Uncommon Systems of Equations

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Abstract. A system of linear equations L over $\mathbb{F}_q$ is *common* if the number of monochromatic solutions to L in any two-colouring of $\mathbb{F}_q^n$ is asymptotically at least the number of monochromatic solutions in a random two-colouring of $\mathbb{F}_q^n$. The line of research on common systems of linear equations was recently initiated by Saad and Wolf. They were motivated by existing results for specific systems (such as Schur triples and arithmetic progressions), as well as extensive research on common and Sidorenko graphs. Building on earlier work, Fox, Pham and Zhao characterised common linear equations. For systems of two or more equations, only sporadic results were known.
We prove that any system containing an arithmetic progression of length four is uncommon, confirming a conjecture of Saad and Wolf. This follows from a stronger result which allows us to deduce the uncommonness of a general system from considering certain one- or two-equation subsystems.

Keywords: Ramsey theory, linear systems, Fourier analysis

1 Introduction

A classical theorem of Goodman states that over all 2-edge-colourings of the complete graph K_n, the number of monochromatic triangles is asymptotically minimised by a random 2-colouring. Erdős conjectured that in Goodman's result, the triangle can be replaced by any fixed clique K_s, and Burr and Rosta extended the conjecture to any fixed graph. Erdős' conjecture was disproved by Thomason, motivating numerous results on *common and Sidorenko graphs*, including the famous Sidorenko conjecture. In the arithmetic setting, Graham, Rödl and Ruciński asked about the minimal number of Schur triples (triples satisfying $x + y - z = 0$) in 2-colourings of $[n] = \{1, 2, \ldots, n\}$. Questions of this type for linear systems of equations were studied more systematically in [1, 7, 4], and we continue this line of research.

Following Fox, Pham and Zhao [4], we work in the finite field model – we fix a finite field $\mathbb{F}_q$, where q is a prime power, and consider a linear homogeneous

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system L on k variables with coefficients in $\mathbb{F}_q$. We say that the system L is *common* if the number of monochromatic solutions in any two-colouring of $\mathbb{F}_q^n$ is asymptotically at least the number of monochromatic solutions in a random two-colouring of $\mathbb{F}_q^n$. Formal definitions will be given later. Let us briefly discuss systems consisting of a single equation $a_1x_1 + \cdots + a_kx_k = 0$ with coefficients $a_i \in \mathbb{F}_q \setminus \{0\}$, which are now completely characterised. Cameron, Cilleruelo and Serra [1] showed that in fact, any such linear equation with an odd number of variables k is common. For even k, Saad and Wolf [7] proved that the equation is common whenever $a_1, \ldots, a_k$ can be partitioned into pairs, each pair summing to zero. They conjectured that when k is even, this sufficient condition is also necessary, which was confirmed by Fox, Pham and Zhao [4].

Much less is known when L consists of more than one equation. Saad and Wolf [7] showed that arithmetic progressions of length four (4-APs) over $\mathbb{F}_5$ are uncommon, and conjectured that any system *containing* a 4-AP is uncommon. Their conjecture can be seen as an analogue of the famous result of Jagger, Št'ovíček and Thomason [6], showing that any graph containing a K_4 is *uncommon*. Fox, Pham and Zhao [4] asked for a characterisation of common systems of equations, hoping that it might lead to a better understanding of the analogous properties for graphs and hypergraphs, but noted that they do not have a guess for such a characterisation.

Confirming the conjecture of Saad and Wolf, we show that any system L containing a 4-AP is uncommon. This result follows from a more general theorem which provides a sufficient condition for a system to be uncommon, based on certain one- or two-equation *subsystems* of L. Using this theorem, we display two large classes of uncommon systems. The reduction to one- or two-equation systems opens up avenues for using discrete Fourier analysis in studying systems with two or more equations.

We also give examples of common systems based on intricate relations between the condensed equations, indicating that a characterisation of common systems might be rather elusive.

2 Results

Before stating our results, let us introduce some notation. In a slight abuse of notation, we identify a system L with an $m \times k$ matrix L, so that the solution set of L in $\mathbb{F}_q^n$ is

$$\mathrm{sol}(L) = \{\mathbf{x} = (x_1, \ldots, x_k) \in (\mathbb{F}_q^n)^k : L\mathbf{x}^T = 0\}.$$

We state the definitions and results in terms of functions $f : \mathbb{F}_q^n \to \mathbb{R}$, rather than subsets of $\mathbb{F}_q^n$. This is standard in arithmetic combinatorics, since a function can be used to sample a *random* subset of $\mathbb{F}_q^n$, and thus the commonness property for sets is equivalent to its functional version. This correspondence between functions and sets is explained in more detail in [4]. The *density* of solutions to a system L in f is

$$\Lambda_L(f) = \frac{1}{|\mathrm{sol}(L)|} \sum_{\mathbf{x} \in \mathrm{sol}(L)} f(x_1)f(x_2)\dots f(x_k).$$

We refer to a system L with k variables and m equations as an $m \times k$ system or a k-variable system, and we have $m \leq k$ throughout. An $m \times k$ system is non-degenerate if its rank is m and there are no variables x_i and x_j such that the equation $x_i = x_j$ can be derived from the system. A non-degenerate $m \times k$ system is *common* if for every $f : \mathbb{F}_q^n \to [0, 1]$

$$\Lambda_L(f) + \Lambda_L(1 - f) \geq 2^{1-k}.$$

Note that the right-hand side is the expected density of monochromatic solutions in a random two-colouring of $\mathbb{F}_q^n$, and if f is the indicator function of a set A, then $\Lambda_L(f)$ is the density of solutions in A. Hence this definition corresponds to the intuitive definition given above. Any degenerate system can be easily reduced to the corresponding non-degenerate system, so we restrict our attention to non-degenerate systems throughout the note.

Consider a k-variable system L and a 4-variable system M (such as a 4-AP). We say that L *contains* M if there are coordinates $a, b, c, d \in [k]$, such that whenever $(x_1, \dots, x_k) \in (\mathbb{F}_q^n)^k$ is a solution to L, (x_a, x_b, x_c, x_d) is a solution to M. This is equivalent to saying that the equations for M (with relabelled variables) can be derived from L using elementary row operations. We can now state our first result, which confirms a conjecture of Saad and Wolf [7], when the system M is taken to be a 4-AP.

Theorem 1. *Let M be a non-degenerate 2×4 system. Any non-degenerate system containing M is uncommon.*

Even the fact that a four-variable system M itself is uncommon is a new result, and finding a function ψ which certifies that (for any M) is not straightforward. Indeed, previously it was only known that 4-APs are uncommon over $\mathbb{F}_5$ and $\mathbb{Z}_N$ [7, 5], and the functions used there rely on the geometric structure of 4-APs. For a system L containing a 4-variable system M, we start with the above-mentioned function ψ, and turn it into a 'uniform' function using a trick due to Gowers [5], which in some sense isolates the contribution of the system M.

For our second result, we will introduce the notion of condensed equations of a system L, which turn out to be the crucial equations 'forcing' the uncommonness of L. A specific example can be found in Section 2.1. In reducing the properties of L to its condensed equations, we build on the key idea from [4], where a random function f is specified by sampling its Fourier coefficients.

Let L be an $m \times k$ matrix, which corresponds to an $m \times k$ system of m equations on k variables. We call a set $B \subseteq [k]$ *generic* if the matrix obtained from L by removing the columns corresponding to B has rank m. We define $s(L)$ to be the minimal order of a non-generic set. For example, when L is a 4-AP, we have $s(L) = 3$ as all column sets of order two are generic. Our next theorem deals with the systems L with $s(L)$ even, for which we define a collection of *critical* sets

$$\mathcal{C}(L) = \{B \subseteq [k] : |B| = s(L) \text{ and } B \text{ is not generic}\}.$$

Note that for $B \in \mathcal{C}(L)$, the rank of the matrix obtained from L after removing the columns corresponding to B is $m-1$. Hence there is a unique equation L_B (up to rescaling) derived from L by eliminating the variables x_i for $i \notin B$. This equation is called the *condensed equation* for B, denoted L_B. The following theorem describes a rather general class of uncommon systems L with even $s(L)$.

Theorem 2. *Let L be a system with $s(L)$ even. Suppose that for every set $B \in \mathcal{C}(L)$, the condensed equation L_B is uncommon. Then the system L is uncommon.*

Recall that a single equation L_B of even length is only common if its coefficients can be partitioned into pairs, each summing to zero. Thus in some sense, a 'typical' equation is uncommon, so we may say that a 'typical' system with $s(L)$ even is uncommon. The hypothesis that $s(L)$ is even is more than an artefact of our proofs, and is implicitly present in the results of [1, 4, 5].

2.1 A general theorem and an example

We will now describe our main theorem whose consequences are Theorem 1 and Theorem 2. For this purpose, we need to generalise our notion of condensed systems. Recall that $s(L)$ is the minimal order of a non-generic set B. We define a collection of sets

$$\mathcal{C}(L) = \begin{cases} \{B \subseteq [k] : |B| = s(L) \text{ and } B \text{ is not generic}\}, & \text{if } s(L) \text{ is even,} \\ \{B \subseteq [k] : |B| = s(L)+1 \text{ and } B \text{ is not generic}\}, & \text{if } s(L) \text{ is odd} \end{cases}.$$

Each set $B \in \mathcal{C}(L)$ corresponds to a condensed system L_B consisting of one or two equations. We do not define L_B here, but it has the key property that any solution $(x_i : i \in B)$ to L_B extends to a solution to L.

Recall the definition of $\Lambda_L(f)$. Our main theorem reduces the uncommonness of an $m \times k$ system L to the 'cumulative' uncommonness of its condensed systems.

Theorem 3. *Let L be a non-degenerate $m \times k$ system over $\mathbb{F}_q$. L is uncommon whenever there is a positive integer n and a function $f : \mathbb{F}_q^n \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ with $\mathbb{E}f = 0$ and*

$$\sum_{B \in \mathcal{C}(L)} \Lambda_{L_B}(f) < 0.$$

We finish with examples of common systems which will hopefully motivate further research and unveil some subtle phenomena. For instance, unlike in the single-equation case [4], the multiplicative structure of the field plays an important role in commonness.

Example 1. We consider a class $\mathcal{L}(q)$ consisting of 2×5 systems L over $\mathbb{F}_q$ with $s(L) = 4$. (Note that $s(L) = 4$ is equivalent to the property that all 2×2 determinants of L are non-zero). In this case, we can also deduce the commonness of L from considering its condensed equations. Systems in $\mathcal{L}(q)$ have five critical sets $\mathcal{C}(L) = \{B \subset [5] : |B| = 4\}$ and five corresponding condensed equations.

1. Let M be the system whose matrix is
$$\begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & -1 & 0 & -2 \end{pmatrix}.$$
The remaining condensed equations are
$$\begin{pmatrix} 0 & -1 & -2 & 1 & 2 \\ 2 & -3 & 0 & -1 & 2 \\ -1 & 0 & -3 & 2 & 2. \end{pmatrix}.$$
If $q \in \{5, 7\}$ the system is common as 3 can be written as -2 or -2^2 respectively, so the coefficients 'align' in a peculiar way for an application of Cauchy's inequality. For $q > 7$, the system is uncommon.
2. The system L generated by the equations $2x_1 + x_2 + 3x_4 - 6x_5 = 0$ and $x_1 + 2x_2 + 3x_3 - 6x_5 = 0$ is common over all fields $\mathbb{F}_q$ with $q \geq 5$. We suspect that there are no 'similar' systems L
3. If all five condensed equations of $L \in \mathcal{L}(q)$ are uncommon, the system is uncommon by Theorem 2. There is also an abundance of uncommon systems with one common condensed equation. One example is $x_1 + 3x_2 - x_3 - 3x_4 = 0$, $x_1 - 2x_2 - 3x_3 + 4x_5 = 0$.

3 Remarks and open problems

There are numerous avenues for further exploration. We select several of the problems which we find most interesting, and state only the simplest open case.

Systems with many uncommon condensed equations. Theorem 2 states that if $s(L)$ is even and all the condensed equations are uncommon, then L is uncommon. Our computational tests confirm the intuition that the conclusion holds even if the 'majority' of the condensed equations are uncommon. In the following conjecture, we propose such a class of two-equation systems.

Conjecture 1. For odd $k \geq 20$, any $2 \times k$ system L with $s(L) = k - 1$ is uncommon.

Partition regularity for linear systems For simplicity, we discuss systems with integer coefficients. A system L is *2-partition-regular* if any 2-colouring of $\mathbb{Z}$ contains a monochromatic solution to L. The famous theorem of Rado characterises partition-regular systems for many colours, but 2-partition regularity seems to be much less understood. Two classes of known 2-partition regular systems are (i) single equations with at least three variables and (ii) translation-invariant systems [2].

Question 1. Is there a system L with $s(L) \geq 3$ which is not 2-partition-regular?

Commonness and translation-invariance Translation-invariance is a sufficient condition for 2-partition regularity, but certainly not necessary (e.g. all one-equation systems of odd length are common). Still, it seems difficult to construct a larger common systems which is not translation-invariant.

Question 2. Is there a system with at least least two equations which is common, but not translation-invariant?

How uncommon can an equation be? Even if a system is uncommon, it is still natural to enquire about the minimum density of monochromatic solutions. For single equations, this minimum density can be expressed as an apparently simple optimisation problem in terms of Fourier coefficients (see, e.g., equation (3) in [4]). This leads to the following question.

Question 3. Let $\mathcal{L}_k$ be the collection of equations of length k over $\mathbb{F}_q$. What is the asymptotically minimal density of monochromatic solutions to L, over all colourings of $\mathbb{F}_q^n$ and all $L \in \mathcal{L}_{2k}$?

Note that for odd k and all $L \in \mathcal{L}_k$, the density of monochromatic solutions depends only on the size of the colour classes. The analogous question for graphs has also been investigated [3].

Finally, many of the previous results have been generalised to the setting of arbitrary abelian groups [1, 8]. We have not attempted to extend our results in this direction.

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