On Some Results on Nonlocal Neutral Integro-Differential Equations with Deviated argument and Non-instantaneous Impulses

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Abstract

In this paper, we consider a non-instantaneous impulsive system represented by nonlocal and nonlinear differential equation with deviated argument in Banach space. We used semigroup of linear operators and fixed point method to study the existence and uniqueness of the solutions of the non-instantaneous impulsive system. Finally, we give examples to illustrate the application of these abstract results.

Keywords: Neutral equations, Equations with impulses, Non-instantaneous impulse condition, Integro-differential equations, Fixed point theorem, Semigroup theory, deviated argument.

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1 Introduction

Impulsive differential equations are utilized to represent numerous practical dynamical systems including evolutionary process characterized by abrupt changes of the state at specific moments. Such process are naturally seen in biology, optimal control in economics, physics engineering, etc. Because of their importance, many authors have established the solvability of impulsive differential equations. The qualitative theory of impulsive differential equation was initiated by V. Milman and A. Myshkis in 1960s. There has been substantial advancements in impulsive theory, in recent times, particularly in the area of impulsive differential equations with fixed moments; see the monographs [1, 2, 6, 7, 18–20, 23, 24, 29–31, 33, 34, 36] and the references cited there in.

The study of abstract differential equations with non-instantaneous impulses was initiated recently by Hernandez and O'Regan in [12]. In the abstract model analyzed in [12], the impulses are triggered abruptly at the instants $'ti'$ and their action remains during a finite time interval of the form $[ti, si]$. As pointed out in [12], there are many different motivations for the investigation of this kind of problems. To continue the development in [12], the more realistic situation in which the impulsive action is not instantaneous but depends on its accumulation over the entire time interval in which acts was studied

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in [28]. Lately, Pierri et al [27] study the existence results of some abstract differential equations with non-instantaneous impulses with the help of fractional powers of operators and semigroup theory.

However, due to theoretical and practical difficulties, the study of impulsive differential equations with deviating arguments has been developed rather gradually. Mean while, Guobing et al [35] established the existence solution of periodic boundary value problems for a class of impulsive neutral differential equations with multi-deviation arguments. However the existence of piecewise continuous mild solutions to impulsive functional differential equations with iterated deviating arguments was obtained by Kumar et al [15]. Furthermore the approximation of a solution to a class of evolution equations with a deviated argument was derived by kumar [16]. Many authors studied about existence and uniqueness results concerning the P - mild solutions. On the other hand Kumar et al [17] derived the P - mild solutions for Impulsive Integro-Differential Equations with a Deviating Argument. Existence results for impulsive neutral functional differential equations with infinite delay was analyzed by [3].

The plentiful applications of differential equations with deviating arguments has motivated the rapid development of the theory of differential equations with deviating arguments and their generalization in the recent years see [9, 10, 15, 32]. Extension of the theory of differential equations with deviating argument as well as stimuli of developments within various fields of science and technology contribute to the need for further development. This theory in recent years has attracted the attention of vast number of researchers, interested in both in the theory and its applications. For more details, we refer [4, 5, 10, 21]. For further references about functional differential equations, we refer [8, 13, 14] and [11].

Our objective here is to give existence results for the given problem by using semigroup theory. In section 2, we recall some preliminary results and definitions which will be utilized throughout this manuscript. In section 3, we present and prove the existence of solutions for the given problem. Our approach here is based on Krasnoselskii's fixed point theorem [22, theorem 1]. Finally in section 4, an application is provided to illustrate the obtained results.

2 Problem formation and preliminaries

In this section, we recall some preliminary results and necessary conditions to prove the problem. We consider the impulsive Neutral Integro-Differential Equations with Deviated argument and Non-instantaneous Impulses of the model

$$\frac{d}{dt}\Big[w(t) - G_1\left(t, w_t, (\mathscr{H}_1 w)(t)\right)\Big] = Aw(t) + F\left(t, w(t), w(h(w(t), t))(t)\right) + G_2\left(t, w_t, (\mathscr{H}_2 w)(t)\right), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, \dots \delta,\ J = [0, T] \tag{2.1}$$

$$w(t) = h_i(t, w(t_i^-)) \text{ for } t \in (t_i, s_i],\ i = 1, 2, \dots \delta, \tag{2.2}$$

$$w(0) = \varphi = w_0 \in \mathscr{B}_h, \tag{2.3}$$

where $w(t)$ is the state function and the operator A is the infinitesimal generator of a analytic semigroup $\{T(t)\}_{t\geq 0}$ in a Banach space X having norm $\|\cdot\|$ and M_1 is a positive constant to ensure that

$\|T(t)\| \leq M_1, G_j : I \times \mathscr{B}_h \times X \to X, j = 1,2$ are given X-valued functions, $\mathscr{H}_j, j = 1,2$ are described as

$$(\mathscr{H}_j w)(t) = \int_0^t e_j(t,s,w_s)ds,$$

where $e_j : \mathscr{D} \times \mathscr{B}_h \to X$, $j = 1,2$; $\mathscr{D} = \{(t,s) \in J \times J : 0 \leq s \leq t \leq T\}$ are suitable functions, and $\mathscr{B}_h$ is a phase space characterized in preliminaries. $w(t_i^-) = w(t_i)$ and $w(t_i^+)$ exists for $i = 1,2\ldots,m$. $h_i(t, w(t_i^-))$ represent the non instantaneous impulses during the interval $(t_i, s_i]$ where $i = 1,2,\cdots,m$ so impulses at t_i^- have some duration on $(t_i, s_i]$. Here $0 = s_0 = t_0 < t_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = T$. We give existence theorems for solutions satisfying $w(0) = w_0$, when F and h are continuous and uniformly locally Lipschitz on all of their variables.For almost any continuous function w characterized on $(-\infty, T]$ and any $t \geq 0$, we represent by w_t the part of $\mathscr{B}_h$ characterized by $w_t(\theta) = w(t+\theta)$ for $\theta \in (-\infty, 0]$ and $\|T(t-s) \leq M_1\|$. The following $w_t(\cdot)$ denote the history of the state from time $-\infty$, up to the current time t.

To consider the impulsive systems (2.1)-(2.3), we need to define the following concepts. Let X be a Banach space provided with norm $\|\cdot\|$. Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X. Then it is possible to determine the fractional power A^α for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A)^\alpha$, being dense in X. If X_α represents the space $D(A)^\alpha$ endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha)$$

then the following properties are well known. With this discussion, we recall fundamental properties of fractional powers A^α from Pazy [26].

Definition 2.1. *Let $T(t)$ is an analytic semigroup with the infinitesimal generator A and If $0 \in \varrho(A)$, then*

(i) Let $0 < \alpha \leq 1$, then X_α is a Banach space.

(ii) If $0 < \beta \leq \alpha$, then the embedding $X_\alpha \subset X_\beta$ is compact whenever the resolvent operator of A is compact.

(iii) For each $\alpha > 0$, we can find a positive constant M_α to ensure that

$$\| A^\alpha T(t) \| \leq \frac{M_\alpha}{t^\alpha}, \quad 0 < t \leq T.$$

We denote by $PC([0,T], X)$ the space of piecewise continuous function from $[0,T]$ into X. $g : PC(J,X) \to X$ are given functions satisfying certain assumptions. In particular, we introduce the space PC formed by all piecewise continuous functions $w : [0,T] \to X$ such that $w(\cdot)$ is continuous at $t \neq t_k$, $w(t_k^-) = w(t_k)$ and $w(t_k^+)$ exists for $k = 1,2\ldots,m$. We assume that PC is a Banach space , endowed with the norm $\|w\|_{PC} = \sup_{s\in[0,T]} \|w(s)\|_{PC}$. It is clear that $(PC, \|\cdot\|_{PC})$ is a Banach space. $PC((0,T], X)$=$\{w : (0,T] \to X$ such that $w_k \in C((t_k, t_{k+1}], X), k = 0,1,2,\ldots,m$ and there exist $w({t_k}^+)$ and $w({t_k}^-)$ with $w(t_k) = w({t_k}^-), k = 0,1,2\ldots,m\}$, We define $C_L(J,X) = \{y \in PC((0,T], X) : \|y(t) - y(s)\| \leq \widetilde{K}|t-s| \,\forall\, t,s \in [0,T]\}$ where $\widetilde{K}$ is some positive constant, is a Banach space endowed with Piecewise norm. It should be fixed that, once the delay is infinite, then we need to discuss about the theoretical phase space $\mathscr{B}_h$ in a useful way. In this manuscript, we consider phase spaces $\mathscr{B}_h, \mathscr{B}'_h$ which are same as described in [25].

We present the abstract phase space $\mathscr{B}_h$. Suppose $h : (-\infty, 0] \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^{0} h(t)dt < +\infty$ and for any $a > 0$, we define $\mathscr{B} = \{\psi : [-a, 0] \to X \quad \text{such that } \psi(t) \text{ is bounded and measurable}\}$ and equip the space $\mathscr{B}$ with norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\| \quad \text{and} \quad \psi \in \mathscr{B}$. Let us define $\mathscr{B}_h = \{\psi : (-\infty, 0] \to X$ such that for any $c > 0, \psi|_{[-c,0]} \in \mathscr{B}$ and $\int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < +\infty$. If $\mathscr{B}_h$ is endowed with the norm $\|\psi\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds$ for every $\psi \in \mathscr{B}_h$, then it is clear that $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is a Banach space. Now, we consider the space $\mathscr{B}'_h = PC((-\infty, T], X) = \{w : (-\infty, T] \to X$ such that $w_k \in C(J_k, X)$ and there exist $w(t_k{}^+)$ and $w(t_k{}^-)$ with $w(t_k) = w(t_k{}^-), w_0 = \phi \in \mathscr{B}_h, k = 0, 1, 2 \ldots m\}$, where w_k is the restriction of w to $J_k = (t_k, t_{k+1}]$, set$\|\cdot\|_{\mathscr{B}'_h}$ be the seminorm in $\mathscr{B}'_h$ defined by $\|w\|_{\mathscr{B}'_h} = \|\phi\|_{\mathscr{B}_h} + \sup\{|w(s)| : s \in [0, T]\}, w \in \mathscr{B}'_h$.

A semi-normed linear space of functions $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is the phase space which mapping $(-\infty, 0]$ into X, and satisfying the ensuing rudimentary axioms as a consequence of Hale and Kato (see case point in [11, 13]).

If $w : (-\infty, T] \to X, T > 0$, is continuous on $[0, T]$ and $w_0 \in \mathscr{B}_h$, then the following conditions hold for every $t \in J$:

(P_1) w_t is in $\mathscr{B}_h$;

(P_2) $\|w(t)\|_X \leq H\|w_t\|_{\mathscr{B}_h}$;

(P_3) $\|w_t\|_{\mathscr{B}_h} \leq D_1(t) \sup\{\|w(s)\| : 0 \leq s \leq t\} + D_2(t)\|z_0\|_{\mathscr{B}_h}$, where $H > 0$ is a constant and $D_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $D_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded, and D_1, D_2 are independent of $w(\cdot)$. For our convenience, denote $D_1^* = \sup_{s\in[0,T]} D_1(s), \quad D_2^* = \sup_{s\in[0,T]} D_2(s)$.

In order to study the existence results for the problem (2.1) to (2.3), we need to list the following hypotheses:

(H1) The map $\mathscr{F} : (J_1 \times X \times X) \to X$ is a continuous function. Then a constant $K_f > 0$ exists and $J_1 = \cup_{i=0}^{m}[s_i, t_{i+1}]$ such that $\|\mathscr{F}(t, \varphi_1, x) - \mathscr{F}(s, \varphi_2, y)\| \leq K_f\left[|t - s| + \|\varphi_1 - \varphi_2\| + \|x - y\|\right]$, for all (φ_1, x), $(\varphi_2, y) \in X$ and $t, s \in J_i$. Also there exists a constant $\widetilde{K_f}$ such that $\|\mathscr{F}(t, \varphi, x)\| \leq \widetilde{K_f}$, $\forall \varphi, x \in X$ and $t \in J_1$.

(H2) The function $q : X \times [0, \infty) \to [0, \infty)$ is continuous function and there are positive constants K_a such that $\|q(x, s) - q(y, s)\| \leq K_a\left[\|x - y\|\right]$, for all $x, y \in X, t \in [0, T]$.

(H3) The function $G_1 : J \times \mathscr{B}_h \times X \to X$ is continuous and there exist positive constants $\beta \in (0, 1)$, $K_{G_1} > 0$, $\widetilde{K}_{G_1} > 0$ and $K^*_{G_1} > 0$ such that G_1 is X_β-valued and for all $(t, \varphi_j) \in J \times \mathscr{B}_h$, $x, y \in X, \quad j = 1, 2$;

(i) $\|A^\beta G_1(t, \varphi_1, x) - A^\beta G_1(t, \varphi_2, y)\|_X \leq K_{G_1}\|\varphi_1 - \varphi_2\|_{\mathscr{B}_h} + \widetilde{K}_{G_1}\|x - y\|_X, \ x, y \in X$;

(ii) $\|A^\beta G_1(t, \varphi, 0)\|_X \leq K_{G_1}\|\varphi\|_{\mathscr{B}_h} + K^*_{G_1}$, $K^*_{G_1} = \max_{t\in J} \|A^\beta G_1(t, 0, 0)\|_X$.

(H4) There exists positive constants $K_{G_2} > 0$, $\widetilde{K}_{G_2} > 0$ and $K^*_{G_2} > 0$, the function $G_2 : J \times \mathscr{B}_h \times X \to X$ is continuous for all $(t, \varphi_j) \in J \times \mathscr{B}_h, \quad j = 1, 2$;

(i) $\|G_2(t,\varphi_1,x) - G_2(t,\varphi_2,y)\|_X \le K_{G_2}\|\varphi_1 - \varphi_2\|_{\mathscr{B}_h} + \widetilde{K}_{G_2}\|x-y\|_X,\ x,y \in X;$

(ii) $\|A^\beta G_2(t,\varphi,0)\|_X \le K_{G_2}\|\varphi\|_{\mathscr{B}_h} + K^*_{G_2}, K^*_{G_2} = \max_{t\in J}\|G_2(t,0,0)\|_X.$

(H5) There exist positive constants $K_{h_i}, i = 1,2,\cdots N$, such that $\|h_i(t,\varphi_1) - h_i(t,\varphi_2)\|_X \le K_{h_i}\|\varphi_1 - \varphi_2\|_{\mathscr{B}_h}$, for each $t \in (t_i, s_i]$ and all $\varphi_1, \varphi_2 \in \mathscr{B}_h$.

(H6) The functions $e_j : \mathscr{D} \times \mathscr{B}_h \to X$ are continuous and there exist constants $K_{e_j} > 0, K^*_{e_j} > 0$, to ensure that $\left\|e_j(t,s,\varphi) - e_j(t,s,\psi)\right\|_X \le K_{e_j}\|\varphi - \psi\|_{\mathscr{B}_h}, (t,s) \in \mathscr{D}, (\varphi,\psi) \in \mathscr{B}^2_h, j = 1,2$; and $K^*_{e_j} = \max_{t\in J}\|e_j(t,s,0)\|_X,\ j = 1,2.$

(H7) The following inequalities holds: Let

$$\begin{aligned}
&\|\Phi^1_i u(t) + \Phi^2_i u(t)\| \\
&\le M_1\left[K_{h_i}(D^*_1 q + c_n) + K^*_{h_i}\right] + \left(M_0(1+M_1) + \frac{M_{1-\beta}T^\beta}{\beta}\right)\left[(K_{G_1} + \widetilde{K}_{G_1}TK_{e_1})D^*_1 q + p_1\right] \\
&+ M_1(t_{i+1} - s_i)\widetilde{W_f} + M_0M_1T\left[(K_{G_2} + \widetilde{K}_{G_2}TK_{e_2})D^*_1 q + p_2\right], \le q,\ t \in [0,T],
\end{aligned}$$

where $p_1 = [(K_{G_1} + \widetilde{K}_{G_1}TK_{e_1})c_n + \widetilde{K}_{G_1}TK^*_{e_1} + K^*_{G_1}], p_2 = [(K_{G_2} + \widetilde{K}_{G_2}TK_{e_2})c_n + \widetilde{K}_{G_2}TK^*_{e_2} + K^*_{G_2}]$

Definition 2.2. *A function $w : (0,T] \to X$ is called a mild solution of the impulsive model (2.1)-(2.3) if it satisfies the following conditions $w(0) = w_0$, the non instantaneous impulse conditions $w(t) = h_i(t,w_t)$ for $t \in (t_i, s_i]$ for each $i = 1,2,\ldots,N$ and $w(t)$ is the solution of the following integral equations*

$$w(t) = \begin{cases}
T(t)[(\varphi(0)) - G_1(0,\varphi,0)] + G_1\left(t, w_t, \int_0^t e_1(t,s,w_s)ds\right) \\
\quad + \int_0^t AT(t-s)G_1\left(s, w_s, \int_0^s e_1(s,\tau,w_\tau)d\tau\right)ds \\
\quad + \int_0^t T(t-s)F\left(t, w(t), w(h(w(t),t))\right)ds \\
\quad + \int_0^t T(t-s)G_2\left(s, w_s, \int_0^s e_2(s,\tau,w_\tau)d\tau\right)ds, \quad t \in [0,t_1], \\
h_i(t,w_t),\ t \in (t_i, s_i], \text{ for } i = 1,2,\cdots,N, \\
T(t-s_i)\left[h_i(s_i, w_{s_i}) - G_1\left(s_i, w_{s_i}, \int_0^{s_i} e_1(s_i,\tau,w_\tau)d\tau\right)\right] \\
\quad + G_1\left(t, w_t, \int_0^t e_1(t,s,w_s)ds\right) \\
\quad + \int_{s_i}^t AT(t-s)G_1\left(s, w_s, \int_0^s e_1(s,\tau,w_\tau)d\tau\right)ds \\
\quad + \int_0^t T(t-s)F\left(t, w(t), w(h(w(t),t))\right)ds \\
\quad + \int_{s_i}^t T(t-s)G_2\left(s, w_s, \int_0^s e_2(s,\tau,w_\tau)d\tau\right)ds, \quad t \in [s_i, t_{i+1}]
\end{cases} \tag{2.4}$$

3 Existence results

In this section we present the existence results for the structure (2.1)-(2.3) under Krasnoselskii's fixed point theorem.

Theorem 3.1. *Assume that the hypotheses (H1)-(H7) hold. Then the problem (2.1)-(2.3) has a unique solution on J. Then*

$$\Lambda^* = D_1^*\bigg\{M_1 K_{h_i} + \bigg((1+M_1)M_0 + \frac{M_{1-\beta}T^\beta}{\beta}\bigg)\Big[K_{G_1} + \widetilde{K}_{G_1} T K_{e_1}\Big] + M_0 M_1 D_1^*\Big[K_{G_2} + \widetilde{K}_{G_2} T K_{e_2}\Big]\bigg\} < 1$$

Proof. The problem (2.1)-(2.3) will be transformed into a fixed point problem. Consider the operator $\Phi : \mathscr{B}_h' \to \mathscr{B}_h'$. Then

$$(\Phi w)(t) = \begin{cases} T(t)[\varphi(0) - G_1(0,\varphi,0)] + G_1\left(t, w_t, \int_0^t e_1(t,s,w_s)ds\right) \\ \quad + \int_0^t AT(t-s)G_1\left(s, w_s, \int_0^s e_1(s,\tau,w_\tau)d\tau\right)ds \\ \quad + \int_0^t T(t-s)F\left(t, w(t), w(h(w(t),t))\right)ds, \\ \quad + \int_0^t T(t-s)G_2\left(s, w_s, \int_0^s e_2(s,\tau,w_\tau)d\tau\right)ds, \quad t \in [0,t_1], \\ h_i(t,w_t),\ t \in (t_i, s_i], \text{ for } i = 1,2,\cdots,N, \\ T(t-s_i)\left[h_i(s_i, w_{s_i}) - G_1\left(s_i, w_{s_i}, \int_0^{s_i} e_1(s_i,\tau,w_\tau)d\tau\right)\right] \\ \quad + G_1\left(t, w_t, \int_0^t e_1(t,s,w_s)ds\right) \\ \quad + \int_{s_i}^t AT(t-s)G_1\left(s, w_s, \int_0^s e_1(s,\tau,w_\tau)d\tau\right)ds \\ \quad + \int_{s_i}^t T(t-s)F\left(t, w(t), w(h(w(t),t))\right)ds \\ \quad + \int_{s_i}^t T(t-s)G_2\left(s, w_s, \int_0^s e_2(s,\tau,w_\tau)d\tau\right)ds, \quad t \in [s_i, t_{i+1}]. \end{cases} \tag{3.1}$$

Obviously the fixed points of the operator Φ are mild solutions of the model (2.1)-(2.3). The function $v(\cdot) : (-\infty, T] \to X$ is defined by

$$v(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0]; \\ T(t)\varphi(0), & t \in J, \end{cases}$$

then $v_0 = \varphi$. For each function $u \in C(J,\mathbb{R})$ with $u(0) = 0$, we assign as $\widetilde{u}$ is defined by

$$\widetilde{u}(t) = \begin{cases} 0, & t \le 0; \\ u(t), & t \in J. \end{cases}$$

If $w(\cdot)$ fulfills (2.4), we can easily split it as $w(t) = u(t) + v(t)$, for all $t \in J$, this means $w_t = u_t + v_t$.

$$u(t)=\begin{cases} -T(t)G_1(0,\varphi,0)+G_1\left(t,u_t+v_t,\int_0^t e_1(t,s,u_s+v_s)ds\right) \\ \quad +\int_0^t AT(t-s)G_1\left(s,u_s+v_s,\int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau\right)ds \\ \quad +\int_0^t T(t-s)F\left(s,(u+v)(s),(u+v)(h((u+v)(s),s))\right)ds \\ \quad +\int_0^t T(t-s)G_2\left(s,u_s+v_s,\int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds, \qquad t\in[0,t_1], \\ h_i(t,u_t+v_t),\ t\in(t_i,s_i],\ \text{for } i=1,2,\cdots,N, \\ T(t-s_i)\Big[h_i(s_i,u_{s_i}+v_{s_i})-G_1\left(s_i,u_{s_i}+v_{s_i},\int_0^{s_i} e_1(s_i,\tau,u_\tau+v_\tau)d\tau\right)\Big] \\ \quad +G_1\left(t,u_t+v_t,\int_0^t e_1(t,s,u_s+v_s)ds\right) \\ \quad +\int_{s_i}^t AT(t-s)G_1\left(s,u_s+v_s,\int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau\right)ds \\ \quad +\int_{s_i}^t T(t-s)F\left(s,(u+v)(s),(u+v)(h((u+v)(t),t)\right)ds \\ \quad +\int_{s_i}^t T(t-s)G_2\left(s,u_s+v_s,\int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds \qquad t\in[s_i,t_{i+1}]. \end{cases} \tag{3.2}$$

Let $\mathscr{B}_h''=\{u\in\mathscr{B}_h' \colon u_0=0\in\mathscr{B}_h\}$. Let $\|\cdot\|_{\mathscr{B}_h''}$ be the seminorm in $\mathscr{B}_h''$ described by

$$\|u\|_{\mathscr{B}_h''}=\sup_{t\in I}\|u(t)\|_X+\|u_0\|_{\mathscr{B}_h}=\sup_{t\in I}\|u(t)\|_X,\quad u\in\mathscr{B}_h''$$

as a result $(\mathscr{B}_h'',\|\cdot\|_{\mathscr{B}_h''})$ is a Banach space. Consider $B_q=\{u\in\mathscr{B}_h'':\|u\|_X\le q\}$ for some $q\ge 0$; then for each q, $B_q\subset\mathscr{B}_h''$ is clearly a bounded closed convex set. For $u\in B_q$, from the phase space axioms (P_1)-(P_3),

$$\begin{aligned} &\|u_s+v_s\|_{\mathscr{B}_h} \\ &\le \|u_s\|_{\mathscr{B}_h}+\|v_s\|_{\mathscr{B}_h} \\ &\le D_1^*\sup_{(0\le\tau\le u_s+v_s)}\|u(\tau)\|_X+D_2^*\|u_0\|_{\mathscr{B}_h}+D_1^*\sup_{(0\le\tau\le u_s+v_s)}\|v(\tau)\|+D_2^*\|v_0\|_{\mathscr{B}_h} \\ &\le D_1^*\sup_{(0\le\tau\le s)}\|u(\tau)\|_X+D_1^*\|T(t)\|_{L(X)}\|\varphi(0)\|_{\mathscr{B}_h}+D_2^*\|\varphi\|_{\mathscr{B}_h} \\ &\le D_1^*\|u\|_X+(D_1^*M_1+D_2^*)\|\varphi\|_{\mathscr{B}_h} \\ &\le D_1^*q+c_n, \end{aligned}$$

In the event that $\|u\|_X<q$, $q>0$.

$$\|u_s+v_s\|_{\mathscr{B}_h}\le D_1^*q+c_n, \tag{3.3}$$

where $c_n=(D_1^*M_1+D_2^*)\|\varphi\|_{\mathscr{B}_h}$.
We introduce the operator $\overline{\Phi}:\mathscr{B}_h''\to\mathscr{B}_h''$ where $\overline{\Phi}$ maps $B_q(0,\mathscr{B}_h'')$ into $B_q(0,\mathscr{B}_h'')$. For any $u(\cdot)\in\mathscr{B}_h''$,

$$(\overline{\Phi}u)(t) = \begin{cases} -T(t)G_1(0,\varphi,0) + G_1\left(t, u_t+v_t, \int_0^t e_1(t,s,u_s+v_s)ds\right) \\ \quad + \int_0^t AT(t-s)G_1\left(s, u_s+v_s, \int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau\right)ds \\ \quad + \int_0^t T(t-s)F\left(t,(u+v)(t),(u+v)(h((u+v)(t),t)\right)ds \\ \quad + \int_0^t T(t-s)G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds, \qquad t\in[0,t_1], \\ h_i(t,u_t+v_t), \qquad t\in(t_i,s_i], \text{ where } i=1,2,\cdots,N, \\ T(t-s_i)\left[h_i(s_i,u_{s_i}+v_{s_i}) - G_1\left(s_i, u_{s_i}+v_{s_i}, \int_0^{s_i} e_1(s_i,\tau,u_\tau+v_\tau)d\tau\right)\right] \\ \quad +G_1\left(t, u_t+v_t, \int_0^t e_1(t,s,u_s+v_s)ds\right) \\ \quad + \int_{s_i}^t AT(t-s)G_1\left(s, u_s+v_s, \int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau\right)ds \\ \quad + \int_{s_i}^t T(t-s)F\left(t,(u+v)(t),(u+v)(h((u+v)(t),t)\right) \\ \quad + \int_{s_i}^t T(t-s)G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds, \qquad t\in[s_i,t_{i+1}]. \end{cases} \tag{3.4}$$

From this, it is known that the operator Φ has a fixed point if and only if $\overline{\Phi}$ has a fixed point. Let us prove that $\overline{\Phi}$ has a fixed point. Before we prove the main results, first we calculate the following estimations.

Remark 3.1. *By utilizing (3.1), and (H1)-(H6), we obtain*

$$J_1 = \|T(t)\|_{L(X)}\|G_1(0,\varphi,0)\|_X \le M_1M_0\left[K_{G_1}\|\varphi\|_{\mathscr{B}_h} + K^*_{G_1}\right], \textit{where } M_0 = \|A^{-\beta}\|;$$

$$J_2 = \left\|G_1\left(t,u_t+v_t,\int_0^t e_1(t,s,u_s+v_s)ds\right)\right\| \le M_0\left[(K_{G_1}+\widetilde{K}_{G_1}TK_{e_1})D_1^*q + p_1\right];$$

$$J_3 = \left\|\int_0^t AT(t-s)G_1\left(s,u_s+v_s,\int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau\right)ds\right\|$$
$$\le \frac{M_{1-\beta}T^\beta}{\beta}\left[(K_{G_1}+\widetilde{K}_{G_1}TK_{e_1})D_1^*q+p_1\right]; \textit{where } p_1 = [(K_{G_1}+\widetilde{K}_{G_1}TK_{e_1})c_n + \widetilde{K}_{G_1}TK^*_{e_1} + K^*_{G_1}],$$

$$J_4 = \left\|\int_0^t T(t-s)F\left(s,(u+v)(s),(u+v)(h((u+v)(s),s)\right)ds\right\| \le M_1T\widetilde{K_f}$$

$$J_5 = \left\|\int_0^t T(t-s)G_2\left(s,u_s+v_s,\int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds\right\|_X$$
$$\le M_0M_1T\left[(K_{G_2}+\widetilde{K}_{G_2}TK_{e_2})D_1^*q+p_2\right]; \textit{where } p_2 = [(K_{G_2}+\widetilde{K}_{G_2}TK_{e_2})c_n + \widetilde{K}_{G_2}TK^*_{e_2}+K^*_{G_2}].$$

$$J_6 = \|h_i(t,u_t+v_t)\| \le K_{h_i}(D_1^*q+c_n)+K^*_{h_i},\ t\in(t_i,s_i].$$

$$J_7 = \|h_i(s_i,u_{s_i}+v_{s_i})\| \le K_{h_i}(D_1^*q+c_n)+K^*_{h_i},\ t\in(s_i,t_{i+1}]$$

$$J_8 = \left\|T(t-s_i)G_1\left(s_i,u_{s_i}+v_{s_i},\int_0^{s_i} e_1(s_i,\tau,u_\tau+v_\tau)d\tau\right)\right\|$$
$$\le M_0M_1\left[(K_{G_1}+\widetilde{K}_{G_1}TK_{e_1})D_1^*q+p_1\right],\ t\in(s_i,t_{i+1}].$$

$$J_9 = \left\| \int_{s_i}^t AT(t-s)G_1 \left(s, u_s+v_s, \int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau \right) ds \right\|$$
$$\leq \frac{M_{1-\beta}(t_{i+1}-s_i)^\beta}{\beta} \left[(K_{G_1}+\widetilde{K}_{G_1}TK_{e_1})D_1^* q \right] + \frac{M_{1-\beta}(t_{i+1}-s_i)^\beta}{\beta} p_1,\ t \in (s_i,t_{i+1}].$$
$$J_{10} = \left\| \int_{s_i}^t T(t-s)F(s,(u+v)(s),(u+v)(h((u+v)(s),s)))ds \right\| \leq M_1 \widetilde{K_f}(t_{i+1}-s_i),\ t \in (s_i,t_{i+1}].$$
$$J_{11} = \left\| \int_{s_i}^t T(t-s)G_2 \left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau \right) ds \right\|$$
$$\leq M_0 M_1 (t_{i+1}-s_i) \left[(K_{G_2}+\widetilde{K}_{G_2}TK_{e_2})D_1^* q + p_3 \right],\ t \in (s_i,t_{i+1}].$$
$$J_{12} = \left\| G_1 \left(t, u_t+v_t, \int_0^t e_1(t,s,u_s+v_s)ds \right) - G_1 \left(t, \overline{u}_t+v_t, \int_0^t e_1(t,s,\overline{u}_s+v_s)ds \right) \right\|$$
$$\leq M_0 \left[K_{G_1}D_1^* + \widetilde{K}_{G_1}TK_{e_1}D_1^* \right] \|u-\overline{u}\|_{\mathscr{B}_h''},$$
$$J_{13} = \left\| \int_0^t AT(t-s)G_1 \left(s, u_s+v_s, \int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau \right) ds \right.$$
$$\left. - \int_0^t AT(t-s)G_1 \left(s, \overline{u}_s+v_s, \int_0^s e_1(s,\tau,(\overline{u}_\tau+v_\tau))d\tau \right) ds \right\|$$
$$\leq \frac{M_{1-\beta}T^\beta}{\beta} D_1^* \left[K_{G_1} + \widetilde{K}_{G_1}TK_{e_1} \right] \|u-\overline{u}\|_{\mathscr{B}_h''}.$$
$$J_{14} = \left\| \int_0^t T(t-s)G_2 \left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau \right) ds \right.$$
$$\left. - \int_0^t T(t-s)G_2 \left(s, \overline{u}_s+v_s, \int_0^s e_2(s,\tau,(\overline{u}_\tau+v_\tau))d\tau \right) ds \right\|$$
$$\leq M_1 T D_1^* \left[K_{G_2} + \widetilde{K}_{G_2}TK_{e_2} \right] \|u-\overline{u}\|_{\mathscr{B}_h''}.$$
$$J_{15} = \|h_i(t,u_t+v_t) - h_i(t,\overline{u}_t+v_t)\| \leq D_1^* K_{h_i} \|u-\overline{u}\|_{\mathscr{B}_h''},\ t \in (t_i,s_i].$$
$$J_{16} = \left\| h_i(s_i,u_{s_i}+v_{s_i}) - h_i(s_i,(\overline{u}_{s_i}+v_{s_i})) \right\| \leq D_1^* K_{h_i} \|u-\overline{u}\|_{\mathscr{B}_h''},\ t \in (s_i,t_{i+1}].$$
$$J_{17} = \left\| G_1 \left(s_i, u_{s_i}+v_{s_i}, \int_0^{s_i} e_1(s_i,\tau,u_\tau+v_\tau)d\tau \right) \right.$$
$$\left. - G_1 \left(s_i, (\overline{u}_{s_i}+v_{s_i}), \int_0^{s_i} e_1(s_i,\tau,(\overline{u}_\tau+v_\tau))d\tau \right) \right\|$$
$$\leq M_0 D_1^* \left[K_{G_1} + \widetilde{K}_{G_1}TK_{e_1} \right] \|u-\overline{u}\|_{\mathscr{B}_h''},\ t \in (s_i,t_{i+1}].$$
$$J_{18} = \left\| \int_{s_i}^t AT(t-s)G_1 \left(s, u_s+v_s, \int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau \right) ds \right.$$
$$\left. - \int_{s_i}^t AT(t-s)G_1 \left(s, \overline{u}_s+v_s, \int_0^s e_1(s,\tau,(\overline{u}_\tau+v_\tau))d\tau \right) ds \right\|$$
$$\leq \frac{M_{1-\beta}(t_{i+1}-s_i)^\beta}{\beta} D_1^* \left[K_{G_1} + \widetilde{K}_{G_1}TK_{e_1} \right] \|u-\overline{u}\|_{\mathscr{B}_h''},\ t \in (s_i,t_{i+1}].$$
$$J_{19} = \left\| \int_{s_i}^t T(t-s)G_2 \left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau \right) ds \right.$$
$$\left. - \int_{s_i}^t T(t-s)G_2 \left(s, \overline{u}_s+v_s, \int_0^s e_2(s,\tau,(\overline{u}_\tau+v_\tau))d\tau \right) ds \right\|$$

$$\leq M_0 M_1(t_{i+1}-s_i)D_1^*\Big[K_{G_2}+\widetilde{K}_{G_2}TK_{e_2}\Big]\|u-\overline{u}\|_{\mathscr{B}_h''},\ t\in(s_i,t_{i+1}].$$

$$J_{20}=\Big\|\int_0^t T(t-s)F(s,(\bar u+v)(s),(\bar u+v)(h((\bar u+v)(s),s)))ds - \int_0^t T(t-s)F(s,(u+v)(s),(u+v)(h((u+v)(s),s)))ds\Big\|$$

Now, we estimate

$$\begin{aligned}\Big\|(\bar u+v)(h((\bar u+v)(s),s))-(u+v)(h((u+v)(s),s))\Big\| &\leq \|(\bar u+v)(h((\bar u+v)(s),s))-(u+v)((h(\bar u+v)(s),s))\|\\ &\quad+\|(\bar u+v)(h(\bar u+v)(s),s))-(u+v)(h((u+v)(s),s))\| \end{aligned}\tag{3.5}$$

But we have

$$\|(\bar u+v)(s)-(u+v)(s)\|=\sup_{h((\bar u+v)(s),s)\in[0,T]}\|\big((\bar u+v)(h((\bar u+v)(s),s))-(u+v)(h((\bar u+v)(s),s))\|\tag{3.6}$$

$$\|(\bar u-u)\|=\sup_{h((\bar u+v)(s),s)\in[0,T]}\|\big((\bar u+v)(h((\bar u+v)(s),s))-(u+v)(h((\bar u+v)(s),s))\|\tag{3.7}$$

$$\begin{aligned}\|\big((\bar u+v)(h((\bar u+v)(s),s))-(u+v)(h((\bar u+v)(s),s))\| &\leq \sup_{t\in[0,t_1]}\|(\bar u+v)(s)-(u+v)(s)\|\\ &\leq \widetilde{K^*}_f\|\bar u-u\|\end{aligned}$$

Therefore the inequality (3.6) *becomes*

$$\|(\bar u+v)(h((\bar u+v)(s),s))-(u+v)(h((u+v)(s),s))\|\leq\|\bar u-u\|+\widetilde{K^*}_f\|\bar u-v\|\tag{3.8}$$

$$\Big\|\int_0^t T(t-s)F(s,(\bar u+v)(s),(\bar u+v)(h((\bar u+v)(s),s)))ds - \int_0^t T(t-s)F(s,(u+v)(s),(u+v)(h((u+v)(s),s)))ds\Big\|\leq M_1TK_fD_1^*[2\|\bar u-u\|+\widetilde{K^*}_f\|\bar u-v\|]$$

Now, we enter the main proof of the theorem. To apply Krasnoselskii's fixed point theorem, we introduce the decomposition $\overline{\Phi}=\sum_{i=0}^{N}\Phi_i^1+\sum_{i=0}^{N}\Phi_i^2$

$$(\Phi_i^1u)(t)=\begin{cases} -T(t)G_1(0,\varphi,0)+G_1\left(t,u_t+v_t,\int_0^t e_1(t,s,u_s+v_s)ds\right)\\ +\int_0^t AT(t-s)G_1\left(s,u_s+v_s,\int_0^s e_1(s,\tau,u_\tau+v_\tau)\right)ds\\ +\int_0^t T(t-s)F\left(s,(u+v)(s),(u+v)(h((u+v)(s),s))\right)ds,\ t\in[0,t_1], i=1,2,\cdots,N,\\ 0,\qquad t\in[t_i,t_{i+1}],\ i\geq 0,\\ h_i(t,u_t+v_t),\qquad t\in(t_i,s_i],\ i=1,2,\cdots,N,\\ T(t-s_i)\Big[h_i(s_i,u_{s_i}+v_{s_i})-G_1\left(s_i,u_{s_i}+v_{s_i},\int_0^{s_i}e_1(s_i,\tau,u_\tau+v_\tau)d\tau\right)\Big]\end{cases}$$

$$(\Phi_i^1 u)(t) = \begin{cases} +G_1\left(t, u_t + v_t, \int_0^t e_1(t,s,u_s+v_s)ds\right) \\ +\int_{s_i}^t AT(t-s_i)G_1\left(s, u_s+v_s, \int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau\right)ds \\ +\int_{s_i}^t T(t-s_i)F\left(s,(u+v)(s),(u+v)(h((u+v)(s),s))\right)ds \\ t\in[s_i,t_{i+1}] \text{ for } i=1,2,\cdots,N,. \end{cases}$$

$$(\Phi_i^2 u)(t) = \begin{cases} \int_0^t T(t-s)G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds \quad t\in[0,t_1], \\ 0, t\in(t_i,s_i], \\ \int_{s_i}^t T(t-s)G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds, t\in[s_i,t_{i+1}], \\ \text{for } i=1,2,\cdots,N. \end{cases}$$

For better readability, we divide our results into four steps.

Step 1: First we show that $\Phi_i^1 u(t) + \Phi_i^2 u(t) \in B_q$, whenever $u \in B_q$. For all $u \in B_q$ and Remark 3.1, we have

$$\begin{aligned} &\|\Phi_i^1 u(t) + \Phi_i^2 u(t)\| \\ &\le M_1 M_0\left[K_{G_1}\|\varphi\|_{\mathscr{B}_h} + K_{G_1}^*\right] + \left[M_0 + \frac{M_{1-\beta}T^\beta}{\beta}\right]\left[(K_{G_1} + \widetilde{K}_{G_1}TK_{e_1})D_1^* q + p_1\right] \\ &+ M_0 M_1 T\left[(K_{G_2} + \widetilde{K}_{G_2}TK_{e_2})D_1^* q + p_2\right] + M_1 T\widetilde{K_f} \le q,\ t\in[0,t_1], \\ &\|\Phi_i^1 u(t) + \Phi_i^2 u(t)\| \le K_{h_i}(D_1^* q + c_n) + K_{h_i}^* \le q,\ t\in(t_i,s_i]. \\ &\|\Phi_i^1 u(t) + \Phi_i^2 u(t)\| \\ &\le M_1\left[K_{h_i}(D_1^* q + c_n) + K_{h_i}^*\right] + \left(M_0(1+M_1) + \frac{M_{1-\beta}(t_{i+1}-s_i)^\beta}{\beta}\right)\left[(K_{G_1} + \widetilde{K}_{G_1}TK_{e_1})D_1^* q + p_1\right] \\ &+ M_1(t_{i+1}-s_i)\widetilde{K_f} + M_0 M_1(t_{i+1}-s_i)\left[(K_{G_2} + \widetilde{K}_{G_2}TK_{e_2})D_1^* q + p_2\right], \\ &\le q,\ t\in(s_i,t_{i+1}]. \\ &\|\Phi_i^1 u(t) + \Phi_i^2 u(t)\| \\ &\le M_1\left[K_{h_i}(D_1^* q + c_n) + K_{h_i}^*\right] + \left(M_0(1+M_1) + \frac{M_{1-\beta}T^\beta}{\beta}\right)\left[(K_{G_1} + \widetilde{K}_{G_1}TK_{e_1})D_1^* q + p_1\right] \\ &+ M_1(t_{i+1}-s_i)\widetilde{K_f} + M_0 M_1 T\left[(K_{G_2} + \widetilde{K}_{G_2}TK_{e_2})D_1^* q + p_2\right], \\ &\le q,\ t\in[0,T], \end{aligned}$$

where $p_1, p_2, M_1 T\widetilde{K_f}$ are independent of q. Dividing both sides by q, we have $\Phi_i^1 u(t) + \Phi_i^2 u(t) \in B_q$.

Step 2: Next we will show that $\Phi^1 = \sum_{i=0}^{N} \Phi_i^1$ is a contraction.

From the definition of $\Phi^1 u(t), \Phi_i^1 u(t)$ and the assumption of $(H1)-(H6)$, we get

$$
\begin{aligned}
&\|(\Phi_i^1 u)(t) - (\Phi_i^1 \overline{u})(t)\| \\
&\qquad \leq \left\| G_1\left(t, u_t + v_t, \int_0^t e_1(t,s,u_s+v_s)ds\right) \right. \\
&\qquad\quad \left. - G_1\left(t, \overline{u}_t + v_t, \int_0^t e_1(t,s,\overline{u}_s+v_s)ds\right) \right\| \\
&\qquad\quad + \left\| \int_0^t AT(t-s)\left[G_1\left(s, u_s+v_s, \int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau\right)\right.\right. \\
&\qquad\quad \left.\left. - G_1\left(s, \overline{u}_s+v_s, \int_0^s e_1(s,\tau,(\overline{u}_\tau+v_\tau))d\tau\right)\right] ds \right\| \\
&\qquad + \left\| \int_0^t T(t-s)F(s,(\widetilde{u}+v)(s),(\widetilde{u}+v)(h((\widetilde{u}+v)(s),s))) \right. \\
&\qquad\quad \left. - F(s,(u+v)(s),(u+v)(h((u+v)(s),s))) \right\| ds \\
&\qquad \leq D_1^*\left[\left(M_0 + \frac{M_{1-\beta}{t_1}^\beta}{\beta}\right)\left(K_{G_1} + \widetilde{K}_{G_1}TK_{e_1}\right) \right. \\
&\qquad\quad \left. + M_1TK_f[2+\widetilde{K^*}_f] \right] \|\bar{u} - v\|,\ t \in [0,t_1]. \\
&\|(\Phi_i^1 u)(t) - (\Phi_i^1 \overline{u})(t)\| \leq D_1^* K_{h_i}\|u-\overline{u}\|_{\mathscr{B}_h''},\ t \in (t_i, s_i], \\
&\|(\Phi_i^1 u)(t) - (\Phi_i^1 \overline{u})(t)\| \leq D_1^*\left[\left\{ M_1K_{h_i} + \left((1+M_1)M_0 + \frac{M_{1-\beta}(t_{i+1}-s_i)^\beta}{\beta}\right)\left[K_{G_1} + \widetilde{K}_{G_1}TK_{e_1}\right]\right.\right. \\
&\qquad\quad \left.\left. + M_0M_1(t_{i+1}-s_i)D_1^*\left[K_f[2+\widetilde{K^*}_f]\right]\right\}\right\|u-\overline{u}\|_{\mathscr{B}_h''},\ t \in (s_i, t_{i+1}] \\
&\|(\Phi_i^1 u)(t) - (\Phi_i^1 \overline{u})(t)\| \leq D_1^*\left\{ M_1K_{h_i} + \left((1+M_1)M_0 + \frac{M_{1-\beta}T^\beta}{\beta}\right)\left[K_{G_1} + \widetilde{K}_{G_1}TK_{e_1}\right]\right. \\
&\qquad\quad \left. + M_0M_1D_1^*\left[K_{G_2} + \widetilde{K}_{G_2}TK_{e_2}\right]\right\}\|u-\overline{u}\|_{\mathscr{B}_h''},\ t \in [0,T] \\
&\qquad \leq \Lambda^*\|u-\overline{u}\|_{\mathscr{B}_h''},
\end{aligned}
$$

where
$\Lambda^* = D_1^*\left\{ M_1K_{h_i} + \left((1+M_1)M_0 + \frac{M_{1-\beta}T^\beta}{\beta}\right)\left[K_{G_1} + \widetilde{K}_{G_1}TK_{e_1}\right] + M_0M_1D_1^*\left[K_{G_2} + \widetilde{K}_{G_2}TK_{e_2}\right]\right\} < 1$
Hence, $\Phi^1 u(t)$ is a contraction.

Step 3: Next we will prove that Φ_i^2 is compact and continuous. We split the proof into three parts.
(a) Φ^2 is continuous.

Let the sequence u_n such that $u_n \to u$ in $\mathscr{B}_h''$. Then for all $t \in J$, by the definition of $\Phi^2 u(t), \Phi_i^2 u(t)$,

$$
\begin{aligned}
&\|(\Phi_i^2 u^n)(t) - (\Phi_i^2 u)(t)\| \\
&\qquad \leq \left\| \int_0^t T(t-s)\left[G_2\left(s, u_s^n + v_s, \int_0^s e_2(s,\tau,u_\tau^n + v_\tau d\tau\right)\right.\right. \\
&\qquad\quad \left.\left. - G_2\left(s, u_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)\right] ds \right\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq M_1 t_1 \bigg\| G_2\left(s, u_s^n+v_s, \int_0^s e_2(s,\tau,u_\tau^n+v_\tau d\tau\right) \\
&\quad - G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)\bigg\|, \; t\in[0,t_1],
\end{aligned}
$$

$$
\begin{aligned}
&\|(\Phi_i^2 u^n)(t) - (\Phi_i^2 u)(t)\| \\
&\leq \bigg\| \int_{s_i}^t T(t-s)\bigg[G_2\left(s, u_s^n+v_s, \int_0^s e_2(s,\tau,u_\tau^n+v_\tau)d\tau\right) \\
&\quad - G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)\bigg]ds\bigg\| \\
&\leq M_1(t_{i+1}-s_i)\bigg\| G_2\left(s, u_s^n+v_s, \int_0^s e_2(s,\tau,u_\tau^n+v_\tau)d\tau\right) \\
&\quad - G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)\bigg\|, \; t\in[s_i,t_{i+1}]
\end{aligned}
$$

and G_2 is continuous, then we know that $\|(\Phi_i^2 u^n)(t) - (\Phi_i^2 u)(t)\| \to 0$ as $n\to\infty, u_n\to u$, which shows that Φ^2 is continuous.

(b) Φ^2 maps bounded sets into bounded sets in $\mathscr{B}_h''$. It is enough to show that for any $R>0$, there exists $R'>0$ such that for each $u\in B_q = u\in\mathscr{B}_h'' : \|u\|_{PC}\leq R$, we have $\|\Phi^2 u\|_{\mathscr{B}_h''}\leq R'$. From the definition of $\Phi^2 u(t)$

$$
\begin{aligned}
\|(\Phi_i^2 u)(t)\| &\leq\leq M_0 M_1 T\Big[(K_{G_2}+\widetilde{K}_{G_2} T K_{e_2})D_1^* q + p_2\Big], \; t\in[0,t_1], \\
\|(\Phi_i^2 u)(t)\| &\leq M_0 M_1 (t_{i+1}-s_i)\Big[(K_{G_2}+\widetilde{K}_{G_2} T K_{e_2})D_1^* q + p_2\Big], t\in(s_i,t_{i+1}], \\
\|(\Phi_i^2 u)(t)\| &\leq M_0 M_1 \Big[(K_{G_2}+\widetilde{K}_{G_2} T K_{e_2})D_1^* q + p_2\Big], \\
&\leq R, \; t\in[0,T].
\end{aligned}
$$

Then we conclude that Φ^2 maps bounded sets into bounded sets.

(c) Finally, we show that Φ^2 maps bounded sets into equicontinuous sets.

For interval $t\in(s_i,t_{i+1}], s_i\leq l_1\leq l_2\leq t_{i+1}, i=1,\cdots,N$, for every $u(t)\in B_q$, by definition of $\Phi_i^2 u(t)$,

$$
\begin{aligned}
&\|(\Phi_i^2 u)(l_2) - (\Phi_i^2 u)(l_1)\| \\
&\leq \bigg\| \int_{s_i}^{l_2} T(l_2-s) G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds \\
&\quad - \int_{s_i}^{l_1} T(l_1-s) G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds\bigg\| \\
&\leq \int_{l_1}^{l_2} \bigg\| T(l_2-s) G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)\bigg\| ds \\
&\quad + \int_{s_i}^{l_1} \bigg\| [T(l_2-s)-T(l_1-s)] G_2\left(s, u_s+v_s, \int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)\bigg\|
\end{aligned}
$$

$$\leq M_1(l_2-l_1)\bigg[(K_{G_2}+\widetilde{K}_{G_2}TK_{e_2})D_1^*q+p_2\bigg],$$
$$+(l_1-s_i)\|T(l_2-s)-T(l_1-s)\|\bigg[(K_{G_2}+\widetilde{K}_{G_2}TK_{e_2})D_1^*q+p_2\bigg],\ t\in[s_i,t_{i+1}]$$

as $l_1\to l_2$, the right hand side tends to zero is equicontinuous.
step 4: Φ_i^2 maps B_q into a precompact set in $\mathscr{B}_h''$.

Now, we shall prove that Φ_i^2 is relatively compact in Φ_i^2. Obviously Φ_i^2 is relatively compact in $\mathscr{B}_h''$, for $t=0,\ 0<\epsilon<t$ for $u\in B_q$. We define

$$\begin{aligned}(\Phi_i^{2,\epsilon}u)(t)&=\int_0^{t-\epsilon}T(t-s)G_2\left(s,u_s+v_s,\int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds\\&=T(\epsilon)\int_0^{t-\epsilon}T(t-s-\epsilon)G_2\left(s,u_s+v_s,\int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds\end{aligned}$$

For the reason that $T(t)$ is compact operator, $V_\epsilon(t)=\left\{(\Phi_i^{2,\epsilon}u)(t):u\in B_q\right\}$ is relatively compact in X for every ϵ, for every $0<\epsilon<t$, for each $u\in B_q$,

$$\begin{aligned}\left\|(\Phi_i^2u)(t)-(\Phi_i^{2,\epsilon}u)(t)\right\|&\leq\int_{t-\epsilon}^t\left\|T(t-s)G_2\left(s,u_s+v_s,\int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds\right\|\\&=\int_{t-\epsilon}^t M_1\bigg[(K_{G_2}+\widetilde{K}_{G_2}TK_{e_2})D_1^*q+p_2\bigg],\\&\leq\epsilon[\Lambda]\to0\text{ as }\epsilon\to0.\end{aligned}$$

which are relatively compact sets arbitrarily close to the set $V_\epsilon(t),t>0$ as a result $V_\epsilon(t)$ is relatively compact in X. From the above steps, it follows by the Krasnoselskii's fixed point theorem , we get that $\overline{\Phi}$ has at least one fixed point $u(t)\in\mathscr{B}_h''$. With these, the mild solution u is a fixed point of the operator Φ of the problem (2.1)-(2.3). This completes the proof of the theorem. □

4 The Nonlocal Problem

In this section, we consider the nonlocal differential problem. The nonlocal condition is a generalization of the classical initial condition. The study of nonlocal initial value problems are important because they appear in many physical systems. Byszewski (1991) was the first author who studied the existence and uniqueness of mild solutions to the Cauchy problems with non local conditions.

We consider the following nonlocal differential problem with deviated argument in a Banach space X:

$$\frac{d}{dt}\Big[w(t)-G_1\left(t,w_t,(\mathscr{H}_1w)(t)\right)\Big]=Aw(t)+F\left(t,w(t),w(h(w(t),t))(t)\right)+G_2\left(t,w_t,(\mathscr{H}_2w)(t)\right),$$
$$t\in(s_i,t_{i+1}],\ i=0,1,\dots\delta,\ J=[0,T]\tag{4.1}$$
$$w(t)=h_i(t,w(t_i^-))\text{ for }t\in(t_i,s_i],\ i=1,2,\dots\delta,\tag{4.2}$$
$$w(0)=w_0+g(w)\in\mathscr{B}_h,\tag{4.3}$$

where $w(t)$ is the state function and the operator A is the infinitesimal generator of a analytic semigroup $\{T(t)\}_{t\geq 0}$ in a Banach space X having norm $\|\cdot\|$ and M_1 is a positive constant to ensure that $\|T(t)\| \leq M_1, G_j : I \times \mathscr{B}_h \times X \to X, j = 1, 2$ are given X-valued functions, $\mathscr{H}_j, j = 1, 2$ are described as

$$(\mathscr{H}_j w)(t) = \int_0^t e_j(t, s, w_s) ds,$$

where $e_j : \mathscr{D} \times \mathscr{B}_h \to X,\ j = 1, 2;\ \mathscr{D} = \{(t, s) \in J \times J : 0 \leq s \leq t \leq T\}$ are suitable functions. We give existence theorems for solutions satisfying $w(0) = w_0 + g(w)$, when the nonlocal condition $g(w)$ will suitably specified.

Definition 4.1. *A function $w : (0, T] \to X$ is called a mild solution of the impulsive model (4.1)-(4.3) if it satisfies the following conditions $w(0) = w_0 + g(w)$, the non instantaneous impulse conditions $w(t) = h_i(t, w_t)$ for $t \in (t_i, s_i]$ for each $i = 1, 2, \ldots, N$ and $w(t)$ is the solution of the following integral equations*

$$w(t) = \begin{cases} T(t)[(\varphi(0) + g(w)) - G_1(0, \varphi, 0)] + G_1\left(t, w_t, \int_0^t e_1(t, s, w_s) ds\right) \\ \quad + \int_0^t AT(t-s) G_1\left(s, w_s, \int_0^s e_1(s, \tau, w_\tau) d\tau\right) ds \\ \quad + \int_0^t T(t-s) F\left(t, w(t), w(h(w(t), t))\right) ds \\ \quad + \int_0^t T(t-s) G_2\left(s, w_s, \int_0^s e_2(s, \tau, w_\tau) d\tau\right) ds, \quad t \in [0, t_1], \\ h_i(t, w_t),\ t \in (t_i, s_i],\ \textit{for}\ i = 1, 2, \cdots, N, \\ T(t - s_i)\left[h_i(s_i, w_{s_i}) - G_1\left(s_i, w_{s_i}, \int_0^{s_i} e_1(s_i, \tau, w_\tau) d\tau\right)\right] \\ \quad + G_1\left(t, w_t, \int_0^t e_1(t, s, w_s) ds\right) \\ \quad + \int_{s_i}^t AT(t-s) G_1\left(s, w_s, \int_0^s e_1(s, \tau, w_\tau) d\tau\right) ds \\ \quad + \int_0^t T(t-s) F\left(t, w(t), w(h(w(t), t))\right) ds \\ \quad + \int_{s_i}^t T(t-s) G_2\left(s, w_s, \int_0^s e_2(s, \tau, w_\tau) d\tau\right) ds, \quad t \in [s_i, t_{i+1}] \end{cases} \tag{4.4}$$

Further, we need the assumptions on the function $g(w)$ to prove the existence results for the problem $(4.1) - (4.3)$.

(H8) The function $g : C_L(J, X)$ is continuous and there exist a positive constant K_g such that

$$\|g(x_1) - g(x_2)\| \leq K_g \|x_1 - x_2\|.$$

Theorem 4.1. *Assume that the hypotheses (H1)-(H7) and (H8) hold. Then the problem (4.1)-(4.3) has a unique solution on J. Then*

$$\Lambda = D_1^*\left[K_g + \left(M_0 + \frac{M_{1-\beta} {t_1}^\beta}{\beta}\right)\left(K_{G_1} + \widetilde{K}_{G_1} T K_{e_1}\right) + M_0 T K_f [2 + \widetilde{K^*}_f]\right] < 1.$$

Proof. Consider the operator $\Phi^* : \mathscr{B}'_h \to \mathscr{B}'_h$ by

$$(\Phi_i^{*1}u)(t) = \begin{cases} -T(t)[g(u_t+v_t)+G_1(0,\varphi,0)] + G_1\left(t, u_t+v_t, \int_0^t e_1(t,s,u_s+v_s)ds\right) \\ +\int_0^t AT(t-s)G_1\left(s, u_s+v_s, \int_0^s e_1(s,\tau,u_\tau+v_\tau)\right)ds \\ +\int_0^t T(t-s)F\left(s,(u+v)(s),(u+v)(h((u+v)(s),s))\right)ds,\ t\in[0,t_1], i=1,2,\cdots,N, \\ \quad 0, \qquad t\in[t_i,t_{i+1}],\ i\geq 0, \\ \quad h_i(t,u_t+v_t), \qquad t\in(t_i,s_i],\ i=1,2,\cdots,N, \\ \quad T(t-s_i)\left[h_i(s_i,u_{s_i}+v_{s_i}) - G_1\left(s_i,u_{s_i}+v_{s_i},\int_0^{s_i} e_1(s_i,\tau,u_\tau+v_\tau)d\tau\right)\right] \\ +G_1\left(t,u_t+v_t,\int_0^t e_1(t,s,u_s+v_s)ds\right) \\ +\int_{s_i}^t AT(t-s_i)G_1\left(s,u_s+v_s,\int_0^s e_1(s,\tau,u_\tau+v_\tau)d\tau\right)ds \\ +\int_{s_i}^t T(t-s_i)F\left(s,(u+v)(s),(u+v)(h((u+v)(s),s))\right)ds \\ t\in[s_i,t_{i+1}] \text{ for } i=1,2,\cdots,N,. \end{cases}$$

$$(\Phi_i^{*2}u)(t) = \begin{cases} \int_0^t T(t-s)G_2\left(s,u_s+v_s,\int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds \quad t\in[0,t_1], \\ 0, t\in(t_i,s_i], \\ \int_{s_i}^t T(t-s)G_2\left(s,u_s+v_s,\int_0^s e_2(s,\tau,u_\tau+v_\tau)d\tau\right)ds, t\in[s_i,t_{i+1}], \\ \text{for}\, i=1,2,\cdots,N. \end{cases}$$

Let $\mathscr{B}''_h = \{u\in\mathscr{B}'_h \colon u_0 = 0\in\mathscr{B}_h, \|\Phi^*\| < \Lambda\}$. Let $\|\cdot\|_{\mathscr{B}''_h}$ be the seminorm in $\mathscr{B}''_h$ described by

$$\|u\|_{\mathscr{B}''_h} = \sup_{t\in I}\|u(t)\|_X + \|u_0\|_{\mathscr{B}_h} = \sup_{t\in I}\|u(t)\|_X, \quad u\in\mathscr{B}''_h$$

as a result $(\mathscr{B}''_h, \|\cdot\|_{\mathscr{B}''_h})$ is a Banach space. Consider $B_q = \{u\in\mathscr{B}''_h : \|u\|_X \leq q\}$ for some $q\geq 0$; then for each $q, B_q\subset\mathscr{B}''_h$ is clearly a bounded closed convex set. For $u\in B_q$, from the phase space axioms (P_1)-(P_3),

$$\|(\Phi_i^{*1}u)(t) - (\Phi_i^{*1}\overline{u})(t)\| \leq \Lambda\|\bar{u}-v\|.$$

where $\Lambda = D_1^*\left[K_g + \left(M_0 + \frac{M_{1-\beta}t_1{}^\beta}{\beta}\right)\left(K_{G_1} + \widetilde{K}_{G_1}TK_{e_1}\right) + M_0TK_f[2+\widetilde{K^*}_f]\right] < 1$. Hence, $\Phi^1u(t)$ is a contraction. When we apply Banach fixed point theorem, immediately gives unique solution for the problem (4.1)-(4.2). □

5 Conclusion

The research presented in this paper focuses on the existence, uniqueness of solutions to the impulsive systems represented by first order nonlinear differential equations with noninstantaneous impulses and deviated argument. We extended the same problem for nonlocal conditions. We used strongly continuous semigroup of bounded linear operators, Krasnoselski's fixed point theorem and Banach's fixed point theorem to get the existence and uniqueness of the solutions.

6 Application

To epitomize our hypothetical results, now, we consider the following INIDE with infinite delay of the structure To epitomize our hypothetical results, now, we consider the following Impulsive neutral integro-differential equation with deviated argument of the structure

$$\begin{aligned}\frac{d}{dt}\Big\{\Big[z(t,x)+\Big[\int_{-\infty}^{t}a_1(t,x,s-t)Q_1(z(s,x))ds+\int_0^t\int_{-\infty}^{s}K_1(s,\tau)Q_2(z(\tau,x))d\tau ds\Big]\Big\} = \frac{\partial^2}{\partial x^2}\Big\{z(t,x)\\ +\Big[\int_{-\infty}^{t}a_2(t,x,s-t)Q_3(z(s,x))ds+\int_0^t\int_{-\infty}^{s}K_2(s,\tau)Q_4(z(\tau,x))d\tau ds\Big]\Big\}\\ +\Big[\int_{-\infty}^{t}a_3(t,x,s-t)Q_5(z(s,x))ds+\int_0^t\int_{-\infty}^{s}K_3(s,\tau)Q_6(z(\tau,x))d\tau ds\Big]\\ +\mu_1(x,z(t,x))+\mu_2(t,x,z(t+\theta,x)),x\in(0,\pi),t>0 \qquad (6.1)\end{aligned}$$

$$u(t,x)=\varphi(t,x),\quad t\in(-\infty,0],\quad x\in[0,\pi], \qquad (6.2)$$

$$u_i(t,x)=\int_{-\infty}^{t}\eta(t_i-s)u(s,x)ds,\quad (t,x)\in(t_i,s_i],\times[0,\pi],i=1,2,...,N, \qquad (6.3)$$

where $0<t_1<t_2<\cdots<t_n<b$ are prefixed real numbers and $\varphi\in\mathscr{B}_h$ and

$$\mu_1(x,z(t,x))=\int_0^x\mathscr{V}(x,y)z(y,\tilde{z}(t)(b_1|z(t,y)|))dy$$

for all $(t,x)\in[0,\infty)\times[0,\pi]$.

The function μ_2 is measurable in x, locally Hölder continuous in t and $\theta\in[-b,0]$ locally lipschitz continuous in z and uniformly in x.
ϕ is Lipschitz continuous on $[-b,0]$ with Lipschitz constant $\sum_{k=1_\phi}^{m}>0$ and it follows the conditions $\phi(0,0)=0$ and $\phi(0,1)=0$.
In $\mu_1,\tilde{z}$ is locally Hölder continuous in t and it satisfies the condition $\tilde{z}(0)=0$. Here $\varphi(\cdot,\cdot)\in C'([0,\pi]\times[0,\pi],\mathbb{R})$
We defined μ_2 as

$$\mu_2[t,x,z(t+\theta,x)]=\int_{-\infty}^{0}\int_0^{\pi}\mathcal{V}_0(t)P(s,y,x)z(t+s,y)dyds$$

In such a way that the function A which is measurable and

$$\sup_{r\in[-r,\infty)}\int_0^{\pi}\int_0^{\pi}A^2(t,y,w)dydw<\infty$$

Let $X=L^2([0,\pi])$. We characterize $\mathcal{B}_h(T)z=-z''-b(t)z$ and we defined the domain

$$D(\mathcal{B}_h)=\{z(\cdot)\in X: z,z'\text{are absolutely continuous}, z''\in X, z(0)=z(\pi)=0\}$$

Then the family $\mathcal{B}_h(t) : t > 0$ satisfies the suppositions $(P1) - (P4)$. Therefore $\mathcal{B}_h(t)$ develops an evolution operator $S(t, s)$ defined by

$$S(t,s) = T(t-s)exp\bigg(\int_s^t b(\tau)d\tau\bigg)$$

$-\mathcal{B}_h$ will generate a compact semigroup $T(t)$ such that

$$-\mathcal{B}_h(t)z = \sum_{n=1}^{\infty}[-n^2 + b(t)] < y, u_n > u_n, \text{ for } n = 1, 2, ..., \forall y \in D(\mathcal{B}_h),$$

According to eigen value $\lambda_n = n^2, n \in N$

$$u_n(x) = \sqrt{\frac{2}{\pi}}\text{sin (nx)} \text{ for } x \in [0, \pi].$$

We assigned $\mathcal{B}_h^{\alpha}(t_0)$ where $t_0 \in [0, T]$ by

$$\mathcal{B}_h^{\alpha}(t_0)y = \sum_{n=1}^{\infty}(n^{\alpha} - b(t_0))^{\alpha} < y, u_n > u_n$$

so

$$\mathcal{B}_h^{1/2}(t_0)y = \sum_{n=1}^{\infty}\sqrt{n^2 - b(t_0)} < y, u_n > u_n$$

To reformulate our system, we define the function

$$f = \mu_1(x, z(t, x)) + \mu_2(t, x, z(t + \theta, x))$$

and

$$\frac{d}{dt}g = \int_{-\infty}^{t} a_1(t, x, s-t)Q_1(z(s, x))ds + \int_0^t \int_{-\infty}^{s} K_1(s, \tau)Q_2(z(\tau, x))d\tau ds$$

$$G = \int_{-\infty}^{t} a_3(t, x, s-t)Q_5(z(s, x))ds + \int_0^t \int_{-\infty}^{s} K_3(s, \tau)Q_6(z(\tau, x))d\tau ds$$

Obviously $\mathscr{F}, \mathscr{G}, \overline{\mathscr{G}}$ are satisfied the Hypotheses from $(H1) - (H9)$ and $a(z(x,t),t) = \tilde{z}(t)b_1|z(x,t)|$ fulfills $(H2)$, I_i satisfies $(H6)$ and $(H7)$. Apply theorem 3.1, therefore the system 4.1 to 4.4 has a mild solution hence the proof.

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