Deep on Goldbach's Conjecture

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Abstract

Goldbach's conjecture is one of the most difficult unsolved problems in mathematics. This states that every even natural number greater than 2 is the sum of two prime numbers. In 1973, Chen Jingrun proved that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes). In 2015, Tomohiro Yamada, using the Chen's theorem, showed that every even number $> \exp\exp 36$ can be represented as the sum of a prime and a product of at most two primes. In 2002, Ying Chun Cai proved that every sufficiently large even integer N is equal to $p + P_2$, where P_2 is an almost prime with at most two prime factors and $p \leq N^{0.95}$ is a prime number. In this note, we prove that for every even number $N \geq 32$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $N \gg \sigma(m)$ and p is coprime with m, then m is necessarily a prime number when $\sigma(m)$ is the sum-of-divisors function of m, $N = 2 \cdot n$ and $\gg$ means "much greater than". Indeed, this is a trivial and short note very easy to check and understand which is a breakthrough result at the same time.

Keywords: Goldbach's conjecture, Prime numbers, Sum-of-divisors function, Euler's totient function

MSC Classification: 11A41 , 11A25

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n. Define $s(n)$ as $\frac{\sigma(n)}{n}$. In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n. We can state the sum-of-divisors function of n as

$$\sigma(n) = \prod_{p|n} \frac{p^{\nu_p(n)+1} - 1}{p - 1}$$

with the product extending over all prime numbers p which divide n. In addition, the well-known Euler's totient function $\varphi(n)$ can be formulated as

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Chen's theorem states that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes) [1]. Tomohiro Yamada using an explicit version of Chen's theorem showed that every even number greater than $e^{e^{36}} \approx 1.7 \cdot 10^{1872344071119343}$ is the sum of a prime and a product of at most two primes [2]. A natural number is called k-almost prime if it has k prime factors [3]. A natural number is prime if and only if it is 1-almost prime, and semiprime if and only if it is 2-almost prime. Let N be a sufficiently large even integer. Ying Chun Cai proved that the equation

$$N = p + P_2, \quad p \leq N^{0.95},$$

is solvable, where p denotes a prime and P_2 denotes an almost prime with at most two prime factors [3]. In mathematics, two integers a and b are coprime, if the only positive integer that is a divisor of both of them is 1. Putting all together yields the proof of the main theorem.

Theorem 1 *For every even number $N \geq 32$, if there is a prime p and a natural number m such that $n < p < N-1$, $p+m = N$, $N \gg \sigma(m)$ and p is coprime with m, then m is necessarily a prime number when $N = 2 \cdot n$ and $\gg$ means "much greater than".*

2 Proof of Theorem 1

Proof Suppose that there is an even number $N \geq 32$ which is not a sum of two distinct prime numbers. We consider all the pairs of positive integers $(n-k, n+k)$ where $n = \frac{N}{2}$, $k < n$ is a natural number, $n+k$ and $n-k$ are coprime integers and $n+k$ is prime. By definition of the functions $\sigma(x)$ and $\varphi(x)$, we know that

$$2 \cdot N = \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n-k$ is also prime. We notice that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n-k$ is not a prime. Certainly, we see that $(n-k)+(n+k) = N$ and thus, the inequality

$$2 \cdot ((n-k)+(n+k)) + \varphi((n-k) \cdot (n+k)) < \sigma((n-k) \cdot (n+k))$$

holds when $n-k$ is not a prime. That is equivalent to

$$2 \cdot ((n-k)+(n+k)) + \varphi(n-k) \cdot \varphi(n+k) < \sigma(n-k) \cdot \sigma(n+k)$$

since the functions $\sigma(x)$ and $\varphi(x)$ are multiplicative. Let's divide both sides by $(n-k) \cdot (n+k)$ to obtain that

$$2 \cdot \left(\frac{(n-k)+(n+k)}{(n-k) \cdot (n+k)}\right) + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k} < s(n-k) \cdot s(n+k).$$

We know that

$$s(n-k) \cdot s(n+k) > 1$$

since $s(m) > 1$ for every natural number $m > 1$ [4]. Moreover, we could see that

$$2 \cdot \left(\frac{(n-k)+(n+k)}{(n-k) \cdot (n+k)}\right) = \frac{2}{n+k} + \frac{2}{n-k}$$

and therefore,

$$1 > \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}.$$

It is enough to see that

$$1 > \frac{2}{23} + \frac{2}{9} + \frac{2}{3} \geq \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}$$

when $n+k$ is prime and $n-k$ is composite for $N \geq 32$. Under our assumption, every of these pairs of positive integers $(n-k, n+k)$ implies that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

holds when $n = \frac{N}{2}$, $k < n$ is a natural number, $n+k$ and $n-k$ are coprime integers and $n+k$ is prime. Now suppose that $N \gg \sigma(n-k)$, where $\gg$ means "much greater than". Besides, we deduce that

$$2 = \sigma(n+k) - \varphi(n+k)$$

when $n+k$ is prime. Hence, we have

$$(\sigma(n+k) - \varphi(n+k)) \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

that is equivalent to

$$(\sigma(n+k) - \varphi(n+k)) < \frac{\sigma(n-k)}{N} \cdot \sigma(n+k) - \frac{\varphi(n-k)}{N} \cdot \varphi(n+k)$$

and

$$\sigma(n+k) \cdot \left(\frac{1}{\sigma(n-k)} - \frac{1}{N}\right) < \varphi(n+k) \cdot \left(\frac{1}{\sigma(n-k)} - \frac{\varphi(n-k)}{N \cdot \sigma(n-k)}\right).$$

However, we can assure that the previous inequality does not hold when $N \gg \sigma(n-k)$. For that reason, we obtain the desired contradiction. By reductio ad absurdum, the natural number $n-k$ is necessarily prime. ∎

References

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