties of their characteristic matrices. And what is more, $\mathcal{A}$ is $\vee$-disjunctive and has a unary unitary equality determinant (cf. Remark 3.31 with $j = 0$), so Theorems 3.24 and 3.28 are well applicable to C immediately yielding the item (1k) of the Abstract (cf. Subsubsection 7.1.4 for more detail).

Corollary 5.11. *Let $\Sigma' \supseteq \Sigma$ be a signature and C' a four-valued Σ'-expansion of C. Then, C' is defined by a unique Σ'-expansion of $\mathcal{A}$.*

Proof. Then, by Theorem 5.7, C' is defined by a Σ'-expansion $\mathcal{A}'$ of $\mathcal{DM}_4$, in which case C is defined by the Σ-expansion $\mathcal{A}'\restriction\Sigma$ of $\mathcal{DM}_4$, and so $(\mathcal{A}'\restriction\Sigma) = \mathcal{A}$, in view of Theorem 5.10. In this way, Theorem 5.10 completes the argument. □

5.1.1. *Minimal four-valuedness.* As a one more interesting consequence of Proposition 5.4, we have:

Theorem 5.12. *Let M be a class of Σ-matrices. Suppose the logic of M is an expansion of C_{B} (in particular, $\Sigma = \Sigma_0$ and the logic of M is C_{B} itself). Then, $4 \leqslant |B|$, for some $\mathcal{B} \in \mathsf{M}$. In particular, any four-valued expansion of C_{B} (including C_{B} itself) is minimally four-valued.*

Proof. In that case, C_{B} is defined by $\mathsf{M}\restriction\Sigma_0$, and so, by Proposition 5.4, there are some $\mathcal{B} \in \mathsf{M}$ and some submatrix $\mathcal{D}$ of $\mathcal{B}\restriction\Sigma_0$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{D}/\theta$, where $\theta \triangleq \partial(\mathcal{D})$. In this way, $4 = |DM_4| = |D/\theta| \leqslant |D| \leqslant |B|$, as required. □

5.2. **Variable Sharing Property.**

Lemma 5.13. *C is purely-inferential iff $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. First, assume $\{\mathsf{n}\}$ forms a subalgebra of $\mathcal{A}$, in which case $\mathcal{A}\restriction\{\mathsf{n}\}$ is a truth-empty submatrix of $\mathcal{A}$, and so C is purely inferential, in view of (2.6).

Conversely, assume $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{n}) \neq \mathsf{n}$, in which case $(\varphi^{\mathfrak{A}}(\mathsf{n}) \vee^{\mathfrak{A}} {\sim}^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\mathsf{n})) \in D^{\mathcal{A}}$, and so $((x_0 \vee {\sim}x_0) \vee (\varphi \vee {\sim}\varphi)) \in C(\varnothing)$, as required. □

Lemma 5.14. *C has no inconsistent formula iff $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. First, assume $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $(\varphi^{\mathfrak{A}}(\mathsf{b}) \wedge^{\mathfrak{A}} {\sim}^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\mathsf{b})) \notin D^{\mathcal{A}}$, and so $((x_0 \wedge {\sim}x_0) \wedge (\varphi \wedge {\sim}\varphi))$ is an inconsistent formula of C.

Conversely, assume $\{\mathsf{b}\}$ forms a subalgebra of $\mathcal{A}$. Let us prove, by contradiction, that C has no inconsistent formula. For suppose some $\varphi \in \mathrm{Fm}^\omega_\Sigma$ is an inconsistent formula of C, in which case $\varphi \in \mathrm{Fm}^\alpha_\Sigma$, for some $\alpha \in (\omega \setminus 1)$, while $x_\alpha \in C(\varphi)$. Let $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $(V_\alpha \times \{\mathsf{b}\}) \cup (V_{\omega\setminus\alpha} \times \{\mathsf{f}\})$. Then, $h(\varphi) = \mathsf{b} \in D^{\mathcal{A}}$, whereas $h(x_\alpha) = \mathsf{f} \notin D^{\mathcal{A}}$. This contradiction completes the argument. □

Theorem 5.15. *The following are equivalent:*

(i) *C satisfies Variable Sharing Property;*
(ii) *C is purely inferential and has no inconsistent formula;*
(iii) *both $\{\mathsf{n}\}$ and $\{\mathsf{b}\}$ form subalgebras of $\mathfrak{A}$.*

Proof. First, (ii) is a particular case of (i). Next, (ii)⇒(iii) is by Lemmas 5.13 and 5.14.

Finally, assume (iii) holds. Consider any $\alpha \in (\omega \setminus 1)$, any $\phi \in \mathrm{Fm}^\alpha_\Sigma$ and any $\psi \in \mathrm{Fm}^{\omega\setminus\alpha}_\Sigma$. Let $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $(V_\alpha \times \{\mathsf{b}\}) \cup (V_{\omega\setminus\alpha} \times \{\mathsf{n}\})$. Then, $h(\phi) = \mathsf{b} \in D^{\mathcal{A}}$, whereas $h(\psi) = \mathsf{n} \notin D^{\mathcal{A}}$. Thus, $\psi \notin C(\phi)$, and so (i) holds, as required. □

Corollary 5.16 (cf. Theorem 4.2 of [20] for the case $\Sigma = \Sigma_0$)**.** *C has no proper extension satisfying Variable Sharing Property.*

Proof. Consider any extension C' of C satisfying Variable Sharing Property, in which case C, being a sublogic of C', does so as well, and so, by Theorem 5.15(i)⇒(iii), $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$. Moreover, as C' is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.31 with $j = 0$), (5.1) is not satisfied in C', for $1 \in (\omega \setminus 1)$, while $(x_0 \wedge {\sim}x_0) \in \mathrm{Fm}^1_\Sigma$, whereas $(x_1 \vee {\sim}x_1) \in \mathrm{Fm}^{\omega\setminus 1}_\Sigma$. In this way, Corollary 5.3 completes the argument. □

Perhaps, this is the principal maximality of C in addition to the standard one studied below.

5.3. **Maximality.**

Lemma 5.17. *Any proper submatrix $\mathcal{B}$ of $\mathcal{A}$ defines a proper extension C' of C.*

Proof. For consider the following complementary cases:

(1) $\mathsf{b} \in B$.
Then, $\mathsf{n} \notin B$, for $B \neq A$, while $(\mathsf{n} \wedge^{\mathfrak{B}} \mathsf{b}) = \mathsf{f}$, whereas $(\mathsf{n} \vee^{\mathfrak{B}} \mathsf{b}) = \mathsf{t}$. In that case, $(x_0 \vee {\sim} x_0) \in (C'(\varnothing) \setminus C(\varnothing))$.

(2) $\mathsf{b} \notin B$.
Then, $\mathcal{B}$ is not $\sim$-paraconsistent, as opposed to $\mathcal{A}$, and so is C', as opposed to C.

Thus, in any case, $C' \neq C$, as required, in view of (2.6). □

Lemma 5.18. *Let $\mathcal{D} \in \mathbf{S}_*(\mathcal{A})$. Then, providing $D \neq \{\mathsf{n}\}$ (in particular, $\mathcal{D}$ is truth-non-empty), $\{\mathsf{f}, \mathsf{t}\} \subseteq D$, in which case $\mathcal{D}$ is truth-non-empty. In particular, $\mathcal{D}$ is truth-non-empty iff $D \neq \{\mathsf{n}\}$.*

Proof. In that case, we have $(\{\mathsf{f}, \mathsf{n}\} \cap D) \neq \varnothing$. In this way, the fact that $(\mathsf{n} \wedge^{\mathfrak{A}} \mathsf{b}) = \mathsf{f}$, while ${\sim}^{\mathfrak{A}} \mathsf{f} = \mathsf{t}$, whereas ${\sim}^{\mathfrak{A}} \mathsf{t} = \mathsf{f}$, completes the argument. □

Clearly, $\mathcal{A}$ is consistent [and truth-non-empty], and so C is [inferentially] consistent. In this connection, we have:

Theorem 5.19. *C is [inferentially] maximal iff $\mathcal{A}$ has no proper consistent [truth-non-empty] submatrix.*

Proof. First, consider any proper consistent [truth-non-empty] submatrix $\mathcal{B}$ of $\mathcal{A}$. Then, by Lemma 5.17, the logic C' of $\mathcal{B}$ is a[n inferentially] consistent proper extension of C, and so C is not [inferentially] maximal.

Conversely, assume $\mathcal{A}$ has no proper consistent [truth-non-empty] submatrix. Consider any [inferentially] consistent extension C' of C. Then, $x_0 \notin T \triangleq C'(\varnothing[\cup\{x_1\}])[\ni x_1]$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its consistent [truth-non-empty] finitely-generated submatrix $\mathcal{B} = \langle \mathfrak{Fm}^2_{\Sigma}, \mathrm{Fm}^2_{\Sigma} \cap T\rangle$, in view of (2.6). Hence, by Lemma 2.19 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent [truth-non-empty] submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B}/\partial(\mathcal{B}))$, in which case, by (2.6), $\mathcal{D}$ is a consistent model of C', and so, in particular, $I \neq \varnothing$. Moreover, for any $i \in I$, [as $\mathcal{C}_i$ is truth-non-empty] $\mathcal{C}_i = \mathcal{A}$ is truth-non-empty anyway. Hence, by the following claim, both $D \ni a \triangleq (I \times \{\mathsf{f}\})$ and $D \ni b \triangleq (I \times \{\mathsf{t}\})$:

Claim 5.20. *Let I be a finite set, $\overline{\mathcal{C}}$ an I-tuple constituted by consistent truth-non-empty submatrices of $\mathcal{A}$ and $\mathcal{B}$ a subdirect product of $\overline{\mathcal{C}}$. Then, $\{I \times \{\mathsf{f}\}, I \times \{\mathsf{t}\}\} \subseteq B$.*

Proof. In that case, $\mathfrak{B}{\upharpoonright}\Sigma^+$ is a finite lattice, so it has both a zero a and a unit b. Consider any $i \in I$. Then, as $\mathcal{C}_i$ is both consistent and truth-non-empty, by Lemma 5.18, we have $\{\mathsf{f}, \mathsf{t}\} \subseteq C_i$. Therefore, since $\pi_i[B] = C_i$ and $(\pi_i{\upharpoonright}B) \in \hom(\mathfrak{B}{\upharpoonright}\Sigma^+, \mathfrak{C}_i{\upharpoonright}\Sigma^+)$, by Lemma 3.30, we get $\pi_i(a) = \mathsf{f}$ and $\pi_i(b) = \mathsf{t}$. Thus, $B \ni a = (I \times \{\mathsf{f}\})$ and $B \ni b = (I \times \{\mathsf{t}\})$, as required. □

Next, if $\{\mathsf{f}, \mathsf{t}\} \subsetneq A$ did form a subalgebra of $\mathfrak{A}$, $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$ would be a proper consistent [truth-non-empty] submatrix of $\mathcal{A}$. Therefore, there are some $\phi \in \mathrm{Fm}^2_{\Sigma}$ and some $j \in 2$ such that $\phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) = \langle j, 1-j\rangle$. Likewise, if $\{\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}\} \subsetneq A$ did form a subalgebra of $\mathfrak{A}$, $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}\}$ would be a proper consistent [truth-non-empty] submatrix of $\mathcal{A}$. Therefore, there is some $\psi \in \mathrm{Fm}^3_{\Sigma}$ such that $\psi^{\mathfrak{A}}(\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}) = \langle 1-j, j\rangle$. In this way, $\{\phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}), \psi^{\mathfrak{A}}(\mathsf{f}, \phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}), \mathsf{t})\} = \{\mathsf{n}, \mathsf{b}\}$. Then, $D \supseteq \{\phi^{\mathfrak{D}}(a, b), \psi^{\mathfrak{D}}(a, \phi^{\mathfrak{D}}(a, b), b)\} = \{I \times \{\mathsf{n}\}, I \times \{\mathsf{b}\}\}$. Thus, $\{I \times \{c\} \mid c \in A\} \subseteq D$. Hence, as $I \neq \varnothing$, $\{\langle c, I \times \{c\}\rangle \mid c \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.6), C is an extension of C', and so $C' = C$, as required. □

5.4. **Subclassical expansions.**

Lemma 5.21. *Let $\mathcal{B}$ be a (simple) finitely-generated consistent truth-non-empty model of C. Then, the following hold:*

(i) *$\mathcal{B}$ is $\sim$-paraconsistent, if ${\sim}(x_0 \wedge {\sim} x_0)$ is true in $\mathcal{B}$ and $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*

(ii) *providing $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$ is embeddable into $\mathcal{B}/\partial(\mathcal{B})$ (resp., into $\mathcal{B}$ itself).*

Proof. Put $\mathcal{E} \triangleq (\mathcal{B}/\partial(\mathcal{B}))$ (resp., $\mathcal{E} \triangleq \mathcal{B}$). Then, by Lemma 2.19 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent truth-non-empty submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case, by (2.6), $\mathcal{D}$ is consistent, and so, in particular, $I \neq \varnothing$. Hence, by Claim 5.20, both $D \ni a \triangleq (I \times \{\mathsf{f}\})$ and $D \ni b \triangleq (I \times \{\mathsf{t}\})$. Consider the following respective cases:

(i) ${\sim}(x_0 \wedge {\sim} x_0)$ is true in $\mathcal{B}$ and $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}^2_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) \in \{\mathsf{n}, \mathsf{b}\}$. Take any $i \in I \neq \varnothing$. Then, $\{\mathsf{f}, \mathsf{t}\} = \pi_i[\{a, b\}] \subseteq C_i$. Moreover, $(\pi_i{\upharpoonright}D) \in \hom^{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$, in which case, by (2.6) and (2.7), $\mathcal{C}_i$ is a model of ${\sim}(x_0 \wedge {\sim} x_0)$, and so $\mathsf{n} \notin C_i$, for ${\sim}^{\mathfrak{A}}(\mathsf{n} \wedge^{\mathfrak{A}} {\sim}^{\mathfrak{A}} \mathsf{n}) = \mathsf{n} \notin D^{\mathcal{A}}$. And what is more, $\mathfrak{C}_i$ is a subalgebra of $\mathfrak{A}$. Hence, $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) \in C_i$, and so $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) = \mathsf{b}$, for $\mathsf{n} \notin C_i$. Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a, b) = (I \times \{\mathsf{b}\})$, in which case ${\sim}^{\mathfrak{D}} c = c \in D^{\mathcal{D}}$, and so $\mathcal{D}$, being consistent, is $\sim$-paraconsistent, and so is $\mathcal{B}$, in view of (2.6), as required.

(ii) $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$,
in which case $\mathcal{F} \triangleq (\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\})$ is $\sim$-classical, and so simple, in view of Example 3.2 and Lemma 3.4. Finally, as $\{I \times \{d\} \mid d \in F\} \subseteq D$ and $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in F\}$ is an embedding of $\mathcal{F}$ into $\mathcal{D}$, in which case, $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{F}, \mathcal{E})$, and so Corollary 2.14 completes the argument. □

Theorem 5.22. *C is $\sim$-subclassical iff $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case any $\sim$-classical model of C is isomorphic to $\mathcal{A}_{\not\mathsf{n}\not\mathsf{b}} \triangleq \mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$, and so the logic of this submatrix is the only $\sim$-classical extension of C.*

Proof. Let $\mathcal{B}$ be a $\sim$-classical model of C, in which case it is simple (cf. Example 3.2 and Lemma 3.4) and finite (in particular, finitely-generated), but not $\sim$-paraconsistent.

First, consider any $a \in B$. Then, $\{a, \sim^{\mathfrak{B}} a\} \nsubseteq D^{\mathcal{B}}$, for $\mathcal{B}$ is $\sim$-classical, in which case $(a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} a) \notin D^{\mathcal{B}}$, for $\mathcal{B}$ is $\wedge$-conjunctive, because C is so, since $\mathcal{A}$ is so (cf. Remark 3.31 with $j = 0$), and so $\sim^{\mathfrak{B}}(a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} a) \in D^{\mathcal{B}}$, for $\mathcal{B}$ is $\sim$-classical. Thus, $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$. Hence, by Lemma 5.21(i), $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

Conversely, assume $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{D} \triangleq (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})$ is a $\sim$-classical model of C, by (2.6), and is embeddable into $\mathcal{B}$, by Lemma 5.21(ii), and so is isomorphic to it, for they are both two-valued. In this way, (2.6) completes the argument. ☐

In view of Theorem 5.22, the unique $\sim$-classical extension of a $\sim$-subclassical four-valued expansion $\boldsymbol{C}$ of C_{B} is said to be *characteristic for/of* $\boldsymbol{C}$ and denoted by $\boldsymbol{C}^{\mathrm{PC}}$, the maximality nature of which is as follows:

Theorem 5.23. *Let C' be an inferentially consistent (in particular, consistent non-pseudo-axiomatic) extension of C. Suppose $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\}$ is a model of C'.*

Proof. Then, $x_1 \notin C'(x_0) \ni x_0$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C'(x_0)\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\langle \mathfrak{Fm}^{2}_{\Sigma}, \mathrm{Fm}^{2}_{\Sigma} \cap C'(x_0)\rangle$, in view of (2.6). In this way, (2.6) and Lemma 5.21(ii) complete the argument. ☐

Example 5.24. When $\Sigma = \Sigma_0$, $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{B} \triangleq (\mathcal{A} \upharpoonright \{\mathsf{n}\})$ is a consistent truth-empty submatrix of $\mathcal{A}$, and so, by (2.6), the logic C' of $\mathcal{B}$ is a consistent but inferentially inconsistent extension of C. Then, C' is not subclassical, because any classical logic is inferentially consistent, for any classical matrix is both consistent and truth-non-empty. In this way, the reservation "inferentially" cannot be omitted in the formulation of Theorem 5.23. ☐

5.5. **Paraconsistent and paracomplete extensions.** The axiomatic extension of C relatively axiomatized by the *Excluded Middle law* axiom (viz., the Reflexivity axiom for the *material* implication $\sim x_0 \vee x_1$):

$$\sim x_0 \vee x_0 \tag{5.8}$$

is denoted by C^{EM}.

An extension C' of C is said to be *(maximally) [inferentially] paracomplete*, provided $(x_0 \vee \sim x_0) \notin C'(\varnothing[\cup\{x_1\}])$ (and C' has no proper [inferentially] paracomplete extension). Then, a model of C is said to be *[inferentially] paracomplete*, whenever the logic of it is so.

Clearly, a submatrix $\mathcal{B}$ of $\mathcal{A}$ is paracomplete/$\sim$-paraconsistent iff $\mathsf{n} \in B$/both $\mathsf{b} \in B$ and $(B \cap \{\mathsf{n}, \mathsf{f}\}) \neq \varnothing$. In particular, $\mathcal{A}$ is both $\sim$-paraconsistent and paracomplete, and so is C.

By $\mathcal{A}_{-\mathsf{n}}$ we denote the $\sim$-paraconsistent submatrix of $\mathcal{A}$ generated by $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, the logic of it being denoted by $C^{-\mathsf{n}}$. (Clearly, $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}_{\not{\mathsf{n}}} \triangleq (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{b}, \mathsf{t}\})$ is a $\sim$-superclassical and $\wedge$-conjunctive — cf. (2.6) and Remark 3.31 with $j = 0$ — Σ-matrix with underlying algebra being a $(\wedge, \vee)$-lattice, if $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}$, otherwise.)

Lemma 5.25. *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of C. Then, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}_{-\mathsf{n}}$ is embeddable into $\mathcal{D}/\partial(\mathcal{D})$.*

Proof. In that case, there are some $a \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} a \in D^{\mathcal{B}}$ and some $b \in (B \setminus D^{\mathcal{B}})$. Then, in view of (2.6), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a $\sim$-paraconsistent finitely-generated model of C. Hence, by Lemma 2.19 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent submatrices of $\mathcal{A}$, some subdirect product $\mathcal{E}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{D}/\partial(\mathcal{D}))$. Hence, by (2.6), $\mathcal{E}$ is $\sim$-paraconsistent, in which case it is consistent, and so $I \neq \varnothing$. Take any $a \in D^{\mathcal{E}}$ such that $\sim^{\mathfrak{E}} a \in D^{\mathcal{E}}$. Then, by Lemma 4.11, $E \ni a = (I \times \{\mathsf{b}\})$, in which case, for each $i \in I$, $D^{\mathcal{C}_i} \ni \pi_i(a)$, and so $\mathcal{C}_i$ is truth-non-empty. Therefore, by Claim 5.20, we also have both $E \ni b \triangleq (I \times \{\mathsf{f}\})$ and $E \ni c \triangleq (I \times \{\mathsf{t}\})$. Consider the following complementary cases:

1. $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
   Then, $A_{-\mathsf{n}} = A$ and there is some $\varphi \in \mathrm{Fm}^{3}_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{b}, \mathsf{t}) = \mathsf{n}$, in which case $E \ni \varphi^{\mathfrak{E}}(b, a, c) = (I \times \{\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{b}, \mathsf{t})\}) = (I \times \{\mathsf{n}\})$, and so $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$.
2. $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
   Then, $A_{-\mathsf{n}} = \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, and so $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$.

Thus, in any case, $\{I \times \{d\} \mid d \in A_{-\mathsf{n}}\} \subseteq E$. Then, as $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in A_{-\mathsf{n}}\}$ is an embedding of $\mathcal{A}_{-\mathsf{n}}$ into $\mathcal{E}$, in which case $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}_{-\mathsf{n}}, \mathcal{D}/\partial(\mathcal{D}))$, and so Corollary 2.14, Lemmas 3.4, 3.6 and Remark 3.31 with $j = 0$ complete the argument. ☐

Corollary 5.26. *$\mathcal{A}_{-\mathsf{n}}$ is a model of any $\sim$-paraconsistent extension of C. In particular, $C^{-\mathsf{n}}$ is the greatest $\sim$-paraconsistent extension of C, and so maximally $\sim$-paraconsistent, in which case an extension of C is $\sim$-paraconsistent iff it is a sublogic of $C^{-\mathsf{n}}$.*

Proof. Consider any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\})$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a $\sim$-paraconsistent model of C', and so of C. Then, (2.6) and Lemma 5.25 complete the argument. ☐

Lemma 5.27 (cf. Corollary 5.3 of [20] for the case $\Sigma = \Sigma_0$)**.** *Suppose $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$/ $\{\mathsf{f}, \mathsf{t}\}[\cup\{\mathsf{b}\}]$ does [not] form a subalgebra of $\mathfrak{A}$. Then, the logic of $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}}$ is the proper consistent axiomatic extension of C relatively axiomatized by (5.8).*

Proof. In that case, $(\mathrm{Mod}\,(5.8) \cap \mathbf{S}_*(\mathcal{A})) = \mathbf{S}_*(\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}})$. In this way, Corollary 2.21, the consistency of $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}}$ and the fact that (5.8) is not satisfied in $\mathcal{A}$ under $[x_1/\mathsf{n}]$ complete the argument. ☐

The logic of $\mathcal{DM}_{4[01]}\restriction\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is known as the *[bounded] logic of paradox* $LP_{[01]}$ [18] (cf. [21]).

Theorem 5.28. *The following are equivalent:*

(i) *C is maximally $\sim$-paraconsistent;*
(ii) *$C = C^{-\mathsf{n}}$;*
(iii) *$C^{\mathrm{EM}} \neq C^{-\mathsf{n}}$;*
(iv) *$\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*
(v) *C^{EM} is not $\sim$-paraconsistent;*
(vi) *C^{EM} is not maximally $\sim$-paraconsistent;*
(vii) *C^{EM} is either $\sim$-classical, if C is $\sim$-subclassical, or inconsistent, otherwise;*
(viii) *any consistent non-$\sim$-classical extension of C is paracomplete;*
(ix) *any $\sim$-paraconsistent extension of C is paracomplete;*
(x) *no expansion of LP is an extension of C;*
(xi) *C^{EM} is not an expansion of LP.*

Proof. First, (i)⇒(ii) is by (2.6). The converse is by Corollary 5.26. Thus, (i)⇔(ii) holds. Next, (ii)⇒(iii) is by the paracompleteness of C. In addition, (iv)⇒(ii) is immediate.

Further, assume $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $A_{-\mathsf{n}} = A_{\not\mathsf{n}}$, and so, by Lemma 5.27, $C^{\mathrm{EM}} = C^{-\mathsf{n}}$ is an expansion of LP. Thus, both (iii)⇒(iv) and (xi)⇒(iv) hold.

Furthermore, (vi) is a particular case of (v). Likewise, (v) is a particular case of (ix), while (ix) is a particular case of (viii). Moreover, (vi)⇒(iii) is by Corollary 5.26. And what is more, (vii)⇒(viii) is by Theorems 5.22 and 5.23.

Finally, assume (iv) holds. Let S be the set of all non-paracomplete consistent submatrices of $\mathcal{A}$, in which case, by Corollary 2.21, C^{EM} is defined by S. Consider any $\mathcal{B} \in \mathsf{S}$. Since it is not paracomplete, we have $\mathsf{n} \notin B$, in which case $\mathsf{f} \in B$, for it is consistent, and so $\mathsf{t} = \sim^{\mathfrak{A}}\mathsf{f} \in B$. Therefore, by (iv), $\mathsf{b} \notin B$, for $\{\mathsf{f},\mathsf{t}\} \subseteq B \not\ni \mathsf{n}$. Thus, $B = \{\mathsf{f},\mathsf{t}\}$. In this way, by Theorem 5.22, either $\mathsf{S} = \{\mathcal{B}\}$, in which case C^{EM} is $\sim$-classical, if C is $\sim$-subclassical, or $\mathsf{S} = \varnothing$, in which case C^{EM} is inconsistent, otherwise. Thus, (vii) holds.

After all, (xi/x) is a particular case of (x/ix), as required. □

It is Theorem 5.28(i)⇔(iv) that provides a quite useful algebraic criterion of the maximal $\sim$-paraconsistency of C inherited by its four-valued expansions, in view of Corollary 5.11, applications of which are demonstrated in Subsection 7.1.

5.5.1. *The resolutional extension.* By $C^{[\mathrm{EM}+]\mathrm{R}}$ we denote the *resolutional* extension of $C^{[\mathrm{EM}]}$, viz., the one relatively axiomatized by the *Resolution* rule:

$$\{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash x_0. \tag{5.9}$$

Put $\mathsf{S}_{[*]\not\mathsf{b}} \triangleq \{\mathcal{B} \in \mathbf{S}_{[*]}(\mathcal{A}) \mid \mathsf{b} \notin B\}$.

By Lemmas 3.21, 4.22, Corollary 3.17 and Remark 3.31 with $j = 0$, we first have:

Corollary 5.29. *C^{R} is a proper extension of C.*

Theorem 5.30. *$C^{\mathrm{EM+R}}$ is equal to C^{PC}, if C is $\sim$-subclassical, and inconsistent, otherwise.*

Proof. With using Remark 3.31 with $j = 0$, Theorems 3.24, 5.22 and Lemma 4.22. Then, $C^{\mathrm{EM+R}}$ is defined by the set S of all non-paracomplete members of $\mathsf{S}_{*,\not\mathsf{b}}$. In that case, $\mathsf{S} = \{\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}\}$, if $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and $\mathsf{S} = \varnothing$, otherwise, as required. □

By Remark 3.31 with $j = 0$, Theorem 3.24 and Lemma 4.22, we also have:

Lemma 5.31. *C^{R} is defined by $\mathsf{S}_{[*]\not\mathsf{b}}$.*

By Lemmas 5.13 and 5.31, we first have:

Corollary 5.32. *C^{R} is purely inferential iff C is so. In particular, C^{R} is paracomplete, whenever C is purely inferential.*

In addition, we also get:

Corollary 5.33. *Suppose $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, C^{R} is defined by $\mathcal{A}_{\not\mathsf{b}} \triangleq (\mathcal{A}\restriction\{\mathsf{f},\mathsf{n},\mathsf{t}\})$,*

Proof. In that case, $\mathsf{S}_{\not\mathsf{b}} = \mathbf{S}(\mathcal{A}_{\not\mathsf{b}})$, and so (2.6) and Lemma 5.31 complete the argument. □

Theorem 5.34. *The following are equivalent:*

(i) *C^{R} is paracomplete;*
(ii) *there is some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathsf{b} \notin B \ni \mathsf{n}$;*
(iii) *the carrier of the subalgebra of $\mathfrak{A}$ generated by $\{\mathsf{n}\}$ does not contain b;*
(iv) *there is no $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{n}) = \mathsf{b}$.*

Proof. In view of Lemma 5.31, C^{R} is paracomplete iff $\mathsf{S}_{\not\mathsf{b}}$ contains a paracomplete matrix. Thus, (i)⇔(ii) holds. Finally, (ii)⇔(iii)⇔(iv) are immediate. □

Lemma 5.35. *Let $B \subseteq \{\mathsf{b},\mathsf{n}\}$. Suppose $\{\mathsf{f},\mathsf{t}\} \cup B$ forms a specular subalgebra of $\mathfrak{A}$. Then, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. By contradiction. For suppose $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. In that case, there are some $\varsigma \in \Sigma$ of some arity $n \in \omega$ and some $\bar{a} \in \{\mathsf{f},\mathsf{t}\}^n$ such that $\varsigma^{\mathfrak{A}}(\bar{a}) \in B$. Then, $(\mu \circ \bar{a}) = \bar{a}$, while $\mu(\varsigma^{\mathfrak{A}}(\bar{a})) \neq \varsigma^{\mathfrak{A}}(\bar{a})$, in which case $\mu \notin \hom(\mathfrak{A}\restriction(\{\mathsf{f},\mathsf{t}\} \cup B), \mathfrak{A})$, and so this contradiction completes the argument. □

Theorem 5.36. *Suppose* $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ *forms a regular specular subalgebra of* $\mathfrak{A}$*, in which case* $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}_{\not\mathsf{b}}$ *(cf. Lemma 5.35), [while* $\{\mathsf{n}\}$ *does not form a subalgebra of* $\mathfrak{A}_{\not\mathsf{b}}$*] (in particular,* $\Sigma = \Sigma_{0[1]}$*). Then, an extension of* C *is [non-]inferentially paracomplete iff it is a sublogic of* C^{R}*. In particular,* C^{R} *is maximally [non-]inferentially paracomplete.*

Proof. Then, by Corollary 5.33, C^{R} is defined by the truth-non-empty paracomplete (and so inferentially paracomplete) Σ-matrix $\mathcal{A}_{\not\mathsf{b}}$, in which case, in particular, any extension of C, being a sublogic of C^{R}, is inferentially paracomplete, and so paracomplete.

Conversely, consider any [non-]inferentially paracomplete extension C' of C, in which case [since $C'(\varnothing) \supseteq C(\varnothing) \neq \varnothing$, in view of Lemma 5.13] $(x_0 \vee \sim x_0) \notin T \triangleq C'(x_1)$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated inferentially paracomplete submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{2}_{\Sigma}, T \cap \mathrm{Fm}^{2}_{\Sigma}\rangle$, in view of (2.6). Hence, by Lemma 2.19, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$, in which case $(\mathfrak{D}\restriction\Sigma_0) \in \mathsf{DML}$, for $\mathsf{DML} \ni \mathfrak{DM}_4$ is a variety, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \Re(\mathcal{B}))$, in which case, by (2.6), $\mathcal{D}$ is an inferentially paracomplete model of C', and so there are some $a \in D^{\mathcal{D}}$, in which case, for every $i \in I$, $\pi_i(a) \in D^{\mathcal{C}_i}$, for $(\pi_i\restriction D) \in \hom(\mathcal{D}, \mathcal{C}_i)$, and so $\mathcal{C}_i$ is truth-non-empty, and some $b \in (D \setminus D^{\mathcal{D}})$ such that $\sim^{\mathfrak{D}} b \leqslant^{\mathfrak{D}} b$, in which case $b \in \{\mathsf{n},\mathsf{b},\mathsf{t}\}^I$. Put $J \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$ and $K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{n}\} \neq \varnothing$, for $b \notin D^{\mathcal{D}}$. Given any $\bar{a} \in A^3$, put $(a_0|a_1|a_2) \triangleq ((J \times \{a_0\}) \cup (K \times \{a_1\}) \cup ((I \setminus (J \cup K)) \times \{a_2\})) \in A^I$. Then, we have:

$$D \ni b = (\mathsf{t}|\mathsf{n}|\mathsf{b}), \tag{5.10}$$

$$D \ni \sim^{\mathfrak{D}} b = (\mathsf{f}|\mathsf{n}|\mathsf{b}). \tag{5.11}$$

Moreover, by Claim 5.20, we also have:

$$D \ni (\mathsf{f}|\mathsf{f}|\mathsf{f}), \tag{5.12}$$

$$D \ni (\mathsf{t}|\mathsf{t}|\mathsf{t}). \tag{5.13}$$

Consider the following complementary cases:

(1) $J = \varnothing$.
Then, as $K \neq \varnothing$, while $(\mu\restriction A_{\not\mathsf{b}}) \in \hom(\mathfrak{A}_{\not\mathsf{b}}, \mathfrak{A})$, taking (5.10), (5.12) and (5.13) into account, we see that $\{\langle c, (c|c|\mu(c))\rangle \mid c \in A_{\not\mathsf{b}}\}$ is an embedding of $\mathcal{A}_{\not\mathsf{b}}$ into $\mathcal{D}$, in which case, by (2.6), $\mathcal{A}_{\not\mathsf{b}}$ is a model of C', for $\mathcal{D}$ is so.

(2) $J \neq \varnothing$.
Then, as $K \neq \varnothing$, while $(\mu\restriction A_{\not\mathsf{b}}) \in \hom(\mathfrak{A}_{\not\mathsf{b}}, \mathfrak{A})$, taking (5.10), (5.11), (5.12), (5.13) and Lemma 4.32 into account, we see that $\{\langle\langle c, d\rangle, (d|c|\mu(c))\rangle \mid \langle c, d\rangle \in K^{\mathsf{n}}_4\}$ is an embedding of $\mathcal{A}^2\restriction K^{\mathsf{n}}_4$ into $\mathcal{D}$, in which case, by (2.6), $\mathcal{A}^2\restriction K^{\mathsf{n}}_4$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{A}_{\not\mathsf{b}}$.

Thus, in any case, $\mathcal{A}_{\not\mathsf{b}}$ is a model of C', and so $C' \subseteq C^{\mathrm{R}}$, as required. ☐

The logic of $\mathcal{DM}_{4[01]}\restriction\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ is known as *Kleene's [bounded] three-valued logic* $K_{3[01]}$ (cf. [12]).

Theorem 5.37. *The following are equivalent:*

(i) $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ *does not form a subalgebra of* $\mathfrak{A}$*;*
(ii) *[providing* C *is not purely inferential]* C^{R} *is [non-]inferentially either* $\sim$*-classical, if* C *is* $\sim$*-subclassical, or inconsistent, otherwise;*
(iii) *[providing* C *is not purely inferential]* C^{R} *is not [non-]inferentially paracomplete;*
(iv) *the* Σ_0*-fragment of* C^{R} *is not inferentially paracomplete;*
(v) *no expansion of* K_3 *is an extension of* C*;*
(vi) C^{R} *is not an expansion of* K_3*.*

Proof. First, (vi)⇒(i) is by Corollary 5.33.

Moreover, (vi) is a particular case of (v).

Next, assume (i) holds. We use Remark 2.11, Theorem 5.22 and Lemmas 5.13 and 5.31 tacitly. Consider the following four exhaustive cases:

(1) C is both $\sim$-subclassical and not purely inferential.
Then, $\mathsf{S}_{*,\not\mathsf{b}} = \{\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}\}$, in which case C^{R} is $\sim$-classical, and so inferentially so.
(2) C is both purely-inferential and $\sim$-subclassical.
Then, $\mathsf{S}_{*,\not\mathsf{b}} = \{\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}, \mathcal{A}\restriction\{\mathsf{n}\}\}$, in which case C^{R} is inferentially $\sim$-classical.
(3) C is both not $\sim$-subclassical and not purely inferential.
Then, $\mathsf{S}_{*,\not\mathsf{b}} = \varnothing$, in which case C^{R} is inconsistent, and so inferentially so.
(4) C is both purely-inferential and not $\sim$-subclassical.
Then, $\mathsf{S}_{*,\not\mathsf{b}} = \{\mathcal{A}\restriction\{\mathsf{n}\}\}$, in which case C^{R} is inferentially inconsistent.

Thus, (ii) holds.

Further, in view of Theorem 5.22, any [inferentially] $\sim$-classical extension of C is not [inferentially] paracomplete. And what is more, any [inferentially] paracomplete extension of C is clearly [inferentially] consistent. Hence, (ii)⇒(iii) holds.

Furthermore, (iii)⇒(iv) is by the fact that $x_0 \vee \sim x_0$ is a Σ_0-formula.

Finally, by Proposition 2.10, K_3 is non-pseudo-axiomatic. Moreover, it is paracomplete, and so inferentially so. And what is more, (5.9), being satisfied in K_3, is so in any expansion of it. In this way, (iv)⇒(v) holds, as required. ☐

In this connection, it is remarkable that paracomplete analogue of the "maximality" items (i) and (vi) of Theorem 5.28 do not hold, generally speaking, as it ensues from the following generic counterexamples collectively with Subsubsections 7.1.1 and 7.1.3:

Example 5.38. Suppose C is $\sim$-subclassical, i.e., $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (cf. Theorem 5.22). Then, $\mathcal{B} \triangleq (\mathcal{A} \times \mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}$ is truth-non-empty, non-$\sim$-paraconsistent and, by (2.7), paracomplete, for $\mathcal{A}$ is so, in which case the logic of $\mathcal{B}$ is a proper (inferentially) paracomplete extension of C, in view of (2.6) (and Proposition 2.10). □

Example 5.39. Let $\sqsupset$ be a (possibly, secondary) binary connective of Σ. Suppose both $\{\mathsf{f},\mathsf{t}\}$ and $\{\mathsf{f},\mathsf{n}[/\mathsf{b}],\mathsf{t}\}$ form subalgebras of $\mathfrak{A}$, in which case $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$ is a submatrix of $\mathcal{A}_{\not\mathsf{b}}$, $\{\mathcal{A}_{\not\mathsf{b}}[,\mathcal{A}_{\not\mathsf{n}}]\}$ defining $C^{\mathrm{R}}[\cap C^{\mathrm{EM}}]$, in view of Corollary 5.33 [and Theorem 5.28(iii)⇒(iv)], while $C^{\mathrm{R}}[\cap C^{\mathrm{EM}}]$ satisfies $x_0 \sqsupset x_0$, whereas $\{x_0, x_0 \sqsupset x_1\} \vdash x_1$ is true in $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$, in which case $\mathcal{B} \triangleq (\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}))$ is truth-non-empty, paracomplete, in view of (2.7), for $\mathcal{A}_{\not\mathsf{b}}$ is so, and a model of the rule $\{\sim^i x_0 \sqsupset \sim^{1-i} x_0 \mid i \in 2\} \vdash (x_0 \vee \sim x_0)$, in its turn, [being also true in $\mathcal{A}_{\not\mathsf{n}}$ but] not being true in $\mathcal{A}_{\not\mathsf{b}}$ under $[x_0/\mathsf{n}]$, and so, by (2.6) (and Proposition 2.10), the logic of $\{\mathcal{B}[,\mathcal{A}_{\not\mathsf{n}}]\}$ is a proper [both $\sim$-paraconsistent and] (inferentially) paracomplete extension of $C^{\mathrm{R}}[\cap C^{\mathrm{EM}}]$. □

Example 5.39 and Subsubsection 7.1.3 show that the preconditions in the formulation of Theorem 5.36 cannot be omitted. And what is more, as it follows from Theorem 5.36 [resp., Corollary 5.64(ii) below], the condition of existence of implication $\sqsupset$ holding both the Reflexivity axiom in $\{\mathcal{A}_{\not\mathsf{b}}[,\mathcal{A}_{\not\mathsf{n}}]\}$ and the Modus Ponens rule in $\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}$ is essential within Example 5.39.

5.5.1.1. The meet with the least non-paracomplete extension. Next, C is said to be *hereditary*, provided $C^{\mathrm{EM}\times\mathrm{R}} \triangleq (C^{\mathrm{EM}} \cap C^{\mathrm{R}})$ is both $\sim$-paraconsistent and inferentially paracomplete.

Corollary 5.40. *The following are equivalent:*

(i) *C is hereditary;*
(ii) *C^{EM} is $\sim$-paraconsistent, while C^{R} is inferentially paracomplete;*
(iii) *both $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ and $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ form subalgebras of $\mathfrak{A}$;*

in which case:

(1) *$C^{\mathrm{EM}\times\mathrm{R}}$ is:*
  (a) *defined by $\{\mathcal{A}_{\not\mathsf{n}},\mathcal{A}_{\not\mathsf{b}}\}$, and so is inductive, inferentially consistent, non-pseudo-axiomatic and $\vee$-disjunctive, while it is purely inferential iff C is so;*
  (b) *axiomatized by:*

$$\{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash ((x_2 \vee \sim x_2) \vee x_0) \tag{5.14}$$

relatively to C, and so is a proper extension of C;

(2) *$\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, that is, C is $\sim$-subclassical.*

Proof. First, (i)⇔(ii) is by the fact that C^{EM} is not inferentially paracomplete, for it is monotonic and not paracomplete, while C^{R} is not $\sim$-paraconsistent, for it is transitive and inherits (3.3) held in its $\vee$-disjunctive sublogic C (cf. Corollary 3.17 and Remark 3.31 with $j = 0$). Next, (ii)⇔(iii) is by Theorems 5.28 and 5.37. Further, assume (iii) holds. Then, (1)(a) is by Theorem 5.28, Remarks 3.15, 3.31 with $j = 0$, Corollaries 3.17,5.33, Lemma 5.13 and (2.6). Likewise, (1)(b) is by Theorem 3.24, for any submatrix $\mathcal{B}$ of $\mathcal{A}$ satisfies (5.1) iff $\{\mathsf{n},\mathsf{b}\} \nsubseteq B$, that is, $\mathcal{B}$ is a submatrix of either $\mathcal{A}_{\not\mathsf{n}}$ or $\mathcal{A}_{\not\mathsf{b}}$, and the fact that (5.14) is not true in $\mathcal{A}$ under $[x_0/\mathsf{f}, x_1/\mathsf{b}, x_2/\mathsf{n}]$. Finally, (2) is by Theorem 5.22, as required. □

5.5.1.1.1. The self-extensionality of the meet.

Theorem 5.41. *Suppose C is hereditary. Then, the following are equivalent:*

(i) *$C^{\mathrm{EM}\times\mathrm{R}}$ is self-extensional;*
(ii) *$C^{\mathrm{EM}\times\mathrm{R}}$ has the property of Weak Contraposition with respect to $\sim$;*
(iii) *$\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}$ is a model of $C^{\mathrm{EM}\times\mathrm{R}}$;*
(iv) *$\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}$ is isomorphic to $\mathcal{A}_{\not\mathsf{b}/\not\mathsf{n}}$;*
(v) *$\mu\upharpoonright A_{\not\mathsf{n}/\not\mathsf{b}}$ is an isomorphism from $\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}$ onto $\mathcal{A}_{\not\mathsf{b}/\not\mathsf{n}}$;*
(vi) *$\mathfrak{A}_{\not\mathsf{n}/\not\mathsf{b}}$ is specular;*
(vii) *$C^{\mathrm{EM}\times\mathrm{R}}$ is defined by $\{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}, \overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}\}$;*
(viii) *$\mathfrak{A}_{\not\mathsf{n}}$ is isomorphic to $\mathfrak{A}_{\not\mathsf{b}}$;*
(ix) *$(\psi \in C^{\mathrm{EM}\times\mathrm{R}}(\phi)) \Leftrightarrow (\mathfrak{A}_{\not\mathsf{n}/\not\mathsf{b}} \models (\phi \lessapprox \psi))$, for all $\phi,\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$;*
(x) *there is some class K of Σ-algebras satisfying semilattice identities for $\wedge$ such that $(\psi \in C^{\mathrm{EM}\times\mathrm{R}}(\phi)) \Leftrightarrow (\mathsf{K} \models (\phi \lessapprox \psi))$, for all $\phi,\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$;*
(xi) *$(\psi \equiv_{C^{\mathrm{EM}\times\mathrm{R}}} \phi) \Leftrightarrow (\mathfrak{A}_{\not\mathsf{n}/\not\mathsf{b}} \models (\phi \approx \psi))$, for all $\phi,\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$;*
(xii) *there is some class K of Σ-algebras such that $(\psi \equiv_{C^{\mathrm{EM}\times\mathrm{R}}} \phi) \Leftrightarrow (\mathsf{K} \models (\phi \approx \psi))$, for all $\phi,\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$;*

in which case $\mathrm{IV}(C^{\mathrm{EM}\times\mathrm{R}}) = \mathbf{V}(\mathfrak{A}_{\not\mathsf{n}/\not\mathsf{b}})$.

Proof. We use Corollary 5.40 tacitly.

First, the equivalence of (i,ix–xii) as well as the final conclusion are due to Theorem 3.9 and Remark 3.31 with $j = 0$, while (i)⇒(ii) is by (3.15), Corollary 3.17, Claim 4.39 and Remark 3.31 with $j = 0$, whereas (ii)⇒(iii) is by Remark 3.31 and Claim 4.40.

Next, assume (iii) holds. Then, $\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}} \in \mathrm{Mod}(C^{\mathrm{EM}\times\mathrm{R}})$, being finite, is finitely-generated, is consistent, in view of (3.13) true in $\mathfrak{A}$, for $\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}$ is truth-non-empty, and, being a submatrix of $\overleftarrow{\mathcal{A}}$, is both simple and $\vee$-disjunctive, by Lemmas 3.4, 3.6 and Remarks 3.15 and 3.31 with $j = 1$. Hence, by Lemma 2.19, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\{\mathcal{A}_{\not\mathsf{n}},\mathcal{A}_{\not\mathsf{b}}\})^I$, some subdirect product $\mathcal{D}$ of it and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}})$, in which case, by (2.6) and Remark 3.15, $\mathcal{D}$ is both consistent and $\vee$-disjunctive. Moreover, by Lemmas 3.4, 3.6 and Remarks 3.15 and 3.31 with $j = 0$, every $\mathcal{C}_i$, where $i \in I$, is both simple and $\vee$-disjunctive. Therefore, by Corollary 3.16, there is some $i \in I$ such that $h \triangleq (\pi_i \upharpoonright D) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$, in which case, by

Proposition 2.16, we have $(\ker h) = h^{-1}[\Delta_{C_i}] = \partial(\mathcal{D}) = g^{-1}[\Delta_{A_{\not\mathsf{n}/\not\mathsf{b}}}] = (\ker g)$, and so, by Proposition 2.15, $e \triangleq (h \circ g^{-1})$ is an embedding of $\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}$ into $\mathcal{C}_i$, and so into either $\mathcal{A}_{\not\mathsf{n}}$ or $\mathcal{A}_{\not\mathsf{b}}$. On the other hand, $\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}$, $\mathcal{A}_{\not\mathsf{n}}$ and $\mathcal{A}_{\not\mathsf{b}}$ are all three-valued. Therefore, e is an isomorphism from $\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}$ onto either $\mathcal{A}_{\not\mathsf{n}}$ or $\mathcal{A}_{\not\mathsf{b}}$. Finally, $\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}$ is truth/false-singular, while $\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}$ is not so, in which case they are not isomorphic, and so (iv) holds.

Further, (vi)⇔(v) is by the following claim:

Claim 5.42. *Any embedding e of a submatrix $\mathcal{B}$ of $\overleftarrow{\mathcal{A}}$ into $\mathcal{A}$ is equal to $\mu{\restriction}B$.*

Proof. Then, since $(\sim^{\mathfrak{A}} a = a) \Leftrightarrow (a \in \{\mathsf{n},\mathsf{b}\})$, for all $a \in A$, we have both $e[\{\mathsf{n},\mathsf{b}\} \cap B] \subseteq \{\mathsf{n},\mathsf{b}\}$ and, by the injectivity of e, $e[\{f,\mathsf{t}\} \cap B] \subseteq \{\mathsf{f},\mathsf{t}\}$. Moreover, as $\mathsf{n},\mathsf{t} \in D^{\overleftarrow{\mathcal{A}}} \not\ni \mathsf{b},\mathsf{f}$, while $(\{\mathsf{n},\mathsf{b}\} \cap D^{\mathcal{A}}) = \{\mathsf{b}\}$, whereas $(\{\mathsf{f},\mathsf{t}\} \cap D^{\mathcal{A}}) = \{\mathsf{t}\}$, we then get $e(\mathsf{n}) = \mathsf{b}$, if $\mathsf{n} \in B$, $e(\mathsf{t}) = \mathsf{t}$, if $\mathsf{t} \in B$, $e(\mathsf{b}) = \mathsf{n}$, if $\mathsf{b} \in B$, and $e(\mathsf{f}) = \mathsf{f}$, if $\mathsf{f} \in B$, as required. □

Furthermore, we use the fact that $\mathfrak{A}_{\not\mathsf{n}/\not\mathsf{b}}{\restriction}\Sigma^+$ is the three-element chain distributive lattice with $\mathsf{f} \leqslant^{\mathfrak{A}_{\not\mathsf{n}/\not\mathsf{b}}} (\mathsf{b}/\mathsf{n}) \leqslant^{\mathfrak{A}_{\not\mathsf{n}/\not\mathsf{b}}} \mathsf{t}$ tacitly. Then, (vi)⇔(viii) is by the following immediate consequence of it:

Claim 5.43. *Any isomorphism from $\mathfrak{A}_{\not\mathsf{n}}$ onto $\mathfrak{A}_{\not\mathsf{b}}$ is equal to $\mu{\restriction}A_{\not\mathsf{n}}$.*

Finally, (v)⇔(vi) is immediate, while (iv)⇒(vii) is by (2.6), whereas (vii)⇒(ix) is by the Prime Ideal Theorem and the fact $D^{\overleftarrow{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}}} = (\{\mathsf{t}\}/\{\mathsf{n},\mathsf{t}\})$ and $D^{\mathcal{A}_{\not\mathsf{n}/\not\mathsf{b}}} = (\{\mathsf{b},\mathsf{t}\}/\{\mathsf{t}\})$ are exactly all prime filters of $\mathfrak{A}_{\not\mathsf{n}/\not\mathsf{b}}{\restriction}\Sigma^+$. □

Further, by the congruence-distributivity of lattice expansions, Lemma 3.32 and Corollary 2.5, we have:

Lemma 5.44. $\mathrm{Si}(\mathbf{V}(\mathfrak{A})) = \mathbf{IS}_{>1}\mathfrak{A}$.

Note that $(\mathfrak{DM}_4{\restriction}\{\mathsf{f},\mathsf{t}\}) \in \mathsf{BL} \not\ni (\mathfrak{DM}_4{\restriction}\{\mathsf{f},(\mathsf{n}/\mathsf{b}),\mathsf{t}\}) \in \mathsf{KL} \not\ni \mathfrak{DM}_4$. In this way, by Remark 2.3 and Lemma 5.44, we immediately get:

Corollary 5.45. *Let $a \in \{\mathsf{n},\mathsf{b}\}$. Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and $\{\mathsf{f},a,\mathsf{t}\}$ does not form [resp., forms] a subalgebra of $\mathfrak{A}$. Then, there is no non-trivial proper subvariety of $\mathbf{V}(\mathfrak{A})$ other than $\mathbf{V}(\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{t}\})$ relatively axiomatized by (3.17) [and $\mathbf{V}(\mathfrak{A}{\restriction}\{\mathsf{f},a,\mathsf{t}\})$ relatively axiomatized by (3.16)].*

After all, combining Proposition 2.18, Remarks 2.8, 2.9, 3.31 with $j = 0$, Theorems 5.22, 5.23, 5.28, 5.41, Lemma 5.13, Corollaries 5.40, 5.45 and Example 2.12, we eventually get:

Theorem 5.46. *Suppose C is hereditary (as well as purely inferential), while $C^{\mathrm{EM}\times\mathrm{R}}$ is self-extensional. Then, there is no inferentially consistent proper self-extensional (non-pseudo-axiomatic/purely-inferential) extension of $C^{\mathrm{EM}\times\mathrm{R}}$ other than $C^{\mathrm{PC}}_{(/+0)}$, being, in its turn, both so and inductive.*

On the other hand, any logic is either purely-inferential or, otherwise, non-pseudo-axiomatic. Therefore, by Remarks 2.8, 2.11, 3.31 with $j = 0$, 3.15, Corollaries 3.17, 5.40 and Theorems 3.24, 5.22, 5.41 and 5.46, we also get the following interesting non-trivial consequence:

Corollary 5.47. *Suppose C is hereditary, and $C^{\mathrm{EM}\times\mathrm{R}}$ is self-extensional. Then, any extension of $C^{\mathrm{EM}\times\mathrm{R}}$ is $\vee$-disjunctive, whenever it is self-extensional.*

5.5.2. *Miscellaneous extensions.* By $C^{[\mathrm{EM}+]\mathrm{NP}}$ we denote the least non-$\sim$-paraconsistent extension of $C^{[\mathrm{EM}]}$, viz., that which is relatively axiomatized by the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \sim x_0\} \vdash x_1. \tag{5.15}$$

Likewise, by $C^{[\mathrm{EM}+]\mathrm{MP}}$ we denote the extension of $C^{[\mathrm{EM}]}$ relatively axiomatized by the rule:

$$\{x_0, \sim x_0 \vee x_1\} \vdash x_1, \tag{5.16}$$

being nothing but Modus Ponens for the *material* implication $\sim x_0 \vee x_1$. (Clearly, it is a/an sublogic/extension of $C^{[\mathrm{EM}+](\mathrm{R/NP})}$, in view of (3.3) held in C by its $\vee$-disjunctivity (cf. Corollary 3.17 and Remark 3.31 with $j = 0$).) An extension of C is said to be *Kleene*, whenever it satisfies the rule (5.14).

Lemma 5.48. *Let I be a finite set, $\overline{\mathcal{C}} \in \{\mathcal{A}, \overleftarrow{\mathcal{A}}, \overrightarrow{\mathcal{A}}\}^I$, and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent submatrix of $\prod_{i\in I} \mathcal{C}_i$. Then, $\hom(\mathcal{B}, \overrightarrow{\mathcal{A}}) \neq \varnothing$.*

Proof. Consider the following complementary cases:

(1) $\mathcal{B}$ is truth-empty.
Take any $i \in I \neq \varnothing$, for $\mathcal{B}$ is consistent. Then, $h \triangleq (\pi_i{\restriction}B) \in \hom(\mathfrak{B},\mathfrak{A})$. Moreover, $D^{\mathcal{B}} = \varnothing \subseteq h^{-1}[\{\mathsf{t}\}]$. Hence, $h \in \hom(\mathcal{B}, \overrightarrow{\mathcal{A}})$.

(2) $\mathcal{B}$ is not truth-empty.
Then, $B \subseteq A^I$ is finite, for both I and A are so, and so is $D^{\mathcal{B}} \subseteq B$. Hence, since $\mathcal{B}$ is $\wedge$-conjunctive, for every member of $\{\mathcal{A}, \overleftarrow{\mathcal{A}}, \overrightarrow{\mathcal{A}}\}$ is so, $D^{\mathcal{B}}$ has a least element a with respect to $\leqslant^{\mathfrak{B}}$. Therefore, as $\mathcal{B}$ is consistent but not $\sim$-paraconsistent, $\sim^{\mathfrak{B}} a \notin D^{\mathcal{B}}$. Then, there is some $i \in I$, in which case $h \triangleq (\pi_i{\restriction}B) \in \hom(\mathcal{B},\mathcal{C}_i) \subseteq \hom(\mathfrak{B},\mathfrak{A})$, such that $h(\sim^{\mathfrak{B}} a) \notin D^{\mathcal{C}_i}$. If there was some $b \in D^{\mathcal{B}}$, in which case $a \leqslant^{\mathfrak{B}} b$, such that $h(b) \neq \mathsf{t}$, we would have $\mathcal{C}_i \in \{\mathcal{A}, \overleftarrow{\mathcal{A}}\}$ and $(\{\mathsf{b},\mathsf{n}\} \cap D^{\mathcal{C}_i}) \ni h(b) \leqslant^{\mathfrak{A}} h(a) \leqslant^{\mathfrak{A}} h(b)$, in which case we would get $h(a) = h(b)$, and so $h(\sim^{\mathfrak{B}} a) = \sim^{\mathfrak{A}} h(a) = \sim^{\mathfrak{A}} h(b) = h(b) \in D^{\mathcal{C}_i}$. Thus, $h \in \hom(\mathcal{B}, \overrightarrow{\mathcal{A}})$, as required. □

Corollary 5.49. *Let I be a finite set , $\overline{\mathcal{C}} \in \{\mathcal{A}, \overleftarrow{\mathcal{A}}, \overrightarrow{\mathcal{A}}\}^I$, and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent non-paracomplete submatrix of $\prod_{i\in I} \mathcal{C}_i$. Then, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\hom(\mathcal{B}, \mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}) \neq \varnothing$.*

Proof. Then, by Lemma 5.48, there is some $h \in \hom(\mathcal{B}, \overrightarrow{\mathcal{A}}) \neq \varnothing$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, and so $h \in \hom^{\mathrm{S}}(\mathcal{B}, \mathcal{D})$, where $\mathcal{D} \triangleq (\overrightarrow{\mathcal{A}}{\restriction}D)$. Hence, by (2.7), $\mathcal{D}$ is not paracomplete. Therefore, as $x_0 \vee \sim x_0$ is not true in $\overrightarrow{\mathcal{A}}$ under $[x_0/\langle\mathsf{b}/\mathsf{n}\rangle]$, we have $(D \cap \{\mathsf{b}, \mathsf{n}\}) = \varnothing$. On the other hand, $\mathcal{D}$, being non-paracomplete, is truth-non-empty, for $D \neq \varnothing$. Therefore, $\mathsf{t} \in D$, in which case $\mathsf{f} = \sim^{\mathfrak{A}}\mathsf{t} \in D$, and so $D = \{\mathsf{f}, \mathsf{t}\}$, in which case $\mathcal{D} = (\mathcal{A}{\restriction}D)$, as required. □

Theorem 5.50. *Suppose C is [not] maximally $\sim$-paraconsistent. Then, $C^{\mathrm{EM+NP}}$ is consistent iff C is $\sim$-subclassical, in which case $C^{\mathrm{EM+NP}}$ is defined by $[\mathcal{A}_{\not\mathsf{n}} \times]\mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}$.*

Proof. First, assume $C^{\mathrm{EM+NP}}$ is consistent, in which case $x_0 \notin T \triangleq C^{\mathrm{EM+NP}}(\varnothing)$, while, by the structurality of $C^{\mathrm{EM+NP}}$, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of $C^{\mathrm{EM+NP}}$ (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, T \cap \mathrm{Fm}^{1}_{\Sigma}\rangle$, in view of (2.6). Hence, by Lemma 2.19, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, in which case this is a submatrix of $\mathcal{A}^I$, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \Re(\mathcal{B}))$, in which case, by (2.6), $\mathcal{D}$ is a consistent model of $C^{\mathrm{EM+NP}}$, so it is neither $\sim$-paraconsistent nor paracomplete. Thus, by Corollary 5.49 and Theorem 5.22, C is $\sim$-subclassical.

Conversely, assume C is $\sim$-subclassical. Consider the following complementary cases:

- C is maximally $\sim$-paraconsistent.
  Then, by Theorems 5.22 and 5.28(i)⇒(v,xiii), $C^{\mathrm{EM+NP}} = C^{\mathrm{EM}} = C^{\mathrm{PC}}$ is defined by the consistent $\mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}$, and so, in particular, is consistent, as required.
- C is not maximally $\sim$-paraconsistent.
  Then, by Theorem 5.28(iii/iv)⇒(i), C^{EM} is defined by $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}_{\not\mathsf{n}}$. Moreover, by Theorem 5.22, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so of $\mathfrak{A}_{\not\mathsf{n}}$, in which case $\mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}$ is a submatrix of $\mathcal{A}_{\not\mathsf{n}}$, and so, by (2.6), $\mathcal{B} \triangleq (\mathcal{A}_{\not\mathsf{n}} \times \mathcal{A}_{\not\mathsf{n}\not\mathsf{b}})$ is a model of C^{EM}. Moreover, $\{a, \sim^{\mathfrak{A}} a\} \subseteq \{\mathsf{t}\}$, for no $a \in \{\mathsf{f}, \mathsf{t}\}$. Therefore, $\mathcal{B}$ is not $\sim$-paraconsistent, so it is a model of $C^{\mathrm{EM+NP}}$. Conversely, consider any finite set I, any $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A}_{\not\mathsf{n}})^I$ and any subdirect product $\mathcal{D} \in \mathrm{Mod}(C^{\mathrm{EM+NP}})$ of $\overline{\mathcal{C}}$, in which case $\mathcal{D}$ is a non-$\sim$-paraconsistent non-paracomplete submatrix of $\mathcal{A}^I$. Put $J \triangleq \hom(\mathcal{D}, \mathcal{B})$. Consider any $a \in (D \setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent, and so, by Corollary 5.49, there is some $g \in \hom(\mathcal{D}, \mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}) \neq \varnothing$. Moreover, there is some $i \in I$, in which case $f \triangleq (\pi_i{\restriction}D) \in \hom(\mathcal{D}, \mathcal{A}_{\not\mathsf{n}})$, such that $f(a) \notin D^{\mathcal{A}_{\not\mathsf{n}}}$. Then, $h \triangleq (f \times g) \in J$ and $h(a) \notin D^{\mathcal{B}}$. In this way, $(\prod \Delta_J) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B}^J)$. Thus, by (2.6) and Theorem 2.20, $C^{\mathrm{EM+NP}}$ is finitely-defined by the consistent six-valued $\mathcal{B}$, and so is consistent and, being finitary, for the four-valued C is so, is defined by $\mathcal{B}$, as required. □

Remark 5.51. Let C' be a Kleene extension of C (in particular, a non-paracomplete one, in view of (3.3)). Then, we have $\{x_0 \vee x_1, \sim x_0 \vee x_1\} \vdash_{C'} (\sim(x_0 \vee x_1) \vee x_1)$. Therefore, in view of (3.3), C' satisfies (5.9) iff it satisfies (5.16). In particular, $C^{\mathrm{EM+MP}} = C^{\mathrm{EM+R}}$. □

Corollary 5.52. *Let C' be a Kleene (in particular, non-paracomplete, in view of* (3.3)*) extension of C. Suppose C is not maximally $\sim$-paraconsistent, and* (5.16) *is not satisfied in C'. Then, C' is a sublogic of $C^{\mathrm{EM+NP}}$.*

Proof. In that case, by Theorem 5.28(iii|iv)⇒(i), $A_{\not\mathsf{n}} = \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, C^{EM} being defined by the $\sim$-superclassical $\wedge$-conjunctive Σ-matrix $\mathcal{A}_{\not\mathsf{n}}$ (cf. (2.6) and Remark 3.31 with $j = 0$) with underlying algebra being a $(\wedge, \vee)$-lattice. Moreover, by Theorem 2.20, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C') \subseteq \mathrm{Mod}(C)$ of it not being a model of (5.16), in which case it is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.31 with $j = 0$), while $(\mathfrak{D}{\restriction}\Sigma_0) \in \mathsf{DML}$, for $\mathsf{DML} \ni \mathfrak{DM}_4 = (\mathfrak{A}{\restriction}\Sigma_0)$ is a variety. Therefore, there are some $a \in D^{\mathcal{D}} \subseteq \{\mathsf{b}, \mathsf{t}\}^I$, in which case $\sim^{\mathfrak{D}} a \leqslant^{\mathfrak{D}} a$, and some $b \in (D \setminus D^{\mathcal{D}})$ such that $(\sim^{\mathfrak{D}} a \vee^{\mathfrak{D}} b) \in D^{\mathcal{D}}$, in which case $(\sim^{\mathfrak{D}} a \vee^{\mathfrak{D}} b) \leqslant^{\mathfrak{D}} (a \vee^{\mathfrak{D}} b)$, and so $(a \vee^{\mathfrak{D}} b) \in D^{\mathcal{D}}$. Hence, by (5.14), $(b \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} b) = ((b \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} b) \vee^{\mathfrak{D}} b) \in D^{\mathcal{D}}$, in which case $b \in \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}^I$, and so the submatrix $\mathcal{E}$ of $\mathcal{D}$, generated by $\{a, b\}$, is a submatrix of $\mathcal{A}^I_{\not\mathsf{n}}$, for $A_{\not\mathsf{n}}$ forms a subalgebra of $\mathfrak{A}$, in which (5.16) is not true under $[x_0/a, x_1/b]$. Then, by (2.6), $\mathcal{E}$ is a model of both C^{EM} and C'. In this way, Proposition 4.1 and Lemma 4.34 with $\wr = \sim$, $\barwedge = \wedge$, $\veebar = \vee$, $\mathcal{A}_{\not\mathsf{n}}$ instead of $\mathcal{A}$, C^{EM} instead of C and $\mathrm{Cn}^{\omega}_{\mathcal{E}}$ instead of C' complete the argument. □

Theorem 5.53 (cf. [25] for the case $\Sigma = \Sigma_0$)**.** *Suppose C is [not] non-$\sim$-subclassical and (not) maximally $\sim$-paraconsistent. Then, extensions of C^{EM} form the one[two](two)[(four)]-element chain $C^{\mathrm{EM}} = (\mathrm{Cn}^{\omega}_{\mathcal{A}_{\not\mathsf{n}}} \subsetneq)[C^{\mathrm{EM+NP}} = \mathrm{Cn}^{\omega}_{(\mathcal{A}_{\not\mathsf{n}}\times)\mathcal{A}_{\not\mathsf{n}\not\mathsf{b}}} = (\subsetneq)C^{\mathrm{PC}} = C^{\mathrm{EM+\{R/MP\}}} \subsetneq] \mathrm{Cn}^{\omega}_{\varnothing}$. [(Moreover, in case K^{b}_4 forms a subalgebra of $\mathfrak{A}^2$ {in particular, $\Sigma = \Sigma_{0\langle 1\rangle}$}, both proper consistent extensions satisfy same axioms as C^{EM} does, and so are not axiomatic, in which case C^{EM} has no proper consistent axiomatic extension)].*

Proof. By Theorems 5.22, 5.23, 5.28(iii|iv|vi)⇒(i), 5.30, 5.50, Remark 5.51, Corollary 5.52, Lemma 4.32 [(as well as the fact that (5.16) is not true in the consistent truth-non-empty Σ-matrix $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\})$ under $[x_0/\langle\mathsf{b}, \mathsf{t}\rangle, x_1/\langle\mathsf{f}, \mathsf{t}\rangle]$)]. □

In view of Lemma 4.22, Theorem 5.53 shows that $(\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}^{\omega}_{\Sigma}] \veebar x_0)$ cannot be replaced by $\mathcal{C}$ in the item (ii)**b)** of Theorem 3.24, when taking $\mathsf{M} = \{\mathcal{A}_{\not\mathsf{n}}\}$ and $\mathcal{C} = \{(5.15)\}$. In addition, the particular case of Theorem 5.53 with $\Sigma = \Sigma_{01}$ provides the "bounded" extension of [31] void of the rather unnatural restriction by merely non-empty sequents. This point, being essentially beyond the scopes of the present study, is going to be discussed in detail elsewhere.

5.5.2.1. Modus ponens versus truth-singularity.

Lemma 5.54. *Let $\mathcal{B}$ be a truth-singular $\wedge$-conjunctive Σ-matrix. Suppose $(\mathfrak{B}{\restriction}\Sigma_0) \in \mathsf{DML}$. Then, any $b \in D^{\mathcal{B}}$ is a unit of $\mathfrak{B}{\restriction}\Sigma^+$, in which case $\sim^{\mathfrak{B}} b$ is a zero of it, and so $\mathcal{B}$ is a model of* (5.16).

Proof. In that case, $\mathfrak{B}{\restriction}\Sigma^+$ is a distributive lattice and $D^{\mathcal{B}}$ is a filter of it. Then, for any $a \in B$, we have $b \leqslant^{\mathfrak{B}} (a \vee^{\mathfrak{B}} b)$, in which case we get $(a \vee^{\mathfrak{B}} b) \in D^{\mathcal{B}}$, and so $(a \vee^{\mathfrak{B}} b) = b$, as required. □

As the truth-singularity is preserved under $\Re$, by the $\wedge$-conjunctivity of $\mathcal{A}$ (cf. Remark 3.31 with $j=0$), (2.6), Lemmas 5.54, 3.5 and Corollary 2.17, we immediately get:

Corollary 5.55. *Any truth-singular model of C is a model of C^{MP}.*

Lemma 5.56. $\Upsilon_0 \triangleq \{\sim^i x_0 \vee x_1 \mid i \in 2\}$ *is a unitary congruence determinant for any $\wedge$-conjunctive Σ_0-matrix $\mathcal{B}$ such that $\mathfrak{B} \in \mathsf{DML}$.*

Proof. Using the distributivity of $\mathfrak{B}\restriction\Sigma^+$, the $\wedge$-conjunctivity of $\mathcal{B}$ as well as the identities (3.13), (3.14) and (3.15), it is routine checking that $\theta \triangleq \theta^{\mathcal{B}}_{\varepsilon\Upsilon_0} \in \mathrm{Con}(\mathfrak{B})$. Finally, consider any $\langle a,b\rangle \in \theta$. Then, $\mathcal{B} \models (\bigwedge \varepsilon\Upsilon_0)[x_0/a, x_1/b, x_2/(a \wedge^{\mathfrak{B}} b)]$, being a consequence of $\mathcal{B} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon\Upsilon_0)[x_0/a, x_1/b]$, implies $(a \in D^{\mathcal{B}}) \Leftrightarrow (b \in D^{\mathcal{B}})$, as required. ☐

Next, combining Lemmas 2.2, 3.32, 5.44, Remark 2.3 and Corollary 2.7, by the congruence-distributivity of lattice expansions, we get the following quite important non-trivial algebraic inheritance result:

Corollary 5.57. *Let $\mathfrak{B} \in \mathbf{V}(\mathfrak{A})$. Then,* $\mathrm{Con}(\mathfrak{B}) = \mathrm{Con}(\mathfrak{B}\restriction\Sigma_0)$.

In particular, by (2.6), Lemmas 3.6, 3.5, 5.56, Corollaries 2.17, 5.57 and the $\wedge$-conjunctivity of $\mathcal{A}$ (cf. Remark 3.31 with $j=0$), we also have:

Corollary 5.58. Υ_0 *is a unitary congruence[equality] determinant for* $\mathrm{Mod}_{[*]}(C)$.

Note that the following rules are satisfied in C^{MP}, in view of (3.3) and (3.4) held in C by its $\vee$-disjunctivity (cf. Corollary 3.17 and Remark 3.31 with $j=0$):

$$\{x_0, x_1, \sim^i x_0 \vee x_2\} \vdash (\sim^i x_1 \vee x_2), \tag{5.17}$$

where $i \in 2$. In this way, by Corollary 5.58, we get:

Corollary 5.59. *Any $\mathcal{B} \in \mathrm{Mod}_*(C^{\mathrm{MP}})$ is truth-singular.*

Theorem 5.60. C^{MP} *is defined by* $\mathsf{S} \triangleq (\mathrm{Mod}(C) \cap \mathbf{P}^{\mathrm{SD}}(\mathbf{S}^*_*(\overrightarrow{\mathcal{A}})))$, *and so by the class of all truth-singular models of C.*

Proof. As $\overrightarrow{\mathcal{A}}$ is truth-singular, while the truth-singularity is preserved under both $\mathbf{P}$ and $\mathbf{S}$, by Corollary 5.55, we have $\mathsf{S} \subseteq \mathrm{Mod}(C^{\mathrm{MP}})$. Conversely, consider any $\mathcal{B} \in (\mathrm{Mod}_*(C^{\mathrm{MP}}) \cap \Re(\mathbf{P}^{\mathrm{SD}}(\mathbf{S}_*(\mathcal{A}))))$, in which case $\mathcal{B} \in \mathrm{Mod}(C)$, while, by Corollary 5.59, $\mathcal{B}$ is truth-singular, whereas $(\mathfrak{B}\restriction\Sigma_0) \in \mathsf{DML}$, and so, by the $\wedge$-conjunctivity of $\mathcal{A}$ (cf. Remark 3.31 with $j=0$) and Lemma 5.54, $D^{\mathcal{B}} = \{b\}$, whereas b is a unit of $\mathfrak{B}\restriction\Sigma^+$. Moreover, $\mathfrak{B} \in \mathbf{V}(\mathfrak{A})$, in which case, by Remark 2.3 and Lemma 5.44, $\mathfrak{B}$ is isomorphic to a subdirect product of some $\overline{\mathfrak{C}} \in (\mathbf{S}_{>1}\mathfrak{A})^I$, where I is a set, and so there is some embedding e of $\mathfrak{B}$ into $\prod_{i\in I} \mathfrak{C}_i$ such that, for each $i \in I$, $h_i \triangleq (\pi_i \circ e) \in \hom(\mathfrak{B}, \mathfrak{C}_i)$ is surjective, in which case C_i, being non-one-element, contains both t and f, and so, by Lemma 3.30, $h_i(b) = \mathsf{t}$. And what is more, for every $a \in B$ distinct from b, by the injectivity of e, there is some $i \in I$ such that $h_i(a) \neq h_i(b) = \mathsf{t}$. In this way, e is an isomorphism from $\mathcal{B}$ onto the subdirect product $(\prod_{i\in I}\langle \mathfrak{C}_i, \{\mathsf{t}\}\rangle)\restriction(\mathrm{img}\, e)$ of $\langle\langle \mathfrak{C}_i, \{\mathsf{t}\}\rangle\rangle_{i\in I} \in \mathbf{S}^*_*(\overrightarrow{\mathcal{A}})^I$. Hence, by (2.6), we get $\mathcal{B} \in \mathbf{I}(\mathsf{S})$. Then, Theorem 2.20, Corollary 5.55 and (2.6) complete the argument. ☐

5.6. **Self-extensionality.**

Theorem 5.61. *The following are equivalent:*

(i) *C is self-extensional;*
(ii) *C has the property of Weak Contraposition with respect to $\sim$;*
(iii) *$\overleftarrow{\mathcal{A}}$ is a model of C;*
(iv) *C is defined by $\{\mathcal{A}, \overleftarrow{\mathcal{A}}\}$;*
(v) $(\psi \in C(\phi)) \Leftrightarrow (\mathfrak{A} \models (\phi \lessapprox \psi))$, *for all* $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$;
(vi) *there is some class K of Σ-algebras satisfying the idempotencity and commutativity identities for $\wedge$ such that* $(\psi \in C(\phi)) \Leftrightarrow (\mathsf{K} \models (\phi \lessapprox \psi))$, *for all* $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$;
(vii) $(\psi \equiv_C \phi) \Leftrightarrow (\mathfrak{A} \models (\phi \approx \psi))$, *for all* $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$;
(viii) *there is some class K of Σ-algebras such that* $(\psi \equiv_C \phi) \Leftrightarrow (\mathsf{K} \models (\phi \approx \psi))$, *for all* $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$;
(ix) *there is an injective homomorphism from $\overleftarrow{\mathcal{A}}$ to $\mathcal{A}$;*
(x) *$\mathfrak{A}$ is specular;*
(xi) *μ is an isomorphism from $\overleftarrow{\mathcal{A}}$ onto $\mathcal{A}$;*
(xii) *$\overleftarrow{\mathcal{A}}$ is isomorphic to $\mathcal{A}$;*
(xiii) *C is defined by $\overleftarrow{\mathcal{A}}$;*
(xiv) *$\overrightarrow{\mathcal{A}}$ is a model of C;*
(xv) *any $\wedge$-conjunctive truth-non-empty Σ-matrix $\mathcal{B}$ such that $\mathfrak{B} \in \mathbf{V}(\mathfrak{A})$ is a model of C;*

in which case $\mathrm{IV}(C) = \mathbf{V}(\mathfrak{A})$.

Proof. First, the equivalence of (i,v-viii,xv) as well as the final conclusion are due to Theorem 3.9 and Remark 3.31 with $j=0$, while (i)⇒(ii) is by (3.15), Corollary 3.17, Claim 4.39 and Remark 3.31 with $j=0$, whereas (v)⇒(xv) is by Claim 3.10, while (ii)⇒(iii) is by Claim 4.40 and Remark 3.31.

Next, assume (xiv) holds. Consider any $\phi \in \mathrm{Fm}^\omega_\Sigma$, any $\psi \in C(\phi)$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ such that $h(\phi) \in D^{\overleftarrow{\mathcal{A}}}$. Then, by the structurality of C, Corollary 3.17(3.9) and Remark 3.31 with $j=0$, $(\sigma_{+1}(\psi) \vee x_0) \in C(\sigma_{+1}(\phi) \vee x_0)$. Let $g \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $[x_0/\mathsf{b}; x_{i+1}/h(x_i)]_{i\in\omega}$, in which case $(g \circ \sigma_{+1}) = h$, and so we have $g(\sigma_{+1}(\phi) \vee x_0) = (h(\phi) \vee^{\mathfrak{A}} \mathsf{b}) = \mathsf{t}$. Hence, by (xiv), we get $(h(\psi) \vee^{\mathfrak{A}} \mathsf{b}) = g(\sigma_{+1}(\psi) \vee x_0) = \mathsf{t}$. Therefore, we eventually get $h(\psi) \in D^{\overleftarrow{\mathcal{A}}}$. Thus, by Claim 3.10 and Remark 3.31 with $j = 1$, (iii) holds.

On the other hand, $D^{\mathcal{A}}$ and $D^{\overleftarrow{\mathcal{A}}}$ are exactly all non-empty proper prime filters of $\mathfrak{A}\restriction\Sigma^+$ (cf. Remark 3.31). Therefore, (iv)⇒(v) is by the Prime Ideal Theorem for distributive lattices (in particular, for $(\mathfrak{A}\restriction\Sigma^+) = \mathfrak{D}_2^2$).

Now, assume (iii) holds. In that case, (iv) is evident. Moreover, $\overleftarrow{\mathcal{A}}$ is consistent and, being finite, is finitely-generated. In addition, by Lemma 3.4 and Remark 3.31 with $j = 1$, $\overleftarrow{\mathcal{A}}$ is simple and ∨-disjunctive. Then, by Lemma 2.19, there is some finite set I, some I-tuple $\overline{\mathcal{C}}$ of submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \overleftarrow{\mathcal{A}})$, in which case, by Remark 3.15 and (2.6), $\mathcal{D}$ is consistent and ∨-disjunctive, and so, by Corollary 3.16, there is some $i \in I$ such that $h \triangleq (\pi_i\restriction D) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$. Moreover, by Lemmas 3.4, 3.6 and Remark 3.31 with $j = 0$, $\mathcal{C}_i$ is simple. Hence, by Proposition 2.16, $(\ker h) = \partial(\mathcal{D}) = (\ker g)$. Therefore, by Proposition 2.15, $e \triangleq (h \circ g^{-1}) \in \hom_{\mathrm{S}}(\overleftarrow{\mathcal{A}}, \mathcal{C}_i) \subseteq \hom(\overleftarrow{\mathcal{A}}, \mathcal{A})$ is injective, and so (ix) holds.

Furthermore, (ix)⇒(x) is by the following claim:

Claim 5.62. *Any injective homomorphism from $\overleftarrow{\mathcal{A}}$ to $\mathcal{A}$ is specular.*

Proof. Consider any injective $e \in \hom(\overleftarrow{\mathcal{A}}, \mathcal{A})$, in which case $e[D^{\overleftarrow{\mathcal{A}}}] \subseteq D^{\mathcal{A}}$, and so $e[D^{\overleftarrow{\mathcal{A}}}] = D^{\mathcal{A}}$, for e is injective, while $|D^{\overleftarrow{\mathcal{A}}}| = 2 = |D^{\mathcal{A}}|$. Hence, e, being injective, is an embedding of $\overleftarrow{\mathcal{A}}$ into $\mathcal{A}$. In this way, Claim 5.42 completes the argument. □

Finally, (x)⇒(xi) is immediate, while (xii/iii/xiv) is a particular case of (xi/xiii/xv)//, in view of Remark 3.31 with $j = 0$, respectively, whereas (xii)⇒(xiii) is by (2.6). □

As a first immediate *generic* consequence of Theorems 5.22, 5.61(i)⇒(x) and Lemma 5.35 with $B = \{\mathsf{b}, \mathsf{n}\}$ applicable to all bilattice expansions *at once* (cf. Subsubsection 7.1.2), we have:

Corollary 5.63. *Suppose $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, C is not self-extensional. In particular, C is ∼-subclassical, whenever it is self-extensional.*

Corollary 5.64. *Suppose C is self-extensional. Then, $\{\mathsf{n}(,\mathsf{f},\mathsf{t})\}$ forms a subalgebra of $\mathfrak{A}$ iff $\{\mathsf{b}(,\mathsf{f},\mathsf{t})\}$ does so. In particular, the following hold:*

(i) *the following are equivalent:*
  - **a)** *C satisfies Variable Sharing Property;*
  - **b)** *C is purely inferential;*
  - **c)** *C has no inconsistent formula;*
  - **d)** *$\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$;*
  - **e)** *$\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$;*
  - **f)** *there is no $\psi \in \mathrm{Fm}_\Sigma^1$ such that $\psi^{\mathfrak{A}}[A] = \{\mathsf{t}\}$;*
  - **g)** *there is no $\phi \in \mathrm{Fm}_\Sigma^1$ such that $\phi^{\mathfrak{A}}[A] = \{\mathsf{f}\}$.*

(ii) *[providing C is not purely inferential] C^{EM} is{ maximally} ∼-paraconsistent iff C^{R} is [non-]inferentially paracomplete, in which case, when $\mathfrak{A}_{\not{\mathsf{b}}}$ is regular (in particular, $\Sigma = \Sigma_{0[1]}$), C^{R} is maximally [non-]inferentially paracomplete, while any extension of C is both ∼-paraconsistent and [non-]inferentially paracomplete iff it is a sublogic of $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$, in its turn, being an axpansion of $LP \cap K_3$.*

Proof. Since $\mu[\{\mathsf{n}(,\mathsf{f},\mathsf{t})\}] = \{\mathsf{b}(,\mathsf{f},\mathsf{t})\}$, in view of Theorems 5.15(i)⇔(iii), 5.28, 5.36, 5.37, 5.61(i)⇒(x), Lemmas 5.13, 5.14 and Corollary 5.26, it only remains to prove the equivalence of the subitems **f)** and **g)** to others within (i).

First, **f)** is a particular case of **b)**. Next, **f)**⇔**g)** is by the fact $\sim^{\mathfrak{A}}(\mathsf{f}/\mathsf{t}) = (\mathsf{t}/\mathsf{f})$.

Finally, assume $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}_\Sigma^1$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\varphi^{\mathfrak{A}}(\mathsf{n}) = \mu(\varphi^{\mathfrak{A}}(\mathsf{b}))$ and $\varphi^{\mathfrak{A}}[\{\mathsf{f}, \mathsf{t}\}] \subseteq \{\mathsf{f}, \mathsf{t}\}$, by Lemma 5.35 and Theorem 5.61(i)⇒(x), and so $\psi^{\mathfrak{A}}[A] = \{\mathsf{t}\}$, where $\psi \triangleq (x_0 \vee (\varphi \vee \sim\varphi)) \in \mathrm{Fm}_\Sigma^1$. Thus, **f)**⇒**e)** holds, as required. □

Corollary 5.64(i)**b)**⇐**f)**⇔**g)** collectively with Theorem 5.7 imply:

Corollary 5.65. *Any self-extensional four-valued expansion of C_{B} is not purely inferential iff it is definitionally equivalent to an expansion of C_{BB}.*

This clarifies the meaning of the bounded version C_{BB} of C_{B} (a much deeper justification of it is provided by Corollary 7.4 below). Subsubsection 7.1.3 shows that the condition of self-extensionality cannot be omitted in the formulations of Corollaries 5.64 and 5.65. As for Corollary 5.64(ii) (in case $\mathfrak{A}_{\not{\mathsf{b}}}$ is regular), it clarifies the meaning of the self-extensional (in view of Theorems 5.61 and 5.41) meet $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ to be studied far more in Paragraph 7.1.4.1.

5.6.1. *Self-extensional extensions.* After all, combining Propositions 2.18, 2.10, Remarks 2.8, 2.9, 3.31 with $j = 0$, Theorems 5.22, 5.23, 5.28, 5.41, 5.61, Lemma 5.13, Corollaries 5.40, 5.63, 5.45 and Example 2.12, we eventually get:

Theorem 5.66. *Suppose C is self-extensional and [not] maximally ∼-paraconsistent (as well as purely inferential). Then, there is no inferentially consistent proper self-extensional (non-pseudo-axiomatic/purely-inferential) extension of C other than $C^{\mathrm{PC}}_{(/+0)}$ [and $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$], being, in its[their] turn, both so and inductive [while the former being a proper extension of the latter].*

On the other hand, any logic is either purely-inferential or, otherwise, non-pseudo-axiomatic. Therefore, by Remarks 2.8, 2.11, 3.31 with $j = 0$, 3.15, Corollaries 3.17, 5.40, 5.63, 5.64 and Theorems 3.24, 5.22, 5.41 and 5.66, we also get the following interesting non-trivial consequence:

Corollary 5.67. *Suppose C is self-extensional [and maximally $\sim$-paraconsistent]. Then, any extension of C is $\vee$-disjunctive if[f] it is self-extensional.*

5.6.2. *Semantics of miscellaneous extensions versus self-extensionality.* By Theorems 5.60, 5.61(i)⇔(xiv) and (2.6), we first get:

Corollary 5.68. *C is self-extensional iff C^{MP} is defined by $\overrightarrow{\mathcal{A}}$.*

Likewise, we also have the following one more characterization of the self-extensionality of C:

Theorem 5.69. *C is self-extensional iff C^{NP} is defined by $\mathcal{A}\times\overrightarrow{\mathcal{A}}$.*

Proof. We use Theorem 5.61(i)⇔(xiv) tacitly. First, $\Delta_A\times\Delta_A$ is an embedding of $\overrightarrow{\mathcal{A}}$ into $\mathcal{A}\times\overrightarrow{\mathcal{A}}$. In this way, (2.6) yields the "if" part. Conversely, assume C is self-extensional. Then, $\mathcal{A}\times\overrightarrow{\mathcal{A}}$ is a model of C. Moreover, $\{a,\sim^{\mathfrak{A}}a\}\subseteq\{\mathsf{t}\}$, for no $a\in A$. Therefore, $\mathcal{A}\times\overrightarrow{\mathcal{A}}$ is not $\sim$-paraconsistent, so it is a model of C^{NP}. Finally, consider any finite set I, any $\overline{\mathcal{C}}\in\mathbf{S}(\mathcal{A})^I$ and any subdirect product $\mathcal{D}\in\mathrm{Mod}(C')$ of $\overline{\mathcal{C}}$, in which case $\mathcal{D}$ is a non-$\sim$-paraconsistent submatrix of $\mathcal{A}^I$. Put $J\triangleq\hom(\mathcal{D},\mathcal{A}\times\overrightarrow{\mathcal{A}})$. Consider any $a\in(D\setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent, and so, by Lemma 5.48, there is some $g\in\hom(\mathcal{D},\overrightarrow{\mathcal{A}})\neq\varnothing$. Moreover, there is some $i\in I$, in which case $f\triangleq(\pi_i{\upharpoonright}D)\in\hom(\mathcal{D},\mathcal{A})$, such that $f(a)\notin D^{\mathcal{A}}$. Then, $h\triangleq(f\times g)\in J$ and $h(a)\notin D^{\mathcal{A}\times\overrightarrow{\mathcal{A}}}$. In this way, $(\prod\Delta_J)\in\hom_{\mathrm{S}}(\mathcal{D},(\mathcal{A}\times\overrightarrow{\mathcal{A}})^J)$. Thus, by (2.6) and Theorem 2.20, C^{NP} is finitely-defined by $\mathcal{A}\times\overrightarrow{\mathcal{A}}$. Then, the finiteness of A completes the argument. □

5.7. **Axiomatic extensions.**

Lemma 5.70. *Suppose $\mathfrak{A}$ is regular and $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of it. Then, so does $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$.*

Proof. By contradiction. For suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$, in which case there is some $\varphi\in\mathrm{Fm}^3_\Sigma$ such that $A\ni\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{b},\mathsf{t})\notin\{\mathsf{f},\mathsf{b},\mathsf{t}\}=(A\setminus\{\mathsf{n}\})$, and so we have $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{b},\mathsf{t})=\mathsf{n}$. Therefore, as $\mathsf{t}\sqsubseteq\mathsf{b}$, by the regularity of $\mathfrak{A}$ and the reflexivity of $\sqsubseteq$, we get $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t},\mathsf{t})\sqsubseteq\mathsf{n}$. Hence, $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t},\mathsf{t})=\mathsf{n}\notin\{\mathsf{f},\mathsf{t}\}$. This contradicts to the assumption that $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, as required. □

By Theorems 5.22, 5.28(i)⇒(iv), Corollary 5.63 and Lemma 5.70, we first have:

Corollary 5.71. *Suppose C is $\sim$-subclassical (in particular, self-extensional) and maximally $\sim$-paraconsistent. Then, $\mathfrak{A}$ is not regular.*

Lemma 5.72. *Let $\mathcal{B}\in\mathbf{S}(\mathcal{A})$. Suppose $B\cup\{\mathsf{b}\}$ forms a regular subalgebra of $\mathfrak{A}$. Then, $\mathrm{Cn}^\omega_{\mathcal{B}}(\varnothing)\subseteq\mathrm{Cn}^\omega_{\mathcal{A}\upharpoonright(B\cup\{\mathsf{b}\})}(\varnothing)$.*

Proof. Consider any $\varphi\in(\mathrm{Fm}^\omega_\Sigma\setminus\mathrm{Cn}^\omega_{\mathcal{A}\upharpoonright(B\cup\{\mathsf{b}\})}(\varnothing))$, in which case there is some $h\in\hom(\mathfrak{Fm}^\omega_\Sigma,\mathfrak{A}{\upharpoonright}(B\cup\{\mathsf{b}\}))$ such that $h(\varphi)\in\{\mathsf{f},\mathsf{n}\}$. Take any $b\in B\neq\varnothing$. Define a $g:V_\omega\to B$ by setting:

$$g(x_i)\triangleq\begin{cases}b & \text{if } h(x_i)=\mathsf{b},\\ h(x_i) & \text{otherwise,}\end{cases}$$

for all $i\in\omega$. Let $e\in\hom(\mathfrak{Fm}^\omega_\Sigma,\mathfrak{B})$ extend g. Then, $e(x_i)=g(x_i)\sqsubseteq h(x_i)$, for all $i\in\omega$, in which case, by the regularity of $\mathfrak{A}{\upharpoonright}(B\cup\{\mathsf{b}\})$, we have $e(\varphi)\sqsubseteq h(\varphi)$, and so we eventually get $e(\varphi)\in\{\mathsf{f},\mathsf{n}\}$, as required. □

Notice that Lemma 5.72 with $B=\{f,\mathsf{t}\}$ subsumes Lemma 4.32(i)⇒(iii) with $a=\mathsf{b}$.

Theorem 5.73. *[Providing $\mathfrak{A}$ is regular/has no three-element subalgebra] C has a proper consistent axiomatic extension if[f] $\{\mathsf{f},\mathsf{b},\mathsf{t}\}/\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ [in which case the logic of $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}}$ is the only proper consistent axiomatic extension of C and is relatively axiomatized by (5.8)].*

Proof. The "if" part is by Lemma 5.27. [Conversely, consider any $\mathcal{A}\subseteq\mathrm{Fm}_\Sigma$ such that the axiomatic extension C' of C relatively axiomatized by $\mathcal{A}$ is both proper and consistent, in which case $\mathcal{A}\neq\varnothing$, while, by Corollary 2.21, C' is the logic of $\mathsf{S}\triangleq(\mathrm{Mod}(\mathcal{A})\cap\mathbf{S}_*(\mathcal{A}))$, so $\mathcal{A}\notin\mathsf{S}\neq\varnothing$. Take any $\mathcal{B}\in\mathsf{S}$, in which case it is both consistent and, as $\mathcal{A}\neq\varnothing$, truth-non-empty. Hence, by Lemma 5.18, $\{\mathsf{f},\mathsf{t}\}\subseteq B$. Therefore, if n was in B, then $(B\cup\{\mathsf{b}\})$ would be equal to A/B would belong to $\{\{\mathsf{f},\mathsf{n},\mathsf{t}\},A\}$, in which case, by Lemma 5.72/the fact that $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, $\mathcal{A}$ would belong to S. Thus, $B\in\{\{\mathsf{f},\mathsf{t}\},\{\mathsf{f},\mathsf{b},\mathsf{t}\}\}$. Then, by Lemma 5.70/the fact that $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, we conclude that $\{\mathsf{f},\mathsf{b},\mathsf{t}\}/\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. And what is more, in that case, by Lemma 5.72/the fact that $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, being three-element, does not form a subalgebra of $\mathfrak{A}$, we have $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}}\in\mathsf{S}\subseteq\mathbf{S}_*(\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}})$, and so C' is equal to the logic of $\mathcal{A}_{\not{\mathsf{n}}/\not{\mathsf{n}}\not{\mathsf{b}}}$, in view of (2.6). In this way, Lemma 5.27 completes the argument.] □

Subsubsection 7.1.3 collectively with the respective part of the paragraph following Theorem 7.10 show that the optional precondition cannot, generally speaking, be omitted in the formulation of Theorem 5.73.

6. Minimally n-valued maximally paraconsistent subclassical logics versus the logic of paradox

Fix any $n\in(\omega\setminus 3)$. Put $\mathcal{K}_n\triangleq\langle\mathfrak{K}_n,n\setminus 1\rangle$.

Then, the *logic of paradox LP* [18], being defined by the $\wedge$-conjunctive $\sim$-superclassical Σ_0-matrix $\mathcal{DM}_{4,\not{\mathsf{n}}}$, is equally defined by $\mathcal{K}_3$ (cf., e.g., [21]), in view of (2.6), for $e_{3,1}$ is an isomorphism from the latter onto the former, in which case, by Proposition 4.1 and Corollary 4.16, LP is maximally $\sim$-paraconsistent that has been proved *ad hoc* in Theorem 2.1 of [21].

Let $\Sigma_{[+]} \triangleq ([\Sigma^+ \cup]\{\supset, \sim\} \cup \{\nabla_i \mid i \in ((n-1) \setminus 1)\})$, where $\supset$ is binary, while other connectives [beyond Σ^+] are unary, $\mathcal{A}_{[+]}$ the $\Sigma_{[+]}$-matrix such that $A_{[+]} \triangleq n$, $D^{\mathcal{A}_{[+]}} \triangleq (n \setminus 1)$, $\sim^{\mathfrak{A}_{[+]}} \triangleq \sim^{\mathfrak{K}_n}$ [while $(\mathfrak{A}_+ \restriction \Sigma^+) \triangleq \mathfrak{D}_n$] whereas

$$\nabla_i^{\mathfrak{A}_{[+]}}(a) \triangleq \begin{cases} a & \text{if } a \in \{0, n-1\}, \\ i & \text{otherwise,} \end{cases}$$

for all $i \in ((n-1) \setminus 1)$ and all $a \in n$, and

$$(a \supset^{\mathfrak{A}_{[+]}} b) \triangleq \begin{cases} n-1 & \text{if } a \leqslant b, \\ 0 & \text{otherwise,} \end{cases}$$

for all $a, b \in n$, and $C_{[+]}$ the logic of $\mathcal{A}_{[+]}$, in which case it is $\sim$-paraconsistent [as well as both $\wedge$-conjunctive and $\vee$-disjunctive], for $\mathcal{A}_{[+]}$ is so [in view of Corollary 3.17]. Note that $\mathcal{A}_{[+]} \restriction \{0, n-1\}$ is $\sim$-classical, in which case, by (2.6), $C_{[+]}$ is $\sim$-subclassical, so, in particular, $\sim$ is a subclassical negation for $C_{[+]}$.

The following key result "kills two birds (both minimal n-valuedness and maximal paraconsistency of $C_{[+]}$) with one stone":

Lemma 6.1 (Many-Valued Key Lemma). *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of $C_{[+]}$. Then, there is a submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}_{[+]}$ is embeddable into $\mathcal{D}/\partial(\mathcal{D})$.*

Proof. In that case, there are some $a \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} a \in D^{\mathcal{B}}$ and some $b \in (B \setminus D^{\mathcal{B}})$. Let $\mathfrak{D}$ be the subalgebra of $\mathfrak{B}$ generated by $\{a, b\}$. Then, in view of (2.6), the submatrix $\mathcal{D} \triangleq (\mathcal{B} \restriction D)$ of $\mathcal{B}$ is a finitely-generated $\sim$-paraconsistent model of $C_{[+]}$. Therefore, by Lemma 2.19 with $\mathsf{M} = \{\mathcal{A}_{[+]}\}$, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}_{[+]}$, some subdirect product $\mathcal{E}$ of $\overline{\mathcal{C}}$ and some $g \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{D}/\partial(\mathcal{D}))$, in which case, by (2.6), $\mathcal{E}$ is $\sim$-paraconsistent (in particular, consistent), and so $I \neq \varnothing$. Take any $c \in D^{\mathcal{E}}$ such that $\sim^{\mathfrak{E}} c \in D^{\mathcal{E}}$. Then, by Lemma 4.11, $c \in ((n-1) \setminus 1)^I$. Hence, for every $j \in ((n-1) \setminus 1)$, we have $E \ni \nabla_j^{\mathfrak{E}} c = (I \times \{j\})$. Moreover, $E \ni (c \supset^{\mathfrak{E}} c) = (I \times \{n-1\})$ and $E \ni \sim^{\mathfrak{E}}(c \supset^{\mathfrak{E}} c) = (I \times \{0\})$. Thus, $\{I \times \{k\} \mid k \in n\} \subseteq E$, in which case, as $I \neq \varnothing$, $e \triangleq \{\langle k, I \times \{k\} \rangle \mid k \in n\}$ is an embedding of $\mathcal{A}_{[+]}$ into $\mathcal{E}$, and so $(g \circ e) \in \mathrm{hom}_{\mathrm{S}}(\mathcal{A}_{[+]}, \mathcal{D}/\partial(\mathcal{D}))$. Moreover, $\{x_0 \supset x_1, x_1 \supset x_0\}$ is clearly a binary equality determinant for $\mathcal{A}_{[+]}$. In this way, Corollary 2.14 and Lemma 3.4 complete the argument. □

Theorem 6.2. *$C_{[+]}$ is maximally $\sim$-paraconsistent.*

Proof. Consider any $\sim$-paraconsistent extension C' of $C_{[+]}$, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\})$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma_{[+]}}, T \rangle$ is a $\sim$-paraconsistent model of C', and so of $C_{[+]}$. Then, by Lemma 6.1 and (2.6), $\mathcal{A}_{[+]}$ is a model of C', as required. □

Theorem 6.3. *Let M be a class of $\Sigma_{[+]}$-matrices. Suppose $C_{[+]}$ is defined by M. Then, there is some $\mathcal{B} \in \mathsf{M}$ such that $n \leqslant |B|$. In particular, $C_{[+]}$ is minimally n-valued.*

Proof. As $C_{[+]}$ is $\sim$-paraconsistent, there must be some $\sim$-paraconsistent $\mathcal{B} \in \mathsf{M}$, in which case it is a model of $C_{[+]}$, and so, by Lemma 6.1, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}_{[+]}$ is embeddable into $\mathcal{D}/\partial(\mathcal{D})$. Thus, $n = |A_{[+]}| \leqslant |D/\partial(\mathcal{D})| \leqslant |D| \leqslant |B|$, as required. □

On the other hand, we have:

Proposition 6.4. *Let $\Sigma'_{[+]} \triangleq (\Sigma_{[+]} \setminus \{\supset\})$. Then, the $\Sigma'_{[+]}$-fragment of $C_{[+]}$ is defined by a [both $\wedge$-conjunctive and $\vee$-disjunctive] $\sim$-superclassical $\Sigma'_{[+]}$-matrix [being a definitional expansion of $\mathfrak{DM}_{4,\not{n}}$, and so the fragment is a definitional three-valued expansion of LP]. In particular, it is not minimally n-valued, unless $n = 3$.*

Proof. Let $\mathcal{S}_{[+]}$ be the [both $\wedge$-conjunctive and $\vee$-disjunctive] $\sim$-superclassical $\Sigma'_{[+]}$-matrix given by $\sim^{\mathfrak{S}_{[+]}} \mathsf{b} \triangleq \mathsf{b}$ [while $(\mathfrak{S}_{[+]} \restriction \Sigma^+) \triangleq (\mathfrak{D}_2^2 \restriction \{\mathsf{f}, \mathsf{b}, \mathsf{t}\})$], whereas $\nabla_i^{\mathfrak{S}_{[+]}}(a) \triangleq a$, for all $a \in \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ and all $i \in ((n-1) \setminus 1)$ [in which case $\mathcal{S}_+$ is an expansion of $\mathfrak{DM}_{4,\not{n}}$ by diagonal operations, and so a definitional one]. Then, $(\{\langle n-1, \mathsf{t} \rangle, \langle 0, \mathsf{f} \rangle\} \cup (((n-1) \setminus 1) \times \{\mathsf{b}\})) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}_{[+]} \restriction \Sigma'_{[+]}, \mathcal{S}_{[+]})$. In this way, (2.6) completes the argument. □

This highlights the special role of involving the implication connective $\supset$ and shows that the implication-less fragment of $C_{[+]}$ yields nothing more than the logic of paradox had done in this connection. More precisely, LP, being defined by $\mathcal{K}_3$, is equally defined by $\mathcal{K}_n$, in view of (2.6), for $\hbar_n \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}_n, \mathcal{K}_3)$, in which case, in particular, since $\mathcal{A}_+$ is an expansion of $\mathcal{K}_n$ (actually arisen by proper expanding $\mathcal{K}_n$ with providing both minimal n-valuedness and maximal $\sim$-paraconsistency of C_+), C_+ is an expansion of LP, C being its fragment. Thus, LP is an n-valued maximally $\sim$-paraconsistent logic but is not minimally n-valued, unless $n = 3$, as opposed to $C_{[+]}$. This highlights the particular meaning of the present subsection. And what is more, we have:

Corollary 6.5. *$C_{[+]}$ [as well as its Σ'_+-fragment] is not self-extensional.*

Proof. First, notice that $x_0 \equiv_{C_{[+]}} \sim((x_0 \supset x_0) \supset \sim x_0)$, while $((x_0 \supset x_0) \supset x_0) \not\equiv_{C_{[+]}} ((x_0 \supset x_0) \supset \sim((x_0 \supset x_0) \supset \sim x_0))$, because the rule $((x_0 \supset x_0) \supset \sim((x_0 \supset x_0) \supset \sim x_0)) \vdash ((x_0 \supset x_0) \supset x_0)$ is not true in $\mathcal{A}_{[+]}$ under $[x_0/1]$. Therefore, $C_{[+]}$ is not self-extensional. [Finally, Corollary 4.42 and Proposition 6.4 complete the argument.] □

7. Applications and Examples

7.1. **Four-valued expansions of Belnap's logic.** Here, we consider applications of Theorems 5.12, 5.15, 5.19, 5.22, 5.28, 5.61(i)⇔ (x), 5.73, Lemmas 5.13, 5.14 and Corollaries 5.63 and 5.64 normally not mentioning them explicitly and implicitly following the conventions adopted in Section 5.

7.1.1. *Fragments of the classical expansion.* Here, we deal with the basic signature $\Sigma = \Sigma_{\simeq,01} \triangleq (\Sigma_{01} \cup \{\neg\})$, where $\neg$ (classical negation) is unary, and its subsignature $\Sigma' \supseteq \Sigma_0$. Put $\neg^{\mathfrak{A}}\vec{a} \triangleq \langle 1 - a_i \rangle_{i \in 2}$, for all $\vec{a} \in 2^2$. Then, $\mu \in \hom(\mathfrak{A}, \mathfrak{A})$. Moreover, $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}\restriction\Sigma'$ iff $\neg \notin \Sigma'$. Likewise, $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}\restriction\Sigma'$ iff $\Sigma' = \Sigma_0$. In this way, we have:

Corollary 7.1. *Let $\Sigma_0 \subseteq \Sigma' \subseteq \Sigma$. Then, the logic of $\mathcal{A}\restriction\Sigma'$:*

(i) *is self-extensional, and so $\sim$-subclassical;*
(ii) *is maximally $\sim$-paraconsistent iff $\neg \in \Sigma'$;*
(iii) *is purely inferential iff it has no consistent formula iff it satisfies Variable Sharing Property iff $\Sigma' = \Sigma_0$.*

In this way, the classical expansion of C_{B} becomes a first instance of a *minimally four*-valued *maximally* paraconsistent *subclassical* logic (further but non-subclassical ones are provided by the next subsubsection). In this connection, we should like to highlight that, as opposed to the generic examples provided by Subsection 6, the four-valued ones provided by this and the next subsubsections are not definable by false-singular matrices (cf. Corollary 5.5).
7.1.1.1. Specular functional completeness. As usual, *Boolean algebras* are supposed to be of the signature $\Sigma^- \triangleq (\Sigma \setminus \{\sim\})$, the ordinary one over 2 being denoted by $\mathfrak{B}_2$.

Lemma 7.2. *Let $n \in \omega$ and $f : 2^n \to 2$. [Suppose f is $\leqslant$-monotonic.] (Suppose f is 2-idempotent, in which case $n > 0$.) Then, there is some $\vartheta \in \mathrm{Fm}^n_{\Sigma^-[\setminus\{\neg\}](\setminus\{\bot,\top\})}$ such that $g = \vartheta^{\mathfrak{B}_2}$.*

Proof. Then, by the functional completeness of $\mathfrak{B}_2$, there is some $\vartheta \in \mathrm{Fm}^n_{\Sigma^-}$ such that $g = \vartheta^{\mathfrak{B}_2} (\notin \{2^n \times \{i\} \mid i \in 2\})$, in which case, without loss of generality, one can assume that $\vartheta = (\wedge\langle\vec{\varphi}, \top\rangle)$, where, for each $m \in \ell \triangleq (\operatorname{dom}\vec{\varphi}) \in (\omega(\setminus 1))$, $\varphi_m = (\vee\langle(\neg \circ \vec{\phi}^m) * \vec{\psi}^m, \bot\rangle)$, for some $\vec{\phi}^m \in V_n^{k_m}$, some $\vec{\psi}^m \in V_n^{l_m}$ and some $k_m, l_m \in \omega$ such that $((\operatorname{img}\vec{\phi}^m) \cap (\operatorname{img}\vec{\psi}^m)) = \varnothing$ (and $(k_m + l_m) > 0$, so $g = \vartheta'^{\mathfrak{B}_2}$, where $\vartheta' \triangleq (\wedge\vec{\varphi}')$, whereas, for each $m \in (\operatorname{dom}\vec{\varphi}') \triangleq \ell$, $\varphi'_m \triangleq (\vee((\neg \circ \vec{\phi}^m) * \vec{\psi}^m)))$. [Respectively, set $\vartheta'' \triangleq (\wedge\langle\vec{\varphi}'', \top\rangle)$, where, for each $m \in (\operatorname{dom}\vec{\varphi}'') \triangleq \ell$, $\varphi''_m \triangleq (\vee\langle\vec{\psi}^m, \bot\rangle)$. Consider any $\bar{a} \in A^n$ and the following exhaustive cases:

(1) $g(\bar{a}) = 0$,
in which case we have $\vartheta''^{\mathfrak{B}_2}[x_j/a_j]_{j \in n} \leqslant \vartheta^{\mathfrak{B}_2}[x_j/a_j]_{j \in n} = 0$, and so we get $\vartheta''^{\mathfrak{B}_2}[x_j/a_j]_{j \in n} = 0$.
(2) $g(\bar{a}) = 1$,
in which case, for every $m \in \ell$, as $\bar{a} \leqslant \bar{b} \triangleq ((\bar{a}\restriction(n \setminus N)) \cup (N \times \{1\})) \in A^n$, where $N \triangleq \{j \in n \mid x_j \in (\operatorname{img}\vec{\phi}^m)\}$, by the $\leqslant$-monotonicity of g, we have $1 \leqslant g(\bar{b}) \leqslant \varphi_m^{\mathfrak{B}_2}[x_j/b_j]_{j \in n} = \varphi_m''^{\mathfrak{B}_2}[x_j/a_j]_{j \in n}$, and so we get $\vartheta''^{\mathfrak{B}_2}[x_j/a_j]_{j \in n} = 1$.

Thus, $g = \vartheta''^{\mathfrak{B}_2}$. (And what is more, since, in that case, $\ell > 0$ and $l_m > 0$, for each $m \in \ell$, we also have $g = \vartheta'''^{\mathfrak{B}_2}$, where $\vartheta''' \triangleq (\wedge\vec{\varphi}''')$, whereas, for each $m \in (\operatorname{dom}\vec{\varphi}''') \triangleq \ell$, $\varphi'''_m \triangleq (\vee\vec{\psi}^m)$.)] This completes the argument. ☐

Theorem 7.3. *Let $n \in (\omega(\setminus 1))$ and $f : A^n \to A$. Then, f is specular [and regular] (as well as $\{\mathsf{n}, \mathsf{b}\}$-idempotent) iff there is some $\tau \in \mathrm{Fm}^n_{\Sigma[\setminus\{\neg\}](\setminus\{\bot,\top\})}$ such that $f = \tau^{\mathfrak{A}}$.*

Proof. The "if" part is immediate. Conversely, assume f is specular [and regular] (as well as $\{\mathsf{n}, \mathsf{b}\}$-idempotent). Then, $g : 2^{2 \cdot n} \to 2, \bar{a} \mapsto \pi_0(f(\langle\langle a_{2 \cdot j}, 1 - a_{(2 \cdot j)+1}\rangle\rangle_{j \in n}))$ [is $\leqslant$-monotonic (and)] (is 2-idempotent). Therefore, by Lemma 7.2, there is some $\vartheta \in \mathrm{Fm}^{2 \cdot n}_{\Sigma^-[\setminus\{\neg\}](\setminus\{\bot,\top\})}$ such that $g = \vartheta^{\mathfrak{B}_2}$. Put $\tau \triangleq (\vartheta[x_{2 \cdot j}/x_j, x_{(2 \cdot j)+1}/(\sim x_j)]_{j \in n}) \in \mathrm{Fm}^n_{\Sigma[\setminus\{\neg\}](\setminus\{\bot,\top\})}$. Consider any $\bar{c} \in A^n$. Then, since, for each $i \in 2$, we have $\pi_i \in \hom(\mathfrak{A}\restriction\Sigma^-, \mathfrak{B}_2)$, we get $\pi_0(\tau^{\mathfrak{A}}[x_j/c_j]_{j \in n}) = \vartheta^{\mathfrak{B}_2}[x_{2 \cdot j}/\pi_0(c_j), x_{(2 \cdot j)+1}/(1 - \pi_1(c_j))]_{j \in n} = \pi_0(f(\bar{c}))$ and, likewise, as f is specular, $\pi_1(\tau^{\mathfrak{A}}[x_j/c_j]_{j \in n}) = \vartheta^{\mathfrak{B}_2}[x_{2 \cdot j}/\pi_1(c_j), x_{(2 \cdot j)+1}/(1 - \pi_0(c_j))]_{j \in n} = \pi_0(f(\mu \circ \bar{c})) = \pi_0(\mu(f(\bar{c}))) = \pi_1(f(\bar{c}))$, as required. ☐

As an immediate consequence of Theorems 5.61(i)⇔(x), 7.3 and Corollary 5.64(i), we eventually get:

Corollary 7.4. *A four-valued expansion of C_{B} is self-extensional iff it is a fragment of a definitional copy of C. Moreover, [non-]purely-inferential regular self-extensional expansions of C_{B} are exactly definitional copies of $C_{[\mathrm{B}]\mathrm{B}}$.*

This definitely justifies both C_{BB} and its classical expansion C. And what is more, it essentially shows that $C_{[\mathrm{B}]\mathrm{B}}$ actually exhaust all regular self-extensional expansions of C_{B}.

7.1.2. *Bilattice expansions.* Here, it is supposed that $\{\sqcap, \sqcup\} \subseteq \Sigma$, where $\sqcap$ and $\sqcup$ are binary (*knowledge* conjunction and disjunction, respectively), while $(\langle a, b\rangle \sqcap^{\mathfrak{A}} \langle c, d\rangle) = \langle \min(a, c), \max(b, d)\rangle$, for all $a, b, c, d \in 2$, in which case $(\mathsf{f} \sqcap^{\mathfrak{A}} \mathsf{t}) = \mathsf{n}$, whereas $(\langle a, b\rangle \sqcup^{\mathfrak{A}} \langle c, d\rangle) = \langle \max(a, c), \min(b, d)\rangle$, for all $a, b, c, d \in 2$, in which case $(\mathsf{f} \sqcup^{\mathfrak{A}} \mathsf{t}) = \mathsf{b}$. In that case, neither $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ nor $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. And what is more, $\{\mathsf{b}\}$ and $\{\mathsf{n}\}$ are exactly all proper subalgebras of $\mathfrak{A}$ in the *purely-bilattice* case $\Sigma = (\Sigma_0 \cup \{\sqcap, \sqcup\})$, $\mathcal{A}\restriction\{\mathsf{n}\}$ being the only proper consistent submatrix of $\mathcal{A}$, in that case. Hence, we immediately obtain the following universal negative and positive results, respectively:

Corollary 7.5. *Any bilattice expansion of C_{B} is not $\sim$-subclassical, and so not self-extensional.*

Corollary 7.6. *Any [purely-]bilattice expansion of C_{B} [satisfies Variable Sharing Property and] is inferentially maximal, and so both is maximally $\sim$-paraconsistent and has no proper consistent axiomatic extensions.*

And what is more, in case $\Sigma_{01} \subseteq \Sigma$, $\mathcal{A}$ has no proper submatrix at all. Thus, by Theorem 5.19 and Lemma 5.13, we also get:

Corollary 7.7. *C is maximal iff it is not purely inferential if $\Sigma_{01} \subseteq \Sigma$.*

7.1.3. *Implicative expansions.* Here, it is supposed that Σ contains a binary $\supset$ (implication) such that

$$(a \supset^{\mathfrak{A}} b) = \begin{cases} b & \text{if } \pi_0(a) = 1, \\ \mathsf{t} & \text{otherwise,} \end{cases}$$

for all $a, b \in 2^2$ (cf. [23]), in which case $\mathcal{A}$ is $\supset$-implicative. Then, $(\mathsf{n} \supset^{\mathfrak{A}} \mathsf{n}) = \mathsf{t} \neq \mathsf{b} = (\mathsf{b} \supset^{\mathfrak{A}} \mathsf{b})$, so $\mu \notin \hom(\mathfrak{A}, \mathfrak{A})$, in which case we immediately get:

Corollary 7.8. *The logic of $\mathcal{A}$ is neither self-extensional nor purely-inferential, ad so does not satisfy Variable Sharing Property.*

It is remarkable that, as opposed to bilattice expansions, implicative ones are not, generally speaking, covered by Corollary 5.63 because $\{\mathsf{f}(,\mathsf{b}/\mathsf{n}),\mathsf{t}\}$ does form a subalgebra of $\mathfrak{B}_{[01]} \triangleq (\mathfrak{A} {\restriction} (\Sigma_{0[1]} \cup \{\supset\}))$, in which case, by Theorem(s) 5.22 (and 5.28), C is $\sim$-subclassical (and is not maximally $\sim$-paraconsistent), whenever $\Sigma \subseteq (\Sigma_{01} \cup \{\supset\})$. It is also remarkable that $\{\mathsf{b}\}$ does [not] form a subalgebra of $\mathfrak{B}_{[01](\not{\mathsf{n}})}$, while $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{B}_{[01]}$.

7.1.4. *Disjunctive extensions of expansions of Belnap's logic.* In view of Corollary 5.40, C is hereditary iff (under identification of submatrices of $\mathcal{A}$ with the underlying algebras of their carriers)

$$\mathbf{S}^*_*(\mathcal{A}) \supseteq \mathsf{S}_{01} \triangleq \mathbf{S}(\mathcal{DM}_{4,01}) = \{\{\mathsf{f},\mathsf{t},\mathsf{b},\mathsf{n}\},\{\mathsf{f},\mathsf{t},\mathsf{n}\},\{\mathsf{f},\mathsf{t},\mathsf{b}\},\{\mathsf{f},\mathsf{t}\}\}$$

(the inverse inclusion always holds), in which case $C^{\mathrm{EM[+R]}}[= C^{\mathrm{PC}}]$ is defined by $\mathcal{A}_{\not{\mathsf{n}}[\not{\mathsf{b}}]}$, in view of Theorem[s] 5.28 [resp., 5.22 and 5.30], while $C^{(\mathrm{EM}\times)\mathrm{R}}$ is defined by $\{\mathcal{A}_{\not{\mathsf{b}}}\}(\cup\{\mathcal{A}_{\not{\mathsf{n}}}\})$, in view of Corollary 5.33 (resp., 5.40). In particular, (the purely-implicative expansion of) $B_{4[01]}$ is hereditary (cf. Subsubsection 7.1.3). In this connection, note that, in view of Theorem 4.1 of [20], $\vee$-disjunctive extensions of B_4 are exactly *De Morgan logics* in the sense of the reference [Pyn 95a] of [21]. In this way, the present subsubsection subsumes the material announced therein advancing it much towards arbitrary four-valued expansions. Set $\mathsf{S} \triangleq \mathbf{S}_*(\mathcal{DM}_4) = (\mathsf{S}_{01} \cup \{\{\mathsf{n}\}\})$.

Remark 7.9. The mappings $\mathsf{C} \mapsto \mathsf{C}^{\triangledown}_{\mathsf{S}_{[01]}}$ and $\mathsf{C} \mapsto (\mathsf{C} \cap \mathbf{S}^{[*]}_*(\mathcal{A}))$ form a dual Galois retraction between the posets of all lower cones of $\mathbf{S}^{[*]}_*(\mathcal{A})$ and those of $\mathsf{S}_{[01]}$, the former/latter mapping preserving generating subsets/relative axiomatizations. ☐

There are exactly nine [six] lower cones of $\mathsf{S}_{[01]}$ [but those containing $\{\mathsf{n}\}$, viz., including C_1, i.e., the last three ones]:

$$\mathsf{C}_{4[01]} \triangleq \{\{\mathsf{f},\mathsf{t},\mathsf{b},\mathsf{n}\}\}^{\triangledown}_{\mathsf{S}_{[01]}}, \quad \mathsf{C}^{\mathsf{b}}_{3[01]} \triangleq \{\{\mathsf{f},\mathsf{t},\mathsf{b}\}\}^{\triangledown}_{\mathsf{S}_{[01]}}, \quad \mathsf{C}^{\mathsf{n}}_{3[01]} \triangleq \{\{\mathsf{f},\mathsf{t},\mathsf{n}\}\}^{\triangledown}_{\mathsf{S}_{[01]}},$$
$$\mathsf{C}_{3[01]} \triangleq (\mathsf{C}^{\mathsf{b}}_{3[01]} \cup \mathsf{C}^{\mathsf{n}}_{3[01]}), \quad \mathsf{C}_2 \triangleq \{\{\mathsf{f},\mathsf{t}\}\}, \quad \mathsf{C}_0 \triangleq \varnothing,$$
$$\mathsf{C}_1 \triangleq \{\{\mathsf{n}\}\}, \quad \mathsf{C}^{\mathsf{b}}_{3\uplus 1} \triangleq (\mathsf{C}^{\mathsf{b}}_3 \cup \mathsf{C}_1), \quad \mathsf{C}_{2\uplus 1} \triangleq (\mathsf{C}_2 \cup \mathsf{C}_1).$$

Those eight [five] ones, which are proper (viz., distinct from $\mathsf{S}_{[01]} = \mathsf{C}_{4[01]}$) are relatively axiomatized by the following $\Sigma_\sim$-calculi (actually arisen according to the constructive proof of Lemma 3.27, and so demonstrating its practical applicability), respectively:

(7.1) $$(5.8),$$
(7.2) $$(5.15),$$
(7.3) $$(5.1),$$
(7.4) $$\{(5.8),(5.15)\},$$
(7.5) $$x_0,$$
(7.6) $$x_0 \vdash x_1,$$
(7.7) $$x_0 \vdash (x_1 \vee \sim x_1),$$
(7.8) $$\{(5.15),(7.7)\}.$$

And what is more, $\sigma_{+1}(7.6) \vee x_0$ is equivalent to (7.6) under (3.3) and (3.5). Likewise, $\sigma_{+1}(7.7) \vee x_0$ is equivalent to (7.7) under (3.3), (3.4) and (3.5). By IC we denote the inconsistent Σ-logic. Moreover, put $PC_{[01]} \triangleq B^{\mathrm{PC}}_{4[01]}$. In this way, taking Remarks 2.8, 2.11, 7.9, Lemmas 4.22, 5.13, Theorems 3.24, 3.28 and Proposition 2.10 into account, we eventually get:

Theorem 7.10. *Suppose C is (not) hereditary and has a/no theorem. Then, $\vee$-disjunctive [non-pseudo-axiomatic] extensions of C form (a Galois retract — in particular, a sublattice — of) the six/nine[six]-element non-chain distributive lattice depicted at Figure 1 (with not necessarily distinct nodes) with solely solid circles/[with solely solid circles]. Moreover, those of them, whose relative axiomatizations are not given by upper indices, are axiomatized relatively to C by the following calculi:*

(7.9) $$C^{\mathrm{EM}} \cap C^{\mathrm{R}} \; : \; (5.14),$$
(7.10) $$IC \; : \; (7.5),$$
(7.11) $$IC_{+0} \; : \; (7.6),$$
(7.12) $$C^{\mathrm{EM}}_{+0} \; : \; (7.7),$$
(7.13) $$C^{\mathrm{EM+R}}_{+0} \; : \; \{(7.7),(5.9)\}.$$

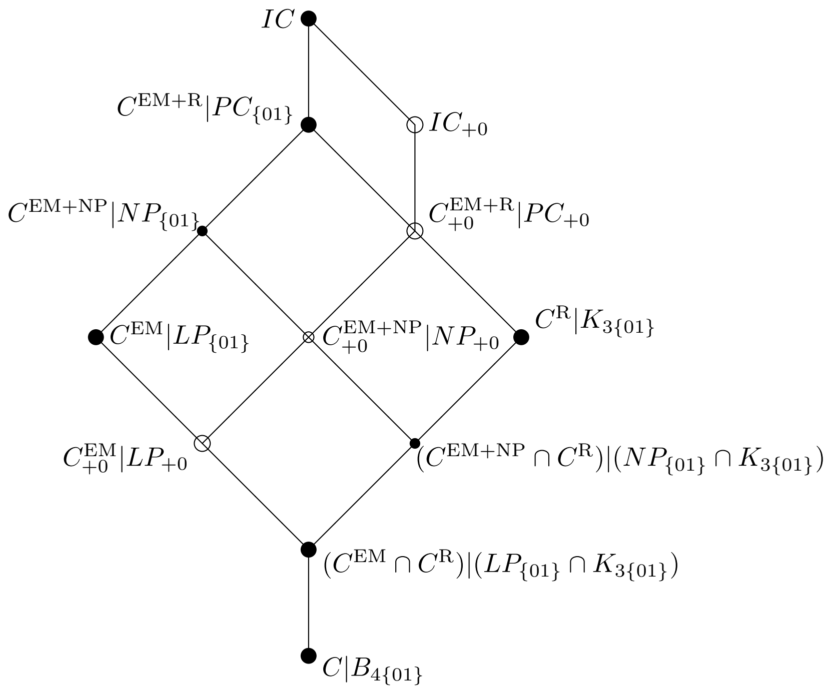

FIGURE 1. The lattice of $\vee$-disjunctive/Kleene extensions of hereditary/strongly hereditary $C|B_{4\{01\}}$ with solely large/non-lowest circles.

In view of Theorems 3.24, 3.28 and Remark 7.9, Theorem 7.10, being immediately applicable to hereditary four-valued expansions of B_4 (in particular, to implicative ones — cf. Subsubsection 7.1.3 — whose $\vee$-disjunctive extensions are exactly axiomatic ones, in view of Remark 3.25, constructively providing, in particular, their finite axiomatic relative axiomatizations), is equally well-applicable to non-hereditary ones, in which case the lattice depicted at Figure 1 is properly degenerated under the corresponding dual Galois retraction. For instance, when dealing with any classically-negative (viz., Boolean) expansion CB_4 (cf. Subsubsection 7.1.1), $\mathbf{S}_*(\mathcal{A})$ becomes equal to $\{A\}[\cup\{\{\mathsf{f},\mathsf{t}\}\}]$, in which case $\vee$-disjunctive (viz., axiomatic; cf. Remark 3.25, for, in that case, $\mathcal{A}$, being both $\neg$-negative and $\vee$-disjunctive, is $(\neg x_0 \vee x_1)$-implicative) extensions of CB_4 form the two[three]-element chain $CB_4 \subsetneq CB_4^{\mathrm{EM(+R)}} = CB_4^{\mathrm{R}} = [CB_4^{\mathrm{PC}} \subsetneq]IC$. Likewise, given any bilattice expansion BL_4 (cf. Subsubsection 7.1.2), $\mathbf{S}_*(\mathcal{A})$ becomes equal to $\{A\}[\cup\{\{\mathsf{n}\}\}]$, in which case $\vee$-disjunctive extensions of BL_4 form the two-[three-]element chain $BL_4[\subsetneq IC_{+0}] \subsetneq IC = BL_4^{\mathrm{EM}}$ with $IC_{[+0]} = BL_4^{\mathrm{R}}$, exhausting *all* extensions of BL_4, in view of its inferential maximality proved in Corollary 7.6.

It is remarkable that, in view of Theorem 5.2 of [20] providing a finite axiomatization of B_4 given by Definition 5.1 therein,[5] Theorem 7.10 yields finite axiomatizations of all $\vee$-disjunctive extensions of B_4 (in particular, of K_3 relatively axiomatized by the Resolution rule (5.9)).

On the other hand, to find *all* extensions of C is a much more complicated problem, a first idea of which having been due to Theorems 5.69, 5.36, 5.53 and Corollaries 5.3 and 5.68. A partial solution of it is presented below.

7.1.4.1. Kleene extensions. Next, C is said to be *strongly hereditary*, provided $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ forms a regular specular subalgebra of $\mathfrak{A}$, in which case, since $\mu \circ \mu$ is diagonal, $\{\mathsf{f},\mathsf{b},\mathsf{t}\} = \mu[\{\mathsf{f},\mathsf{n},\mathsf{t}\}]$ forms a specular subalgebra of $\mathfrak{A}$ as well, and so a regular one, for μ is anti-regular, and so C is hereditary, in view of Corollary 5.40, according to which collectively with Theorem 5.41, C is strongly hereditary iff it is hereditary, $C^{\mathrm{EM}\times\mathrm{R}}$ is self-extensional and $\mathcal{A}_{\not\mathsf{b}}$ is regular. By symmetry between n and b, C is strongly hereditary iff $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a regular specular subalgebra of $\mathfrak{A}$, whenever $\mathfrak{A}$ is both regular and specular, while $\{\mathsf{f},(\mathsf{b}/\mathsf{n}),\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{0[1]}$). According to the following example, equally showing that the framework of strongly hereditary expansions of B_4 is not at all exhausted by solely definitional copies of $B_{4[01]}$, "whenever" cannot be replaced with "iff" in the previous sentence:

Example 7.11. If $\Sigma \triangleq (\Sigma_{0[1]} \cup \{\uplus\})$, where $\uplus$ is binary, and $\uplus^{\mathfrak{A}} \triangleq ((\vee^{\mathfrak{A}} \restriction (A^2_{\not\mathsf{n}} \cup A^2_{\not\mathsf{b}})) \cup \{\langle\langle\mathsf{n},\mathsf{b}\rangle,\mathsf{t}\rangle, \langle\langle\mathsf{b},\mathsf{n}\rangle,\mathsf{f}/\mathsf{b}\rangle\})$, C is strongly hereditary, and $\mathfrak{A}$ is not specular, as opposed to $\mathfrak{DM}_{4[01]}$, and non-regular/regular. □

Throughout the rest of this paragraph, C is supposed to be strongly hereditary. First, as an immediate consequence of Theorems 5.26, 5.28 and 5.36, we have:

Corollary 7.12. $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ *is the greatest both inferentially paracomplete and* $\sim$*-paraconsistent extension of* C.

Corollary 7.13. *Let* I *be a finite set,* $\overline{\mathcal{C}} \in \{\mathcal{A}_{\not\mathsf{b}}, \mathcal{A}_{\not\mathsf{n}}\}^I$*, and* $\mathcal{B}$ *a consistent non-*$\sim$*-paraconsistent submatrix of* $\prod_{i\in I} \mathcal{C}_i$*. Then,* $\hom(\mathcal{B}, \mathcal{A}_{\not\mathsf{b}}) \neq \varnothing$.

Proof. In that case, by Lemma 5.48, there is some $h \in \hom(\mathcal{B}, \langle\mathfrak{A}, \{\mathsf{t}\}\rangle) \neq \varnothing$, in which case $\mathfrak{D} \triangleq (\mathfrak{A}\restriction(\mathrm{img}\, h))$ satisfies (3.16) for $\mathfrak{B}$ does so, because both $\mathfrak{A}_{\not\mathsf{b}}$ and $\mathfrak{A}_{\not\mathsf{n}}$ do so, while $h \in \hom(\mathfrak{B}, \mathfrak{D})$ is surjective. Hence, $\{\mathsf{n},\mathsf{b}\} \nsubseteq D$, for otherwise, (3.16) would not be true in $\mathfrak{D}$ under $[x_0/\mathsf{n}, x_1/\mathsf{b}]$. Thus, $\mathcal{D} \triangleq (\langle\mathfrak{A}, \{\mathsf{t}\}\rangle \restriction D)$ is a submatrix of $\langle\mathfrak{A}, \{\mathsf{t}\}\rangle \restriction A_{\not a}$, for some $a \in \{\mathsf{n},\mathsf{b}\}$, in which case

[5] In this connection, we should also like to take the opportunity to notice that Footnote 3 on p. 443 of [20] has proved absolutely irrelevant and is to be disregarded, simply because Font did never find the Hilbert-style axiomatization of C_{B} independently as he falsely claimed, but rather just plagiarized it, being in the vantage position of learning it from me first.

$h \in \mathrm{hom}(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction A_{\not{a}})$, and so the fact that $\mu \restriction A_{\not{n}}$ is an isomorphism from $\langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction A_{\not{n}}$ onto $(\langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction A_{\not{b}}) = \mathcal{A}_{\not{b}}$ completes the argument. ☐

Lemma 7.14. $(\mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{b}}) \in \mathrm{Mod}(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})$.

Proof. Since, by Theorem 5.50 and Corollary 5.33, $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$ is defined by $\{\mathcal{A}_{\not{b}}, \mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{n}\not{b}}\}$, $\mathcal{A}_{\not{n}} \times (\mathcal{A}_{\not{b}} \times \mathcal{A}_{\not{n}\not{b}})$, being isomorphic to $\mathcal{A}_{\not{b}} \times (\mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{n}\not{b}})$, is a model of $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$, in view of (2.6). Moreover, by Lemma 4.32, $(\mathcal{A}_{\not{b}} \times \mathcal{A}_{\not{n}\not{b}}) \restriction K^{\mathsf{n}}_4$ is a submatrix of $\mathcal{A}_{\not{b}} \times \mathcal{A}_{\not{n}\not{b}}$, in which case $\mathcal{A}_{\not{n}} \times ((\mathcal{A}_{\not{b}} \times \mathcal{A}_{\not{n}\not{b}}) \restriction K^{\mathsf{n}}_4)$ is a submatrix of $\mathcal{A}_{\not{n}} \times (\mathcal{A}_{\not{b}} \times \mathcal{A}_{\not{n}\not{b}})$, and so it is a model of $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$, in view of (2.6). And what is more, by Lemma 4.32, $h \triangleq (\pi_0 \restriction K^{\mathsf{n}}_4) \in \mathrm{hom_S}((\mathcal{A}_{\not{b}} \times \mathcal{A}_{\not{n}\not{b}}) \restriction K^{\mathsf{n}}_4, \mathcal{A}_{\not{b}})$ is surjective, and so is $(\Delta_{A_{\not{n}}} \times h) \in \mathrm{hom_S}(\mathcal{A}_{\not{n}} \times ((\mathcal{A}_{\not{b}} \times \mathcal{A}_{\not{n}\not{b}}) \restriction K^{\mathsf{n}}_4), \mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{b}})$, as required, by (2.6). ☐

Corollary 7.15. *$C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$ is axiomatized by* (5.15) *relatively to $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$.*

Proof. By Corollary 5.33 and Theorem 5.28 [resp., 5.50], $C^{\mathrm{EM[+NP]}} \cap C^{\mathrm{R}}$ is defined by $\{\mathcal{A}_{\not{n}}[\times \mathcal{A}_{\not{n}\not{b}}], \mathcal{A}_{\not{b}}\}$. Consider any model $\mathcal{B} \in \mathbf{S}(\mathbf{P}_\omega(\{\mathcal{A}_{\not{b}}, \mathcal{A}_{\not{n}}\}))$ of (5.15), in which case there is some finite set I, some $\overline{\mathcal{C}} \in \{\mathcal{A}_{\not{b}}, \mathcal{A}_{\not{n}}\}^I$ such that $\mathcal{B}$ is a submatrix of $\prod_{i \in I} \mathcal{C}_i$. Put $J \triangleq \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{b}})$ and $K \triangleq \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not{b}})$. Consider any $a \in (B \setminus D^{\mathcal{B}})$, in which case $\mathcal{B}$ is consistent and there is some $i \in I$ such that $\pi_i(a) \notin D^{\mathcal{C}_i}$. Consider the following complementary cases:

(1) $\mathfrak{C}_i = \mathcal{A}_{\not{n}}$.
Then, by Corollary 7.13, there is some $h \in \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not{b}}) \neq \varnothing$, in which case $g \triangleq ((\pi_i \restriction B) \times h) \in J$ and $g(a) \notin D^{\mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{b}}}$.

(2) $\mathfrak{C}_i \neq \mathcal{A}_{\not{n}}$, in which case $\mathfrak{C}_i = \mathcal{A}_{\not{b}}$, and so $(\pi_i \restriction B) \in K$.

In this way, $((\prod \Delta_J) \times (\prod \Delta_K)) \in \mathrm{hom_S}(\mathcal{B}, (\mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{b}})^J \times \mathcal{A}^K_{\not{b}})$, and so (2.6), Theorem 2.20, Lemma 7.14 and the finiteness of A complete the argument. ☐

By $NP_{[01]}$ we denote the extension of $LP_{[01]}$ relatively axiomatized by (5.15) (cf. [25]).

Theorem 7.16. *Suppose C has a/no theorem. Then, Kleene [non-pseudo-axiomatic] extensions of C form the seven/eleven[seven]-element non-chain distributive lattice depicted at Figure 1 with solely solid circles/[with solely solid circles], both $C^{\mathrm{EM+(NP|R)}}$ and $\{C^{\mathrm{EM+NP}} \cap\} C^{\mathrm{R}}/$ as well as theorem-less proper ones being non-axiomatic extensions of both $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ and C, and so C^{EM} is the only proper axiomatic extension of $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ and, providing either $\mathfrak{A}$ is regular or C has no theorem, of C. Moreover, those of them, which are neither $\vee$-disjunctive nor equal to $C^{\mathrm{EM+NP}}$, are relatively axiomatized as follows:*

$$C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}} \quad by \quad (5.15),$$
$$C^{\mathrm{EM+NP}}_{+0} \quad by \quad \{(5.15), (7.7)\},$$

others inheriting the above axiomatizations relatively to C (cf. Theorem 7.10) with possible relacing (5.9) *by* (5.16).

Proof. We use (2.6), Theorems 5.28, 5.30, 7.10, 5.50, 5.53, Propositions 2.10, 3.20, 5.13, Corollaries 5.33, 7.15, Lemma 5.72 with $\mathcal{B} = \mathcal{A}_{\not{b}}$ and Remarks 2.8 and 5.51 tacitly. First, as C^{EM} is $\sim$-paraconsistent, $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})/C^{\mathrm{EM+NP}}_{+0}/C^{\mathrm{EM+NP}}$ is distinct from $(C^{\mathrm{EM}} \cap C^{\mathrm{R}})/C^{\mathrm{EM}}_{+0}/C^{\mathrm{EM}}$, respectively. Likewise, since (5.16) is not true in $\mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{n}\not{b}}$ under $[x_0/\langle \mathsf{b}, \mathsf{t}\rangle, x_1/\langle \mathsf{f}, \mathsf{t}\rangle]$, $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})/C^{\mathrm{EM+NP}}_{+0}/C^{\mathrm{EM+NP}}$ is distinct from $C^{\mathrm{R}}/C^{\mathrm{EM+R}}_{+0}/C^{\mathrm{EM+R}}$, respectively. Finally, consider any [non-pseudo-axiomatic] extension C' of $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ and the following exhaustive cases [but (3) and (4)]:

(1) $IC \subseteq C'$.
Then, $C' = IC$.

(2) $C^{\mathrm{PC}} \subseteq C'$ but $IC \nsubseteq C'$.
Then, C' is consistent, and so inferentially consistent, for (5.8), being satisfied in C^{PC}, is so in its extension C', in which case, by Theorem 5.22, $C' = C^{\mathrm{PC}}$.

(3) $IC_{+0} \subseteq C'$ but $C^{\mathrm{PC}} \nsubseteq C'$.
Then, IC, being an extension of C^{PC}, is not a sublogic of C', so, by the following claim, C' has no theorem:

Claim 7.17. *Let C'' and C''' be Σ-logics. Suppose $C'' \nsubseteq C'''$ is non-pseudo-axiomatic and $C''_{+0} \subseteq C'''$. Then, C''' has no theorem.*

Proof. By contradiction. For suppose C''' has a theorem, in which case it is non-pseudo-axiomatic, and so, by Remark 2.8, we get $C'' = (C''_{+0})_{-0} \subseteq C'''_{-0} = C'''$. This contradiction completes the proof. ☐

In this way, as $C'_{-0} \subseteq IC$, we have $C' = (C'_{-0})_{+0} \subseteq IC_{+0}$, and so we get $C' = IC_{+0}$.

(4) $C^{\mathrm{PC}}_{+0} \subseteq C'$ but both $C^{\mathrm{PC}} \nsubseteq C'$ and $IC_{+0} \nsubseteq C'$.
Then, by Claim 7.17, C' has no theorem. Moreover, (7.7), being satisfied in C^{PC}_{+0}, is so in its extension C', in which case, by the structurality of C', $(x_1 \vee \sim x_1) \in (\bigcap_{k \in \omega} C'(x_k)) = C'_{-0}(\varnothing)$, and so $C^{\mathrm{PC}} \subseteq C'_{-0}$. On the other hand, $IC = (IC_{+0})_{-0} \nsubseteq C'_{-0}$, so C'_{-0} is consistent, and so inferentially consistent, for it satisfies (5.8). Hence, by Theorem 5.22, $C'_{-0} = C^{\mathrm{PC}}$. In this way, $C' = (C'_{-0})_{+0} = C^{\mathrm{PC}}_{+0}$.

(5) $(C^{\mathrm{PC}}_{+0}[\cup C^{\mathrm{PC}}]) \nsubseteq C'$ but $C^{\mathrm{R}} \subseteq C'$.
Then, [(5.8), and so, in view of the non-pseudo-axiomaticity of C'] (7.7) is not satisfied in C', in which case, by Theorem 5.36, $C' = C^{\mathrm{R}}$.

(6) $C^{\mathrm{R}} \nsubseteq C'$.
Then, (5.16) is not satisfied in C', in which case, by Lemma 5.52, $C' \subseteq C^{\mathrm{EM+NP}}$, and so we have the following exhaustive subcases [but (c) and (d)]:

(a) $C^{\mathrm{EM+NP}} \subseteq C'$.
Then, $C' = C^{\mathrm{EM+NP}}$.

(b) $C^{\mathrm{EM+NP}} \nsubseteq C'$ but $C^{\mathrm{EM}} \subseteq C'$.
Then, C' is $\sim$-paraconsistent, so, by Theorem 5.26, $C' = C^{\mathrm{EM}}$.

(c) $C^{\mathrm{EM+NP}}_{+0} \subseteq C'$ but $C^{\mathrm{EM}} \nsubseteq C'$.
Then, $C^{\mathrm{EM+NP}} \nsubseteq C'$, so, by Claim 7.17, C' has no theorem. Therefore, $C^{\mathrm{EM+NP}} = (C^{\mathrm{EM+NP}}_{+0})_{-0} \subseteq C'_{-0}$, $(C^{\mathrm{EM}} \cap C^{\mathrm{R}}) = (C^{\mathrm{EM}} \cap C^{\mathrm{R}})_{-0} \subseteq C'_{-0}$ and $C^{\mathrm{R}} \nsubseteq C'_{-0}$, for, otherwise, we would have $C^{\mathrm{R}} = (C^{\mathrm{R}})_{+0} \subseteq (C'_{-0})_{+0} = C'$. Hence, by Lemma 5.52, we have $C'_{-0} \subseteq C^{\mathrm{EM+NP}}$, in which case we get $C' = (C'_{-0})_{+0} \subseteq C^{\mathrm{EM+NP}}_{+0}$, and so $C' = C^{\mathrm{EM+NP}}_{+0}$.

(d) $C^{\mathrm{EM}}_{+0} \subseteq C'$ but both $C^{\mathrm{EM}} \nsubseteq C$ and $C^{\mathrm{EM+NP}}_{+0} \nsubseteq C'$.
Then, by Claim 7.17, C' has no theorem. Moreover, (7.7), being satisfied in C^{EM}_{+0}, is so in C', in which case, by the structurality of C', $(x_1 \vee \sim x_1) \in (\bigcap_{k\in\omega} C'(x_k)) = C'_{-0}(\varnothing)$, and so $C^{\mathrm{EM}} \subseteq C'_{-0}$, while $(C^{\mathrm{EM}} \cap C^{\mathrm{R}}) = (C^{\mathrm{EM}} \cap C^{\mathrm{R}})_{-0} \subseteq C'_{-0}$. Also, $C^{\mathrm{EM+NP}} = (C^{\mathrm{EM+NP}}_{+0})_{-0} \nsubseteq C'_{-0}$, so C'_{-0} is $\sim$-paraconsistent. Hence, by Theorem 5.26, $C'_{-0} = C^{\mathrm{EM}}$. In this way, $C' = (C'_{-0})_{+0} = C^{\mathrm{EM}}_{+0}$.

(e) $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}) \subseteq C'$ but $(C^{\mathrm{EM+NP}}_{+0}[\cup C^{\mathrm{EM+NP}}]) \nsubseteq C'$.
Then, [(5.8), and so, in view of the non-pseudo-axiomaticity of C'] (7.7) is not satisfied in C', in which case, by Theorem 5.36, $C' = (C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})$.

(f) $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}) \nsubseteq C'$ and $(C^{\mathrm{EM}}_{+0}[\cup C^{\mathrm{EM}}]) \nsubseteq C'$.
Then, C' is both $\sim$-paraconsistent and inferentially paracomplete [in view of the non-pseudo-axiomaticity of C'], and so, by Corollary 7.12, $C' = (C^{\mathrm{EM}} \cap C^{\mathrm{R}})$. □

As an immediate consequence of Theorems 7.10 and 7.16, as opposed to both $C^{\mathrm{EM}}[\cap C^{\mathrm{R}}]$ and C, we have:

Corollary 7.18. *All extensions of C^{R} are $\vee$-disjunctive.*

Concluding this discussion, we should like to highlight that the technique elaborated here has proved well-applicable to finding all extensions of LP that has been done in [21] with using an advanced algebraic method based upon finding the lattice of all subprevarieties of KL going back to finding that of ones of DML being due to [24]. However, the mentioned method is not applicable to K_3 (as well as to both $LP_{[01]} \cap K_{3[01]}$ and $C_{[\mathrm{B}]\mathrm{B}}$) at all. This highlights the special value of the technique elaborated here.

7.1.4.1.1. Some proper non-Kleene extensions. Finally, we explore some of proper non-Kleene (and so non-$\vee$-disjunctive, in view of Theorem 7.10) extensions of C to be assumed also self-extensional. First of all, notice that (5.14) is not true in $\overrightarrow{\mathcal{A}}$ under $[x_0/\mathsf{n}, x_1/\mathsf{b}, x_2/\mathsf{n}]$. Therefore, by Theorems 5.53 and 5.68, C^{MP} and C^{NP} become first distinct examples of such a kind. (In particular, this shows that Remark 5.51 is not inherited by non-Kleene extensions of C). Moreover, by Theorem 5.53, we get two more distinct proper non-Kleene extensions $C^{\mathrm{EM[+NP]}} \cap C^{\mathrm{MP}}$, for $C^{\mathrm{EM}} \cap C^{\mathrm{MP}}$ is $\sim$-paraconsistent (cf. Theorem 5.28), while $C^{\mathrm{EM+NP}} \cap C^{\mathrm{MP}}$ is an extension of C^{NP}. Then, a one more example of such a kind is as follows:

Theorem 7.19. *$C^{\mathrm{EM}}[\cap C^{\mathrm{R}}] \cap C^{\mathrm{NP}}$ is the proper extension of C relatively axiomatized by the rule* (5.1).

Proof. Let C' be the extension of C relatively axiomatized by the rule (5.1). Since (5.1) is a logical consequence of (5.15) and is true in C_3, $C^{\mathrm{EM}} \cap C^{\mathrm{R}} \cap C^{\mathrm{NP}}$ is an extension of C'. Conversely, consider any $\mathcal{B} \in (\mathrm{Mod}(C') \cap \mathsf{K})$, where $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}(\mathbf{S}_*(\mathcal{A}))$. Assume, (5.15) is not true in $\mathcal{B}$, in which case there is some $a \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{A}} a \in D^{\mathcal{B}}$, and so, by (5.1) [and (3.3)], $(x_0 \vee \sim x_0)[\vee x_1]$ is true in $\mathcal{A}$ [and so is the rule (5.14)]. Thus, $(\mathrm{Mod}(C') \cap \mathsf{K}) \subseteq ((\mathrm{Mod}(C^{\mathrm{NP}}) \cap \mathsf{K}) \cup (\mathrm{Mod}(C^{\mathrm{EM}}[\cap C^{\mathrm{R}}]) \cap \mathsf{K}))$. Hence, by Theorem 2.20, we eventually conclude that $C' = (C^{\mathrm{EM}}[\cap C^{\mathrm{R}}] \cap C^{NP})$. Finally, recall that (5.1) is not true in $\mathcal{A}$ under $[x_0/\mathsf{b}, x_1/\mathsf{n}]$, as required. □

And what is more, we also have:

Theorem 7.20. *The extension of $C^{\mathrm{EM}} \cap C^{\mathrm{MP}}$ relatively axiomatized by* (5.15)*, i.e., the join of $C^{\mathrm{EM}} \cap C^{\mathrm{MP}}$ and C^{NP} is defined by $\{\overrightarrow{\mathcal{A}}, \mathcal{A}_{\not\mathsf{n}} \times \overrightarrow{\mathcal{A}}\}$.*

Proof. By Theorem[s 5.28 and] 5.68, $[C^{\mathrm{EM}} \cap] C^{\mathrm{MP}}$ is defined by $\{\overrightarrow{\mathcal{A}}[, \mathcal{A}_{\not\mathsf{n}}]\}$. In particular, $\overrightarrow{\mathcal{A}}$ is a model of (5.15). Moreover, by (2.6) and Theorem 5.69, $\mathcal{A}_{\not\mathsf{n}} \times \overrightarrow{\mathcal{A}}$, being a submatrix of $\mathcal{A} \times \overrightarrow{\mathcal{A}}$, is a model of (5.15) too. Conversely, consider any finite set I, any $\overline{\mathcal{C}} \in \mathbf{S}_*(\{\overrightarrow{\mathcal{A}}, \mathcal{A}_{\not\mathsf{n}}\})^I$ and any subdirect product $\mathcal{D}$ of it being a model of (5.15). Put $J \triangleq \hom(\mathcal{D}, \mathcal{A}_{\not\mathsf{n}} \times \overrightarrow{\mathcal{A}})$ and $K \triangleq \hom(\mathcal{D}, \overrightarrow{\mathcal{A}})$. Consider any $a \in (D \setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent and there is some $i \in I$, in which case $h \triangleq (\pi_i \restriction D) \in \hom(\mathcal{D}, \mathcal{C}_i)$, such that $h(a) \notin D^{\mathcal{C}_i}$. Consider the following exhaustive cases:

(1) $\mathcal{C}_i = \mathcal{A}_{\not\mathsf{n}}$.
Then, by Lemma 5.48, there is some $g \in \hom(\mathcal{D}, \overrightarrow{\mathcal{A}}) \neq \varnothing$, in which case $f \triangleq (h \times g) \in J$ and $f(a) \notin D^{\mathcal{A}_{\not\mathsf{n}} \times \overrightarrow{\mathcal{A}}}$.

(2) $\mathcal{C}_i = \overrightarrow{\mathcal{A}}$.
Then, $h \in K$.

In this way, $((\prod \Delta_J) \times (\prod \Delta_K)) \in \hom_{\mathrm{S}}(\mathcal{D}, (\mathcal{A}_{\not\mathsf{n}} \times \overrightarrow{\mathcal{A}})^J \times \overrightarrow{\mathcal{A}}^K)$. Hence, by (2.6) and Theorem 2.20, the extension involved is finitely-defined by $\{\overrightarrow{\mathcal{A}}, \mathcal{A}_{\not\mathsf{n}} \times \overrightarrow{\mathcal{A}}\}$. Then, the finiteness of A completes the argument. □

Finally, note that the rule:

$$\{x_0, \sim x_0 \vee x_2\} \vdash ((\sim x_1 \vee x_1) \vee x_2), \tag{7.14}$$

being satisfied in $C^{\mathrm{EM}} \cap C^{\mathrm{MP}}$, in view of (3.3) and (3.4), is not true in $\mathcal{A} \times \overrightarrow{\mathcal{A}}$ under $[x_0/\langle \mathsf{b}, \mathsf{t} \rangle, x_1/\langle \mathsf{n}, \mathsf{t} \rangle, x_2/\langle \mathsf{f}, \mathsf{t} \rangle]$. Therefore, by Theorem 5.69, we get:

Corollary 7.21. *$C^{\mathrm{EM[+NP]}} \cap C^{\mathrm{R/MP}}$ is a proper extension of $(C^{\mathrm{EM}} \cap C^{\mathrm{R}} \cap C^{\mathrm{NP}})[\cup C^{\mathrm{NP}}]$.*

7.2. **Three-valued paraconsistent logics.** Here, we follow Section 4 supposing that $\wr \triangleq \sim \in \Sigma$.

7.2.1. *Three-valued expansions of the logic of paradox.* Here, it is supposed that $\Sigma_0 \subseteq \Sigma$ and $(\mathcal{A}{\restriction}\Sigma_0) = \mathcal{DM}_{4,\not{n}}$ (in which case $\mathcal{A}$ is $\wedge$-conjunctive) that defines LP, so C is an expansion of LP. This covers three-valued expansions of LP by nullary connectives (as regular ones) as well as the *logic of antinomies* LA [2] [resp., $J3$ [5]], defined [up to definitional equivalence] by the $\supset$-implicative $\sim$-superclassical submatrix of the $(\Sigma_{0[1]} \cup \{\supset\})$-matrix given by Subsubsection 7.1.3, where $\supset$ (implication) is binary, the maximal $\sim$-paraconsistency of both of which having been due to [33] collectively with the general part of [25], proved *ad hoc* therein. And what is more, this exhausts *all three-valued* expansions of LP, as it ensues from Corollary 4.7, Corollary 4.16 then yielding the following *universal* result subsuming Theorem 2.1 of [21]:

Corollary 7.22. *Any three-valued expansion of LP is maximally $\sim$-paraconsistent.*

Finally, $\mathfrak{A}$ is clearly a $(\wedge, \vee)$-lattice, so Subsection 4.4.2 (including Theorem 4.35) is well-applicable to C, subsuming corresponding results obtained in [25] and [33] *ad hoc* as well as *immediately* providing a "bounded" expansion of [25]:

Corollary 7.23. *Any [not] non-$\sim$-subclassical three-valued expansion C of LA/ that of LP the underlying algebra of the characteristic matrix $\mathcal{A}$ of which is regular [including $J3/LP_{01}$] has $2[+2]$ extensions forming the chain $C \subsetneq C^{\mathrm{NP}} = [\mathrm{Cn}^{\omega}_{\mathcal{A}\times(\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})} \subsetneq C^{\mathrm{MP}} = C^{\mathrm{PC}} \subsetneq]\, \mathrm{Cn}^{\omega}_{\varnothing}$, C^{NP} being a unique non-axiomatic$|\vee$-disjunctive one/, both proper consistent extensions having same theorems as C having, and so being non-axiomatic, in which case C has no proper axiomatic extensions.*

Likewise, by Corollary 4.42, we immediately have:

Corollary 7.24. *Any three-valued expansion of LP is not self-extensional.*

7.2.2. *Three-valued expansions of Sette's logic.* Let $\Lambda_0 \triangleq \{\supset, \sim\} \subseteq \Sigma$, where $\supset$ (implication) is binary, and $\mathcal{A}$ a $\sim$-superclassical Σ-matrix such that $\sim^{\mathfrak{A}}\mathsf{b} \triangleq \mathsf{t}$, in which case $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$, and

$$(a \supset^{\mathfrak{A}} b) \triangleq \begin{cases} \mathsf{t} & \text{if } (a \neq \mathsf{f}) \Rightarrow (b \neq \mathsf{f}), \\ \mathsf{f} & \text{otherwise}, \end{cases}$$

for all $a, b \in \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, in which case $\mathcal{A}$ is $\supset$-implicative, and so $\vee_{\supset}$-disjunctive, and so is C (cf. Corollary 4.19). In this way, this exhaust all *three-valued* expansions of the logic P^1 [36] of $\mathcal{S}'_3 \triangleq (\mathcal{A}{\restriction}\Lambda_0)$, as it ensues from Corollary 4.7, Theorem 4.13 then yielding the following one more *universal* result:

Corollary 7.25. *Any three-valued expansion of P^1 is maximally $\sim$-paraconsistent.*

This subsumes the maximality result of [36], according to which P^1 itself has no proper $\sim$-paraconsistent *axiomatic* extension, properly strengthened in [19] by proving the fact that the $\sim$-classical logic of $\mathcal{S}'_2 \triangleq (\mathcal{S}'_3{\restriction}\{\mathsf{f},\mathsf{t}\})$ is a unique proper axiomatic extension of $P1$, equally ensuing from Corollary 4.25. This is inherited by three-valued expansions of $P1$.

On the other hand, primary operations of $\mathfrak{S}'_3$ are classically-valued, in which case P^1 is covered by Subsubsection 4.4.3, so, in particular, C, being $\bar{\wedge}$-conjunctive, is equally covered by Corollary 4.16, while, by Corollary 4.17, we have:

Corollary 7.26. *C is $\sim$-subclassical iff $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.*

And what is more, P^1 is a minimal instance of Subsubsection 4.4.3, because the identity $(x_0 \supset x_1) \approx (x_0 \rhd x_1)$ is true in $\mathfrak{S}'_3$, in which case any instance of matrices involved therein id a term-wise definitional copy of an expansion of $\mathcal{S}'_3$, and so any instance of logics involved therein is term-wise definitionally equivalent to a three-valued expansion of P^1.

Finally, in view of the $\bar{\wedge}$-conjunctivity of $\mathcal{S}'_3$ and the fact that all primary operations of $\mathfrak{S}'_3$ are classically-valued, we see that $\mathcal{S}'_3$ is the only consistent truth-non-empty $\bar{\wedge}$-conjunctive Λ_0-matrix with underlying algebra $\mathfrak{S}'_3$. In this way, Remark 4.3 and Proposition 4.50 show that the optional condition of $\mathfrak{A}$'s being a distributive $(\bar{\wedge}, \underline{\vee})$-lattice within the formulation of Theorem 3.9 is essential for (vii[i])$\Rightarrow$(i) therein to hold. Likewise, P^1 collectively with Theorem 4.45 show that, despite of Theorem 4.35, three-valued (even both conjunctive, disjunctive and subclassical) paraconsistent logics with subclassical negation need not have finitely many (even merely finitary) extensions.

7.2.3. *Three-valued expansions of Hałkowska-Zajac' logic.* Let $\Sigma_0 \subseteq \Sigma$ and $\mathcal{A}$ a $\sim$-superclassical Σ-matrix such that $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, while $\wedge^{\mathfrak{A}}$ and $\vee^{\mathfrak{A}}$ are defined as min and max, respectively, but with respect to rather the chain partial ordering $\leqq$ given by $\mathsf{b} \leqq \mathsf{f} \leqq \mathsf{t}$ than the partial ordering given point-wise by the natural one $\leqslant$ on 2, as in the case of the logic of paradox. Then, $\mathcal{A}$ is $\sqsupset$-implicative, where $(x_0 \sqsupset x_1) \triangleq ((\sim x_0 \wedge \sim x_1) \vee x_1)$, in which case it is is $\vee_{\sqsupset}$-disjunctive, and so is C (cf. Corollary 3.17). Moreover, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of the underlying algebra of $\mathcal{HZ} \triangleq (\mathcal{A}{\restriction}\Sigma_0)$, while $\mathcal{HZ}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is $\wedge$-conjunctive. In this way, this exhausts *all three-valued* expansions of the logic HZ [11] of $\mathcal{HZ}$, as it ensues from Corollary 4.7, Theorems 4.13, 4.26[6] and Corollary 4.24 then yielding the following one more *universal* result:

Corollary 7.27. *Any three-valued expansion of HZ is maximally $\sim$-paraconsistent.*

This subsumes the maximality result of [26] concerning HZ alone and proved *ad hoc* therein.

Finally, though $\mathfrak{A}$ is a $(\wedge, \vee)$-lattice, $\mathcal{A}$ is neither $\wedge$-conjunctive nor $\vee$-disjunctive, because $(\mathsf{f} \wedge^{\mathfrak{H}3} \mathsf{b}) = \mathsf{b}$ and $(\mathsf{f} \vee^{\mathfrak{H}3} \mathsf{b}) = \mathsf{f}$. Nevertheless, $\mathfrak{A}$ is a $(\bar{\wedge}, \underline{\vee})$-lattice, where $\bar{\wedge} \triangleq \tilde{\vee}$ and $\underline{\vee} \triangleq \tilde{\wedge}$ (cf. Remark 2.22), because these secondary connectives correspond to min and max, respectively, with regard to the chain partial ordering $\lesssim$ given by $\mathsf{f} \lesssim \mathsf{t} \lesssim \mathsf{b}$, and so $\mathcal{A}$ is both $\bar{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive. (In particular, this case is equally covered by Corollary 4.16.) In this way, Subsection 4.4.2 (including Theorem 4.35) is well-applicable to C, yielding the results obtained in [26] and [33] *ad hoc*:

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[6] It is this case that demonstrates applicability of Theorem 4.26, and so its value.

Corollary 7.28. *Any [not] non-$\sim$-subclassical three-valued expansion C of HZ [in particular, HZ] has* 2[+2] *extensions forming the chain* $C \subsetneq C^{\mathrm{NP}} = [\mathrm{Cn}^{\omega}_{\mathcal{A}\times(\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\})} \subsetneq C^{\mathrm{MP}} = C^{\mathrm{PC}} \subsetneq]\, \mathrm{Cn}^{\omega}_{\varnothing}$, C^{NP} *being a unique non-axiomatic|$\veebar$-disjunctive one.*

Likewise, by Corollary 4.42, we immediately have:

Corollary 7.29. *Any three-valued expansion of HZ is not self-extensional.*

7.2.4. *Three-valued expansions of the dual Stone logic.* Let $\Sigma_0 \subseteq \Sigma$ and $\mathcal{A}$ a $\sim$-superclassical Σ-matrix such that $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$ and $(\mathfrak{A}\restriction\Sigma^+) = (\mathfrak{D}^2_2\restriction A)$, in which case $\mathfrak{A}$ is a $(\wedge,\vee)$-lattice with unit t and zero f, in which case C is covered by Subsubsection 4.4.2, so, by Corollary 4.16, we first have:

Corollary 7.30. *C is maximally $\sim$-paraconsistent.*

And what is more, providing $\Sigma = \Sigma_0$, $h_{\mathsf{b}\to\mathsf{f}}$ is an endomorphism of $\mathfrak{A}$, in which case, by Theorem 4.38, C is self-extensional, so, by Theorem 4.35 and Corollary 4.44, we get:

Corollary 7.31. *Suppose C is [not] non-$\sim$-subclassical [in particular, $\Sigma = \Sigma_0$]/ and $\Sigma = \Sigma_0$. Then, extensions of C form the* (2[+2])*-element chain* $C \subsetneq C^{\mathrm{NP}} = [\mathrm{Cn}^{\omega}_{\mathcal{A}\times(\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\})} \subsetneq] C^{\mathrm{MP}} = [C^{\mathrm{PC}} \subsetneq]\, \mathrm{Cn}^{\omega}_{\varnothing}$ *[C^{NP} being a non-axiomatic and a unique non-$\veebar$-disjunctive/-self-extensional one].*

On the other hand, in that case, the (pre)variety DSA of *dual Stone algebras* generated by $\mathfrak{A}$ is term-wise definitionally equivalent (via the lattice duality) to the variety of *Stone algebras* [9], having exactly three subprevarieties (cf. [22]), and so does DSA, in which case, as opposed to LP, LA, $J3$, HZ as well as the implication-less fragment of Dummett's linear superintuitionistic logic [6] actually dual to C (via both the lattice duality and the truth predicate complement $\complement$), the general study [25] is not applicable to C. This highlights a particular value of Theorem 4.35 and equally concerns the interesting self-extensional three-valued expansion of C studied below. In this connection, it is remarkable that, in view of Lemma 4.37 and Theorem 4.38, any self-extensional instance of Subsection 4.4.2 is a definitional copy of a three-valued expansion of C, so C is a minimal instance of such a kind.

7.2.4.1. The dual version of Gödel's three-valued logic. Here, it is supposed that $\Sigma = (\Sigma_0 \cup \{\supset\})$, while $(a \supset^{\mathfrak{A}} b) \triangleq \min\{c \in A \mid b \leqslant \max(c,a)\}$, for all $a, b \in A$. Then, C is actually dual to Gödel's three-valued logic [8] (via both the lattice duality and the truth predicate complement $\complement$). And what is more, $h_{\mathsf{b}\to\mathsf{f}}$ is an endomorphism of $\mathfrak{A}$, in which case, by Theorem 4.38, C is self-extensional, and so, by Corollary 4.44, we get:

Corollary 7.32. *The extensions of C form the four-element chain* $C \subsetneq C^{\mathrm{NP}} = \mathrm{Cn}^{\omega}_{\mathcal{A}\times(\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\})} \subsetneq C^{\mathrm{MP}} = C^{\mathrm{PC}} \subsetneq \mathrm{Cn}^{\omega}_{\varnothing}$, C^{NP} *being non-axiomatic and the unique non-$\veebar$-disjunctive/non-self-extensional one.*

8. Conclusions

Aside from the quite non-trivial general results and their numerous illustrative applications, the present paper demonstrates the special value of the conception of congruence/equality determinant, initially suggested in [28] just for the sake of construction of *two-side* sequent calculi (like those found in [20] and [23]) for *many*-valued logics as opposed to *many-place* sequent approaches originally going back to independent works [35] and [34] and recently summarized in [27] and [32], where the conception of equality determinant has been advanced towards that of *singularity determinant* within the framework of *many-place* [32] (viz., *signed*; cf. [27]) matrices.

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Department of Digital Automata Theory (100), V.M. Glushkov Institute of Cybernetics, National Academy of Sciences of Ukraine, Glushkov prosp. 40, Kiev, 03680, Ukraine
Email address: pynko@i.ua