Four-Valued Expansions of Belnap'S Logic: Inheriting Basic Peculiarities

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FOUR-VALUED EXPANSIONS OF BELNAP'S LOGIC: INHERITING BASIC PECULIARITIES

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ABSTRACT. The main results of the paper are that:

(1) any four-valued expansion L_4 of Belnap's four-valued logic B_4 (cf. [3]):
   (a) is defined by a unique expansion $\mathcal{M}_4$ of the four-valued matrix $\mathcal{DM}_4$ over the De Morgan truth lattice diamond $\{\mathsf{f},\mathsf{n},\mathsf{b},\mathsf{t}\}$ defining B_4 as such;
   (b) holds Relevance Principle iff it is has neither a theorem nor an inconsistent formula;
   (c) has no proper extension holding Relevance Principle;
   (d) is minimally four-valued;
   (e) is defined by no truth/false-singular matrix;
   (f) has an extension defined by an expansion of a consistent submatrix $\mathcal{B}$ of $\mathcal{DM}_4$ iff the underlying set of $\mathcal{B}$ forms a subalgebra of the underlying algebra $\mathfrak{A}_4$ of $\mathcal{M}_4$;
   (g) is subclassical iff $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}_4$, in which case the logic of $\mathcal{M}_4{\upharpoonright}\{\mathsf{f},\mathsf{t}\}$ defines a unique classical extension of L_4 being also an extension of any inferentially consistent extension of L_4;
   (h) is [inferentially]maximal iff $\mathcal{M}_4$ has no proper consistent submatrix[other than that with carrier $\{\mathsf{n}\}$];
   (i) is maximally paraconsistent iff $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}_4$ iff the proper axiomatic extension L_4^{EM} of L_4 relatively axiomatized by the *Excluded Middle law* axiom is either classical, if L_4 is subclassical, or inconsistent, otherwise, iff L_4^{EM} is not (maximally)paraconsistent iff L_4^{EM} is not an expansion of the logic of paradox $LP = B_4^{\mathrm{EM}}$ and, otherwise, providing L_4 is subclassical and every primary operation of $\mathfrak{A}_4$ is either regular or both b-idempotent and no more than binary, L_4^{EM} has exactly two proper consistent extensions forming a chain, the greatest one being classical and relatively axiomatized by the *Modus ponens* rule for material implication, the least one being relatively axiomatized by the *Ex Contradictione Quodlibet* rule, both being non-axiomatic, whenever $\mathfrak{A}_4$ is regular;
   (j) has no theorem/inconsistent formula iff $\{\mathsf{n}/\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}_4$;
   (k) [providing L_4 has a/no theorem,]L_4 has the distributive lattice of its disjunctive [arbitrary/merely non-pseudo-axiomatic]extensions being dual isomorphic to the one of all lower cones of the set of all [truth-non-empty]consistent submatrices of $\mathcal{M}_4$ (in particular, to be found effectively, whenever the expanded signature is finite) and is a sublattice of the nine[six]-element non-chain distributive lattice of all disjunctive [non-pseudo-axiomatic]extensions of B_4;
   (l) has its proper disjunctive extension L_4^{R} relatively axiomatized by the *Resolution* rule that:
      (i) is paracomplete iff the carrier of the subalgebra of $\mathfrak{A}_4$ generated by $\{\mathsf{n}\}$ does not contain b;
      (ii) is not inferentially paracomplete iff it is inferentially either classical, if L_4 is subclassical, or inconsistent, otherwise, iff $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}_4$ iff L_4^{R} is not an expansion of Kleene's three-valued logic $K_3 = B_4^{\mathrm{R}}$;
   (m) has the entailment relation equal to the set of all inequalities identically true in $\mathfrak{A}_4$ iff L_4 is self-extensional iff it has the Property of Weak Contraposition iff the specular permutation on $\{\mathsf{f},\mathsf{n},\mathsf{b},\mathsf{t}\}$ retaining both f and t but permuting n and b is an endomorphism of $\mathfrak{A}_4$ iff the extension of L_4 relatively axiomatized by the *Modus Ponens[Ex Contradictione Quodlibet]* rule is defined by[the direct product of $\mathcal{M}_4$ and] $\langle\mathfrak{A}_4,\{\mathsf{t}\}\rangle$, in which case:
      (i) L_4 is subclassical;
      (ii) there is either no, if L_4 is maximally paraconsistent, or exactly one, otherwise, non-pseudo-axiomatic consistent non-classical proper self-extensional extension of L_4, any self-extensional extension of L_4 being disjunctive;
      (iii) $\{\mathsf{n}[,\mathsf{f},\mathsf{t}]\}$ forms a subalgebra of $\mathfrak{A}_4$ iff $\{\mathsf{b}[,\mathsf{f},\mathsf{t}]\}$ does so, in which case:
         (A) L_4 holds Relevance Principle iff it has no theorem/inconsistent formula;
         (B) L_4^{EM} is (maximally)paraconsistent iff L_4^{R} is inferentially paracomplete, in which case, providing $\mathfrak{A}_4$ is regular, L_4^{R} is maximally inferentially paracomplete, while any extension of L_4 is both paraconsistent and inferentially paracomplete iff it is a sublogic of $L_4^{\mathrm{EM}} \cap L_4^{\mathrm{R}}$.;
         (C) [providing L_4 has a/no theorem,] disjunctive [arbitrary/merely non-pseudo-axiomatic]extensions of L_4 form the nine[six]-element non-chain distributive lattice isomorphic to that of B_4;
         (D) providing $\mathfrak{A}_4$ is regular[and L_4 has a/no theorem], [arbitrary/merely non-pseudo-axiomatic]extensions of $L_4^{\mathrm{EM}} \cap L_4^{\mathrm{R}}$ form the eleven[seven]-element non-chain distributive lattice, those of L_4^{R} being all disjunctive, proper ones being inferentially either classical or inconsistent, and so not inferentially paracomplete, in which case L_4^{R} is maximally (inferentially)paracomplete, as opposed to its implicative expansions;

(2) any three-valued (disjunctive/conjunctive)paraconsistent logic L_3 with subclassical negation:
   (a) is defined by a(unique disjunctive/conjunctive) *superclassical* matrix over $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, referred to as *characteristic* one of L_3;
   (b) is maximally paraconsistent iff either $\{\mathsf{b}\}$ does not form a subalgebra of the underlying algebra $\mathfrak{A}$ of any characteristic matrix of L_3 or there is a *ternary* b*-relative weak conjunction for* $\mathfrak{A}$, viz., a ternary formula φ such that $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f},\mathsf{t}) = \mathsf{f} = \varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{t},\mathsf{f})$, in which case a characteristic matrix of L_3 is unique(;
   (c) has no proper paraconsistent disjunctive/conjunctive extension/, in which case it is maximally paraconsistent);
   (d) is minimally three-valued;
   (e) is subclassical if(f) $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of the underlying algebra of its characteristic matrix, in which case (L_3 is maximally paraconsistent, while)the logic of the restriction of its characteristic matrix on $\{\mathsf{f},\mathsf{t}\}$ defines a(unique) classical extension of L_3(/, being also an extension of any consistent extension of L_3);

(3) for every $n > 2$, there is a minimally n-valued maximally paraconsistent subclassical[both conjunctive and disjunctive] logic.

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1. Introduction

Perhaps, the principal value of *universal* mathematical investigations consists in discovering uniform transparent points behind particular results originally proved *ad hoc* as well as in providing powerful generic tools enabling one "to kill as much as possible birds with as less as possible stones". This thesis is the main paradigm of the present study.

Belnap's "useful" four-valued logic (cf. [3]) arising as the logic of *first-degree entailment* in relevance logic R (*FDER*, for short) has been naturally expanded by additional connectives in [16]. The present paper pursues the study of such expansions with regard to certain *generic* aspects in addition to those of functional completeness and both sequential and equational axiomatizations comprehensively explored therein collectively with [20].

More precisely, we study how four-valued expansions of FDER (as well as their extensions) inherit certain *remarkable* features of FDER as such. This marks the *primary* framework of the paper. On the other hand, it is closely related to certain more (secondary) issues additionally studied here (especially because this study uses the generic tools initially elaborated for solving exactly the secondary tasks alone and only then applied to primary ones).

First of all, FDER holds Relevance Principle (viz., Variable Sharing Property) in the sense that it holds the entailment $\phi \to \psi$ only if ϕ and ψ have a common propositional variable. This clarifies the items (1b,1c,1j) of the Abstract.

Moreover, the four-valued matrix defining FDER has four proper consistent submatrices, each defining a consistent proper extension of FDER. This explains the item (1f) of the Abstract.

In particular, FDER is *subclassical* in the sense that the classical logic is an extension of it. When exploring this peculiarity within the framework of expansions of FDER, we inevitably deal with formally miscellaneous classical logics as those which are defined by *classical* matrices, that is, two-valued matrices with classical negation. In case such is conjunctive with respect to any (possibly, secondary) binary connective (in particular, is a model of an expansion of FDER), the logic defined by such a matrix is nothing but a definitional copy of the standard classical logic, because any two-valued operation is definable via the classical negation and conjunction. We equally follow this paradigm, when studying three-valued and n-valued paraconsistent logics. This clarifies the items (1g) and (2e) of the Abstract.

The four-valuedness typical of FDER and its expansions also implies their both *[inferential]paracompleteness* (viz., refuting the[inferential version of] *Excluded Middle law* axiom) and *paraconsistency* (viz., refuting the *Ex Contradictione Quodlibet* rule). It is this joint peculiarity of FDER that has predetermined its profound applications to Computer Science and Artificial Intelligence. This inevitably raises the issue of exploring how extensions of(four-valued expansions of) FDER retain such peculiarities (cf. the items (1i,1l) of the Abstract).

In this connection, the issue of strong[inferential] maximality typical of the classical logic in the sense of having no proper [inferentially]consistent extension becomes equally acute as for four-valued expansions of FDER. The thing is that [purely-]bilattice expansions of FDER with[out] truth and falsehood constant are [inferentially]maximal, as it ensues from the general characterization of the maximality (cf. the item (1h) of the Abstract). Taking [18] into account, particular cases of such maximality have actually been proved in [16] *ad hoc*.

And what is more, four-valued expansions of FDER normally (but not at all generally) have three-valued paraconsistent/paracomplete extensions, defined by three-valued submatrices of characteristic four-valued matrices (cf. the items (1f,1i/l) of the Abstract), shown here to be relatively axiomatized by the *Excluded Middle law* axiom/ the *Resolution* rule in that case. Then, their defining three-valued paraconsistent submatrices appear to be conjunctive and *superclassical* in the sense of the reference [Pyn 95b] of [14], according to which any logic defined by such a matrix is *maximally* paraconsistent in the sense of having no proper paraconsistent extension (cf. the items (2b,2c) of the Abstract and historically the paragraph after Theorem 2.1 of [14]).[1] Particular cases of such three-valued maximal paraconsistency have been proved *ad hoc* in [14], [19] as well as in [23] taking [18] into account. On the other hand, as it follows from our characterization of the maximal paraconsistency (cf. the item (1i) of the Abstract), any (including constant-free purely) bilattice expansion is maximally paraconsistent, though is not subclassical, in view the item (1g) of the Abstract, as opposed to the expansion by classical (viz., Boolean) negation.

In this way, we conclude that the maximal paraconsistency is not at all a prerogative of three-valued logics. As a matter of fact, we argue that, for every $n > 2$, there is a *minimally* n-valued (in the sense of not being defined by a matrix with less than n values; cf. the items (1d,2d) of the Abstract in this connection) maximally paraconsistent subclassical logic (cf. the item (3) of the Abstract). In this connection, it is remarkable that existence of non-minimally n-valued maximally paraconsistent subclassical logic has been actually due to [14], because the logic of paradox [11] is equally defined by an n-valued matrix. Among other things, such generic minimally n-valued example is defined by a false-singular matrix, as opposed to four-valued expansions of FDER (cf. the item (1e) of the Abstract).

Furthermore, FDER is disjunctive. This raises the problem of finding all disjunctive extensions of (four-valued expansions of)FDER (cf. the item (1k) of the Abstract). (Although, likewise, FDER is conjunctive, the conjunctivity is immediately inherited by extensions, so this point is just taken for granted.)

After all, a one more quite remarkable peculiarity of FDER is that its entailment relation is defined (semi)lattice-wise in the sense that FDER holds the entailment $\phi \to \psi$ iff the inequality $\phi \lessapprox \psi$ (viz., the equality $(\phi \wedge \psi) \approx \phi$) is identically true in the diamond De Morgan lattice, i.e, in the variety of De Morgan lattices. Within the framework of four-valued expansions of FDER, this property appears to be equivalent to the so-called *self-extensionality* (cf. Theorem 4.53(i)$\Leftrightarrow$(v)), profound study of which has been due to [15] that has provided a generic algebraic (more specifically, lattice-theoretic) approach to conjunctive non-pseudo-axiomatic self-extensional logics (cf. Section 4.1 therein) properly enhanced here by omitting the stipulation "non-pseudo-axiomatic". Recall that a propositional logic is said to be *self-extensional*, provided its interderivability relation is a congruence of the formula algebra, in which case any fragment of it is self-extensional as well (cf. [15]), while the converse is far from being generally valid. Any axiomatic extension of the intuitionistic logic as well as any inferentially consistent

[1]Though being prepared and announced by 1995, the fundamental material of the both references [Pyn 95a] and [Pyn 95b] of [14] has never been published for a quarter of century. This is why we take the opportunity to eventually present them here.

two-valued logic (including the classical one and its fragments) is self-extensional. This explains the meaning of the item (1m) of the Abstract.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set Theory, Lattice Theory, Universal Algebra, Model Theory and Mathematical Logic not specified here explicitly, to be found, e.g., in [2], [4], [6], [8] and [9]). Section 2 is a concise summary of basic issues underlying the paper, most of which have actually become a part of logical and algebraic folklore. Section 3 is devoted to certain key preliminary issues concerning false-singular matrices, disjunctivity, equality determinants and De Morgan lattices. In Section 4 we formulate and prove main results of the paper concerning solely four-valued expansions of FDER. Section 5 is entirely devoted to the issue of (especially, maximal) paraconsistency within both three-valued and generic n-valued framework. Then, in Section 6, we exemplify the previous three sections by applying them to three general classes of expansions, including those introduced in [16], with providing quick argumentations/refutations of their properties under consideration and finding all disjunctive extensions of (first of all, self-extensional non-maximally paraconsistent) expansions of FDER as well as all extensions of the unique proper non-classical self-extensional non-pseudo-axiomatic extension of any regular self-extensional non-maximally paraconsistent expansion of FDER (in particular, FDER itself), as well as to certain well-known three-valued paraconsistent logics. Finally, Section 7 is a brief summary of principal contributions and open problems of the paper as well as an outline of further related work.

2. Basic issues

Notations like img, dom, ker, hom, π_i and Con and related notions are supposed to be clear.

2.1. **Set-theoretical background.** We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞.

Likewise, functions are viewed as binary relations, the left/right components of their elements being treated as their arguments/values, respectively. Then, to retain both the conventional prefix writing of functions and the fact that $(f \circ g)(a) = f(g(a))$, we have just preferred to invert the conventional order of relation composition components. In particular, given two binary relations R and Q, we put $R[Q] \triangleq (R \circ Q \circ R^{-1})$.

In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S[of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. A subset $T \subseteq S$ is said to be *proper*, if $T \neq S$. Further, given any equivalence relation θ on S, as usual, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq [a]_\theta \triangleq \theta[\{a\}]$, for all $a \in S$, in which case $\ker \nu_\theta = \theta$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component (viz., the value under argument s), where $s \in S$, being written as either t_s or t^s. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise, that is, $R \triangleq \{\langle \bar{a}, \bar{b}\rangle \in (A^S \times B^S) \mid \forall s \in S : a_s \; R \; b_s\}$. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any $\bar{f} \in (\prod_{s\in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a)\rangle_{s\in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, set $\Delta_S \triangleq \{\langle a, a\rangle | a \in S\}$, relations of such a kind being referred to as *diagonal*, and $S^+ \triangleq \bigcup_{i\in(\omega\setminus 1)} S^i$. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = \operatorname{dom} \bar{a}$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}\restriction(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

Given any $f : S \to S$, by induction on any $n \in \omega$, define $f^n : S \to S$, by setting:

$$f^n(a) \triangleq \begin{cases} a & \text{if } n = 0, \\ f(f^{n-1}(a)) & \text{otherwise.} \end{cases}$$

for all $a \in S$. Finally, given any $R \subseteq S^2$, $\mathrm{Tr}(R) \triangleq \{\langle \pi_0(\pi_0(\bar{r})), \pi_1(\pi_{l-1}(\bar{r}))\rangle | \bar{r} \in R^l, l \in (\omega \setminus 1)\}$ is the least transitive binary relation on S including R, referred to as the *transitive closure of* R.

In general, we use the following standard notations going back to [3]:

$$\mathsf{t} \triangleq \langle 1, 1\rangle, \qquad \mathsf{f} \triangleq \langle 0, 0\rangle,$$
$$\mathsf{b} \triangleq \langle 1, 0\rangle, \qquad \mathsf{n} \triangleq \langle 0, 1\rangle.$$

In addition, the mapping $\mu : 2^2 \to 2^2, \langle a, b\rangle \mapsto \langle b, a\rangle$ is said to be *specular*, in which case $\mu^{-1} = \mu$, so μ is bijective, i.e., a permutation on 2^2.

Let A be a set. An *anti-chain of* any $S \subseteq \wp(A)$ is any $N \subseteq S$ such that $\max(N) = N$. Likewise, a *lower cone of* S is any $L \subseteq S$ such that, for each $X \in L$, $(\wp(X) \cap S) \subseteq L$. This is said to be *generated by* a $G \subseteq L$, whenever $L = (G)^{\triangledown}_S \triangleq (S \cap \bigcup\{\wp(X) \mid X \in G\})$ (the subscript S is normally omitted, whenever it is clear from the context). (Clearly, in case A is finite, the mappings $N \mapsto (N)^{\triangledown}_S$ and $L \mapsto \max(L)$ are inverse to one another bijections between the sets of all antichains and lower cones of S.) A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$. A subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. Further, any $X \in T \subseteq \wp(A)$ is said to be K*-meet-irreducible (in/of* T*)*, where $K \subseteq \infty$, provided it belongs to every $U \in \wp_K(T)$ such that $(A \cap \bigcap U) = X$ (in which case $X \neq A$, whenever $0 \in K$), the set of all them being denoted by $\mathrm{MI}^K(T)$.[2] A *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$, in which case the poset $\langle \mathcal{C}, \subseteq \cap\, \mathcal{C}^2\rangle$

[2] In general, any mention of K is normally omitted, whenever $K = \infty$. Likewise, "finitely-/pairwise-" means "ω-/$\{2\}$-", respectively.

to be identified with $\mathcal{C}$ alone is a complete lattice with meet $A \cap \bigcap$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure)basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. An *operator over* A is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent] {transitive} ⟨inductive⟩*, provided, for all $(B,)D \in \wp(A)$⟨resp., any upward-directed $U \subseteq \wp(A)$⟩, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)\langle O(\bigcup U) \subseteq \bigcup O[U]\rangle$. A *closure operator over* A is any monotonic idempotent transitive operator C over A, in which case $\operatorname{img} C$ is a closure system over A, determining C uniquely, because, for every closure basis $\mathcal{B}$ of $\operatorname{img} C$ (including $\operatorname{img} C$ itself) and each $X \subseteq A$, it holds that $C(X) = (A \cap \bigcap\{Y \in \mathcal{B} | X \subseteq Y\})$, called *dual to* C and vice versa. (Clearly, C is inductive iff $\operatorname{img} C$ is so.)

Remark 2.1. As a consequence of Zorn's Lemma, according to which any inductive non-empty set has a maximal element, given any inductive closure system $\mathcal{C}$, $\mathrm{MI}(\mathcal{C})$ is a closure basis of $\mathcal{C}$, and so is $\mathrm{MI}^K(\mathcal{C}) \supseteq \mathrm{MI}(\mathcal{C})$, where $K \subseteq \infty$. □

2.2. **Algebraic background.** Unless otherwise specified, abstract algebras are denoted by Fraktur letters (possibly, with indices/prefixes/suffixes), their carriers (viz., underlying sets) being denoted by corresponding Italic letters (with same indices/prefixes/suffixes, if any).

Let $\mathfrak{A}$ be an algebra. Then, $\mathrm{Con}(\mathfrak{A})$ is an inductive closure system over A^2, in which case $\mathfrak{A}$ is said to be *simple/congruence-distributive*, whenever the lattice $\mathrm{Con}(\mathfrak{A})$ is two-element/distributive. Next, $\mathfrak{A}$ is said to be *subdirectly irreducible*, provided $\Delta_A \in \mathrm{MI}(\mathrm{Con}(\mathfrak{A}))$, in which case $|A| > 1$. (Clearly, any simple algebra is subdirectly irreducible.)

A *(propositional) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with by default throughout the paper) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional) connectives.* Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$, put $V_\alpha \triangleq \{x_\beta | \beta \in \alpha\}$ and $(\forall_\alpha) \triangleq (\forall V_\alpha)$. Then, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^\alpha_\Sigma$ freely-generated by the set V_α, elements of which being viewed as *(propositional) variables of rank* α, referred to as the *formula* Σ*-algebra of rank* α, its endomorphisms/elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ (viz., Σ-terms of rank α) being called *(propositional)* Σ*-substitutions/-formulas of rank* α. A Σ*-equation/identity of rank* α is then any couple of the form $\phi \approx \psi$, where $\phi, \psi \in \mathrm{Fm}^\alpha_\Sigma$, to be identified with the ordered pair $\langle\phi, \psi\rangle$, the set of all them being denoted by $\mathrm{Eq}^\alpha_\Sigma$. (In general, the reservation "of rank α" is normally omitted, whenever $\alpha = \omega$.) Given any $[m,]n \in \omega$, by $\sigma_{[m:]+n}$ we denote the Σ-substitution extending $[x_i/x_{i+n}]_{i\in(\omega[\setminus m])}$.

The *variety axiomatized by* a given $\mathcal{I} \subseteq \mathrm{Eq}^\omega_\Sigma$ is the class of all Σ-algebras satisfying each identity in $\mathcal{I}$. A $\theta \in \mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$ is said to be *fully invariant*, provided $\sigma[\theta] \subseteq \theta$, for every Σ-substitution σ, in which case θ is the set of all Σ-identities satisfied in the variety axiomatized by θ. Conversely, the set θ_{V} of all Σ-identities satisfied in a variety V (clearly, axiomatized by θ_{V}) is a fully invariant congruence of $\mathfrak{Fm}^\omega_\Sigma$. In this way, the closure system of all fully invariant congruences of $\mathfrak{Fm}^\omega_\Sigma$ is dual isomorphic to the lattice of all varieties of Σ-algebras.

A class K of Σ-algebras is said to be *congruence-distributive*, whenever every member of it is so. In general, the class of all [non-one-element]subalgebras/homomorphic images/isomorphic copies of members of K is denoted by $(\mathbf{S}/\mathbf{H}/\mathbf{I})_{[>1]}\mathsf{K}$, respectively. Likewise, the class of all subdirectly irreducible members of K is denoted by $\mathrm{Si}(\mathsf{K})$. Finally, the variety *generated by* K (viz., the least one including K), being clearly axiomatized by the set of all Σ-identities true in K, is denoted by $\mathbf{V}(\mathsf{K})$. The variety $\mathbf{V}(\varnothing)$, constituted by all one-element Σ-algebras, is said to be *trivial.*

Let I be a set, $\overline{\mathfrak{A}}$ an I-tuple of Σ-algebras and $\mathfrak{B}$ a subalgebra of $\mathfrak{C} \triangleq \prod_{i\in I} \mathfrak{A}_i$. Given any *[prime]filter* $\mathcal{F}$ *on* I (viz., a non-empty[proper prime] filter of the lattice $\langle\wp(I), \cap, \cup\rangle$), we then have $\theta^B_{\mathcal{F}} \triangleq \{\langle\bar{a}, \bar{b}\rangle \in B^2 \mid \{i \in I \mid a_i = b_i\} \in \mathcal{F}\} \in \mathrm{Con}(\mathfrak{B})$, congruences of such a kind being referred to as *[prime]filtral*[, in which case:

$$(\mathfrak{C}/\theta^C_{\mathcal{F}}) \in \mathbf{I}(\operatorname{img} \overline{\mathfrak{A}}), \tag{2.1}$$

whenever both $\operatorname{img} \overline{\mathfrak{A}}$ and all members of it are finite].

Recall the following useful well-known facts:

Lemma 2.2. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* Σ*-algebras and* $h \in \hom(\mathfrak{A}, \mathfrak{B})$*.(Suppose* $(\operatorname{img} h) = B$*.) Then, for every* $\vartheta \in \mathrm{Con}(\mathfrak{B})$, $h^{-1}[\vartheta] \in \{\theta \in \mathrm{Con}(\mathfrak{A}) \mid (\ker h) \subseteq \theta\}$*(, whereas* $h[h^{-1}[\vartheta]] = \vartheta$*, while, conversely, for every* $\theta \in \mathrm{Con}(\mathfrak{A})$ *such that* $(\ker h) \subseteq \theta$, $h[\theta] \in \mathrm{Con}(\mathfrak{B})$*, whereas* $h^{-1}[h[\theta]] = \theta$*).*

Remark 2.3 (cf., e.g., Theorem 1.3 of [10]). In view of Remark 2.1, given any member $\mathfrak{A}$ of a variety V, $\Theta \triangleq \mathrm{MI}(\mathrm{Con}(\mathfrak{A}))$ is a basis of the inductive closure system $\mathrm{Con}(\mathfrak{A})$ over A^2, each $(\mathfrak{A}/\theta) \in \mathsf{V}$, where $\theta \in \Theta$, being subdirectly irreducible, in view of Lemma 2.2, in which case $\Delta_A = (A^2 \cap \bigcap \Theta)$, so $e \triangleq (\prod_{\theta\in\Theta} \nu_\theta) : A \to (\prod_{\theta\in\Theta}(A/\theta))$ is an embedding of $\mathfrak{A}$ into $\prod_{\theta\in\Theta}(\mathfrak{A}/\theta)$, and so is an isomorphism from $\mathfrak{A}$ onto the subdirect product $(\prod_{\theta\in\Theta}(\mathfrak{A}/\theta))\restriction(\operatorname{img} e)$ of the tuple $\langle\mathfrak{A}/\theta\rangle_{\theta\in\Theta}$ constituted by subdirectly irreducible members of V. In particular, $\mathsf{V} = \mathbf{V}(\mathrm{Si}(\mathsf{V}))$. □

Lemma 2.4 (cf., e.g., the proof of Theorem 2.6 of [10])**.** *Let* I *be a set,* $\overline{\mathfrak{A}}$ *an* I*-tuple of* Σ*-algebras,* $\mathfrak{B}$ *a congruence-distributive subalgebra of* $\prod_{i\in I} \mathfrak{A}_i$ *and* $\theta \in \mathrm{MI}(\mathrm{Con}(\mathfrak{B}))$*. Then, there is some prime filter* $\mathcal{F}$ *on* I *such that* $\theta^B_{\mathcal{F}} \subseteq \theta$*.*

Then, combining (2.1), Lemmas 2.2, 2.4 and the Algebra Homomorphism Theorem, we get:

Corollary 2.5 (cf., e.g., Theorem 2.6 of [10])**.** *Let* K *be a finite class of finite* Σ*-algebras. Suppose* $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$ *is congruence-distributive. Then,* $\mathrm{Si}(\mathsf{V}) \subseteq \mathbf{H}_{>1}\mathbf{S}_{>1}\mathsf{K}$*. In particular,* $\mathrm{Si}(\mathsf{V}) = \mathbf{IS}_{>1}\mathsf{K}$*, whenever every member of* $\mathbf{S}_{>1}\mathsf{K}$ *is simple, in which case every member of* $\mathrm{Si}(\mathsf{V})$ *is simple.*

And what is more, we also have:

Corollary 2.6 (Congruence filtrality)**.** *Let* K *be a finite class of finite* Σ*-algebras,* I *a set,* $\overline{\mathfrak{A}} \in \mathsf{K}^I$ *and* $\mathfrak{B}$ *a congruence-distributive subalgebra of* $\mathfrak{C} \triangleq \prod_{i\in I} \mathfrak{A}_i$*. Suppose every member of* $\mathbf{S}_{>1}\mathsf{K}$ *is simple. Then, each element of* $\mathrm{Con}(\mathfrak{B})$ *is filtral.*

Proof. Consider any $\theta \in \mathrm{MI}(\mathrm{Con}(\mathfrak{B}))$, in which case $\theta \neq B^2$. Then, by Lemma 2.4, there is some prime filter $\mathcal{F}$ on I such that $\mathrm{Con}(\mathfrak{B}) \ni \vartheta \triangleq \theta^B_{\mathcal{F}} \subseteq \theta$, in which case we have $\eta \triangleq \theta^C_{\mathcal{F}} \in \mathrm{Con}(\mathfrak{C})$, while $B^2 \neq \vartheta = (B^2 \cap \eta) = \ker(\nu_\eta \restriction \Delta_B)$, and so, by the Algebra Homomorphism Theorem and (2.1), we get $(\mathfrak{B}/\vartheta) \in \mathbf{IS}_{>1}(\mathfrak{C}/\eta) \subseteq \mathbf{IS}_{>1}\mathbf{I}\mathsf{K} \subseteq \mathbf{IS}_{>1}\mathsf{K}$. Hence, by Lemma 2.2, we eventually get $\theta = \vartheta$. Thus, each element of $\mathrm{MI}(\mathrm{Con}(\mathfrak{B}))$ is filtral. In this way, Remark 2.1 and the fact that the set of all filters on I is a closure system over $\wp(I)$, while the mapping $\mathcal{F} \mapsto \theta^{B}_{\mathcal{F}}$ preserves intersections, complete the argument. □

By Corollary 2.6, we then immediately get:

Corollary 2.7 (Congruence inheritance). *Let $\Sigma' \subseteq \Sigma$, K a finite class of finite Σ-algebras, I a set, $\overline{\mathfrak{A}} \in \mathsf{K}^I$ and $\mathfrak{B}$ a subalgebra of $\prod_{i\in I} \mathfrak{A}_i$. Suppose every member of $\mathbf{S}_{>1}(\mathsf{K}{\restriction}\Sigma')$ is simple and $\mathfrak{B}{\restriction}\Sigma'$ is congruence-distributive. Then,* $\mathrm{Con}(\mathfrak{B}) = \mathrm{Con}(\mathfrak{B}{\restriction}\Sigma')$.

2.3. **Propositional logics and matrices.** A Σ-*rule* is any couple $\langle \Gamma, \varphi\rangle$, where $(\Gamma \cup \{\varphi\}) \in \wp_\omega(\mathrm{Fm}^\omega_\Sigma)$, normally written in the standard sequent form $\Gamma \vdash \varphi$, φ/any element of Γ being referred to as the/a *conclusion/premise of* it. A *(substitutional)Σ-instance* of it is then any Σ-rule of the form $\sigma(\Gamma \vdash \varphi) \triangleq (\sigma[\Gamma] \vdash \sigma(\varphi))$, where σ is a Σ-substitution. As usual, Σ-rules without premises are called Σ-*axioms* and are identified with their conclusions. A[n] *[axiomatic]Σ-calculus* is any set $\mathcal{C}$ of Σ-rules[-axioms], the set of all Σ-instances of its elements being denoted by $\mathrm{SI}_\Sigma(\mathcal{C})$. Then, $\Gamma \vdash \varphi$ is said to be *derivable in* $\mathcal{C}$, if there is a $\mathcal{C}$-*derivation of* it, i.e., a proof of φ (in the conventional proof-theoretical sense) by means of axioms and rules in $\Gamma \cup \mathrm{SI}_\Sigma(\mathcal{C})$.

A *(propositional) Σ-logic* is any closure operator C over $\mathrm{Fm}^\omega_\Sigma$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$ and all $\sigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)$, or, equivalently, $\mathrm{img}\, C$ is closed under inverse Σ-substitutions (we sometimes write $X \vdash_C Y$ for $C(X) \supseteq Y$). A(n) *(in)consistent set of* C is any $X \subseteq \mathrm{Fm}^\omega_\Sigma$ such that $C(X) \neq (=) \mathrm{Fm}^\omega_\Sigma$. Then, C is said to be *[inferentially](in)consistent*, provided $\varnothing[\cup\{x_0\}]$ is a(n in)consistent set of C or, equivalently, in view of the structurality of C, $x_1 \notin (\in) C(\varnothing[\cup\{x_0\}])$. A Σ-rule $\Gamma \vdash \varphi$ is said to be *satisfied in* C, provided $\varphi \in C(\Gamma)$. A *[proper]extension of* C is any Σ-logic $C' \supseteq C$[distinct from C], in which case C is said to be a *[proper]sublogic of* C'. Then, an extension C' of C is said to be *axiomatized by* a Σ-calculus $\mathcal{C}$ *relatively to* C, provided it is the least extension of C satisfying each rule of $\mathcal{C}$. The extension $\mathrm{Cn}_\mathcal{C}$ of the diagonal Σ-logic relatively axiomatized by $\mathcal{C}$ is said to be *axiomatized by* $\mathcal{C}$ and is referred to as the *consequence of* $\mathcal{C}$, in which case it is inductive and satisfies any Σ-rule iff this is derivable in $\mathcal{C}$. (Conversely, any inductive Σ-logic is axiomatized by the set of all Σ-rules satisfied in it.) An extension C' of C is said to be *axiomatic*, whenever it is relatively axiomatized by an axiomatic Σ-calculus $\mathcal{A}$, in which case, for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$, it holds that $C'(X) = C(X \cup \mathrm{SI}_\Sigma(\mathcal{A}))$. Next, C is said to be *[inferentially]maximal*, whenever it is [inferentially]consistent and has no proper [inferentially]consistent extension. Further, C is said to be *[weakly]$\diamond$-conjunctive* (cf. [15]), where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $C(\phi \diamond \psi)[\supseteq] = C(\{\phi, \psi\})$, for all $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$. Next, C is said to *have the Property of Weak Contraposition with respect to* a (possibly, secondary) unary connective $\wr$ of Σ (cf. [13]), provided $(\psi \in C(\phi)) \Rightarrow (\wr\phi \in C(\wr\psi))$, for all $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$. Likewise, C is said to be *[maximally]$\wr$-paraconsistent*, provided $x_1 \notin C(\{x_0, \wr x_0\})$[and C has no proper $\wr$-paraconsistent extension]. Furthermore, C is said to be *non-pseudo-axiomatic* (cf. [15]), provided $\bigcap_{k\in\omega} C(x_k) \subseteq C(\varnothing)$ (the converse inclusion always holds by the monotonicity of C) or, equivalently (taking the structurality of C and the finiteness of the set of all variables occurring in any Σ-formula into account), $(\mathrm{img}\, C) \setminus \{\varnothing\}$ is a basis of $\mathrm{img}\, C$. Likewise, it is said to be *purely-inferential*, provided $C(\varnothing) = \varnothing$ or, equivalently, $\varnothing \in (\mathrm{img}\, C)$. In addition, C is said to *hold Relevance Principle* (viz., *Variable Sharing Property*; cf. [13]), provided, for every $\alpha \in (\omega \setminus 1)$, all $\phi \in \mathrm{Fm}^\alpha_\Sigma$ and all $\psi \in \mathrm{Fm}^{\omega\setminus\alpha}_\Sigma$, $\psi \notin C(\phi)$, in which case C neither is purely-inferential nor has an inconsistent formula. Finally, C is said to be *self-extensional* (cf. [15]), provided $\equiv_C \triangleq (\mathrm{Eq}^\omega_\Sigma \cap (\ker C)) \in \mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$, in which case, by the sructurality of C, $\equiv_C$ is fully invariant, the corresponding variety being called the *intrinsic variety of* C and denoted by $\mathrm{IV}(C)$.

Remark 2.8. The following hold:

(i) given a Σ-logic C, the following hold:

a) $(\mathrm{img}\, C) \cup \{\varnothing\}$ is a closure system over $\mathrm{Fm}^\omega_\Sigma$ closed under inverse substitutions, so the dual closure operator C_{+0} over $\mathrm{Fm}^\omega_\Sigma$ is the greatest purely-inferential sublogic of C, called the *purely-inferential version/counterpart of* C, in which case $\equiv_C = \equiv_{C_{+0}}$;

b) Taking the structurality of C and the finiteness of the set of all variables occurring in any Σ-formula into account, it is routine checking that the closure operator C_{-0} over $\mathrm{Fm}^\omega_\Sigma$ dual to the closure system over $\mathrm{Fm}^\omega_\Sigma$ with basis $(\mathrm{img}\, C) \setminus \{\varnothing\}$, in which case $C_{-0}(X) = C(X)$, for all non-empty $X \subseteq \mathrm{Fm}^\omega_\Sigma$, and $C_{-0}(\varnothing) = (\bigcap\{C(\varphi) \mid \varphi \in \mathrm{Fm}^\omega_\Sigma\})$, is structural, and so C_{-0} is the least non-pseudo-axiomatic extension of C, called the *non-pseudo-axiomatic version/counterpart of* C, in which case $\equiv_C = \equiv_{C_{-0}}$;

(ii) Verifying inclusion/equality of dual closure systems, it is then easy to see that the mappings

$$\begin{aligned} C &\mapsto C_{+0}, \\ C &\mapsto C_{-0}, \end{aligned}$$

are inverse to one isomorphisms between the poset of all non-pseudo-axiomatic(self-extensional) Σ-logics ordered by $\subseteq$ and that of all purely-inferential ones. □

Remark 2.9 (cf. Theorem 4.8 of [15] for the "non-pseudo-axiomatic" case). Since any inductive non-pseudo-axiomatic conjunctive logic C'' is uniquely determined by $\equiv_{C''}$, while the conjunctivity is retained by extensions, in view of Remark 2.8, we immediately conclude that, given any inductive non-pseudo-axiomatic/purely-inferential conjunctive self-extensional Σ-logic C, the mapping $C' \mapsto \mathrm{IV}(C')$ is a dual embedding of the poset of all inductive non-pseudo-axiomatic/purely-inferential self-extensional extensions of C into the lattice of all subvarieties of $\mathrm{IV}(C)$. □

Since any logic is either purely-inferential or, otherwise, non-pseudo-axiomatic, Remark 2.9 actually enhances Theorem 4.8 of [15] beyond non-pseudo-axiomatic logics.

A *(propositional) Σ-matrix* (cf. [7]) is any couple of the form $\mathcal{A} = \langle \mathfrak{A}, D^\mathcal{A}\rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $D^\mathcal{A} \subseteq A$ is called the *truth predicate of* $\mathcal{A}$, elements of which being referred to as *distinguished values of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters (possibly, with indices/prefixes/suffixes), their underlying algebras being denoted by corresponding Fraktur letters (with same indices/prefixes/suffixes, if any).) This is said to be *n-valued/truth[-non]-empty/(in)consistent/false-singular/ truth-singular*, where $n \in \omega$, provided $|A| = n/D^{\mathcal{A}} = [\neq]\varnothing/D^{\mathcal{A}} \neq (=)A/|A \setminus D^{\mathcal{A}}| \in 2/|D^{\mathcal{A}}| \in 2$. Next, given any $\Sigma' \subseteq \Sigma$, put $(\mathcal{A}{\upharpoonright}\Sigma') \triangleq \langle \mathfrak{A}{\upharpoonright}\Sigma', D^{\mathcal{A}}\rangle$, in which case $\mathcal{A}$ is said to be an *expansion of* $\mathcal{A}{\upharpoonright}\Sigma'$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.) Finally, the Σ-matrix $\complement(\mathcal{A}) \triangleq \langle \mathfrak{A}, A \setminus D^{\mathcal{A}}\rangle$ is refereed to as *complementary to/of* $\mathcal{A}$.

A Σ-matrix $\mathcal{A}$ is said to be *finite/generated by* a $B \subseteq A$, whenever $\mathfrak{A}$ is so. Then, it is said to be *K-generated*, where $K \subseteq \infty$, whenever it is generated by some $B \in \wp_K(A)$.

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [8]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any Σ-rule $\Gamma \vdash \phi$ being viewed as the Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$.

Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$ and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ defined by $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) \triangleq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap\{h^{-1}[D^{\mathcal{A}}] | \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}), h[X] \subseteq D^{\mathcal{A}}\}$, for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, in which case we have:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.2}$$

because $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}) = \{h{\upharpoonright}\mathrm{Fm}^{\alpha}_{\Sigma} | h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})\}$, for any Σ-algebra $\mathfrak{A}$, as $A \neq \varnothing$. (Note that $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(\varnothing) = \varnothing$, whenever M has a truth-empty member. Moreover, using either the ultra-product technique (cf. [8]) or the topological one (cf. [7]), $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ is shown to be inductive, whenever both M and all members of it are finite.) Then, $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic called the one *of* M. A Σ-logic C is said to be *K-defined by* M, where $K \subseteq \infty$, provided $C(X) = \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)$, for all $X \in \wp_K(\mathrm{Fm}^{\omega}_{\Sigma})$. A Σ-logic is said to be *[minimally]n-valued*, where $n \in \omega$, whenever it is defined by an n-valued Σ-matrix[but by no m-valued one with $m < n$]. A Σ-matrix $\mathcal{A}$ is said to be *$\wr$-paraconsistent*, where $\wr$ is a (possibly, secondary) unary connective of Σ, whenever the logic of $\mathcal{A}$ is so. (Clearly, the logic of any class of matrices is [inferentially]consistent iff the class contains a consistent [truth-non-empty]member.)

Remark 2.10. In view of Remark 2.8(i)**a)**, given any class M of Σ-matrices and any non-empty class S of truth-empty Σ-matrices, the logic of $\mathsf{S} \cup \mathsf{M}$ is the purely-inferential version of the logic of M. □

Example 2.11. Let $\mathcal{A}$ be a two-valued consistent truth-non-empty Σ-matrix and C the logic of $\mathcal{A}$. Then, $\equiv_C$ is the set of all Σ-identities true in $\mathfrak{A}$, i.e., in $\mathbf{V}(\mathfrak{A})$, in which case C is self-extensional, while $\mathrm{IV}(C) = \mathbf{V}(\mathfrak{A})$. □

A Σ-matrix $\mathcal{A}$ is said to be a *model of* a Σ-logic C, provided $C \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}$, the class of all [simple of]them being denoted by $\mathrm{Mod}_{[*]}(C)$. Next, $\mathcal{A}$ is said to be *[weakly]$\diamond$-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $(\{a, b\} \subseteq D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, $\mathrm{Cn}^{\omega}_{\mathcal{A}}$ is [weakly]$\diamond$-conjunctive. Then, $\mathcal{A}$ is said to be *[weakly]$\diamond$-disjunctive*, whenever $\complement(\mathcal{A})$ is [weakly]$\diamond$-conjunctive.

Given any[axiomatic] Σ-calculus $\mathcal{C}$, members of $\mathrm{Mod}(\mathcal{C}) \triangleq \mathrm{Mod}(\mathrm{Cn}_{\mathcal{C}})$ are called its *models* as well. This fits well the above model-theoretic conventions, according to which, in particular, given a class M of Σ-matrices, $\mathsf{M} \cap \mathrm{Mod}(\mathcal{C})$ is referred to as the *relative equality-free first-order [positive]Horn model subclass of* M *relatively axiomatized by* $\mathcal{C}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict)[surjective]homomorphism from* $\mathcal{A}$ *[on/]to* $\mathcal{B}$ is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that[$h[A] = B$ and] $D^{\mathcal{A}} \subseteq (=) h^{-1}[D^{\mathcal{B}}]$,([in which case $\mathcal{A}/\mathcal{B}$ is said to be a *strict surjective homomorphic counterimage/image of* $\mathcal{B}/\mathcal{A}$,]) the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$. Note that:

$$\hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B}) = \hom_{\mathrm{S}}(\complement(\mathcal{A}), \complement(\mathcal{B})). \tag{2.3}$$

And what is more, we have:

$$(\forall h \in \hom(\mathfrak{A}, \mathfrak{B}) : [((\mathrm{img}\, h) = B) \Rightarrow](\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B}) \supseteq [=]\{h \circ g | g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}), \tag{2.4}$$

and so we get:

$$(\exists h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{B}} \subseteq [=]\, \mathrm{Cn}^{\alpha}_{\mathcal{A}}), \tag{2.5}$$

$$(\exists h \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.6}$$

for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$. Then, $\mathcal{A}$ is said to be a *[proper]submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$[and $\mathcal{A} \neq \mathcal{B}$], in which case we set $(\mathcal{B}{\upharpoonright}A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from* $\mathcal{A}$ *into/onto* $\mathcal{B}$, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}/$, viz., an *isomorphic copy of* $\mathcal{B}$.

Given a class M of Σ-matrices, the class of all (truth-non-empty)[consistent]submatrices/isomorphic copies/strict surjective homomorphic {counter}images of members of M is denoted by $(\mathbf{S}^{(*)}_{[*]}/\mathbf{I}/\mathbf{H}^{\{-1\}})(\mathsf{M})$, respectively.

Proposition 2.12. *Let* M *be a class of* Σ*-matrices and* $\mathcal{A}$ *an axiomatic* Σ*-calculus. Then, the axiomatic extension of the logic of* M *relatively axiomatized by* $\mathcal{A}$ *is defined by* $\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A})$.

Proof. Put $\mathsf{S} \triangleq (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A}))$. Consider any $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$.

First, assume $\varphi \in \mathrm{Cn}^{\omega}_{\mathsf{M}}(\Gamma \cup \mathrm{SI}_{\Sigma}(\mathcal{A}))$. Consider any $\mathcal{A} \in \mathsf{S}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $\Gamma \subseteq h^{-1}[D^{\mathcal{A}}]$, in which case there is some $\mathcal{B} \in \mathsf{M}$ such that $\mathcal{A}$ is a submatrix of $\mathcal{B}$, and so $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{B})$ and $\Gamma \subseteq h^{-1}[D^{\mathcal{B}}]$. Moreover, for every Σ-substitution σ, $(h \circ \sigma) \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, in which case $\mathrm{SI}_{\Sigma}(\mathcal{A}) \subseteq h^{-1}[D^{\mathcal{A}}] \subseteq h^{-1}[D^{\mathcal{B}}]$, and so $\varphi \in h^{-1}[(\mathrm{img}\, h) \cap D^{\mathcal{B}}] \subseteq h^{-1}[A \cap D^{\mathcal{B}}] = h^{-1}[D^{\mathcal{A}}]$.

Conversely, assume $\varphi \notin \mathrm{Cn}^{\omega}_{\mathsf{M}}(\Gamma \cup \mathrm{SI}_{\Sigma}(\mathcal{A}))$. Then, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{B})$ such that $(\Gamma \cup \mathrm{SI}_{\Sigma}(\mathcal{A})) \subseteq h^{-1}[D^{\mathcal{B}}] \not\ni \varphi$, in which case $\mathfrak{A} \triangleq (\mathfrak{B}{\upharpoonright}(\mathrm{img}\, h))$ is a subalgebra of $\mathfrak{B}$, and so $\mathcal{A} \triangleq (\mathcal{B}{\upharpoonright}A)$ is a submatrix of $\mathcal{B}$, $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ is surjective and $(\Gamma \cup \mathrm{SI}_{\Sigma}(\mathcal{A})) \subseteq h^{-1}[D^{\mathcal{B}}] = h^{-1}[A \cap D^{\mathcal{B}}] = h^{-1}[D^{\mathcal{A}}] \not\ni \varphi$. Finally, consider any $\psi \in \mathcal{A}$ and any $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, as $(\mathrm{img}\, h) = A$, by (2.4), there is some Σ-substitution σ such that $g = (h \circ \sigma)$, in which case

$g(\psi) = h(\sigma(\psi)) \in h[\mathrm{SI}_\Sigma(\mathcal{A})] \subseteq D^{\mathcal{A}}$, and so ψ is true in $\mathcal{A}$. Thus, $\mathcal{A}$, being consistent, for $h(\varphi) \in (A \setminus D^{\mathcal{A}})$, belongs to S, as required. □

Let $\mathcal{A}$ be a Σ-matrix. Elements of $\mathrm{Con}(\mathcal{A}) \triangleq \{\theta \in \mathrm{Con}(\mathfrak{A}) | \theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}\} \ni \Delta_A$ are called *congruences of* $\mathcal{A}$. Given any $\varnothing \neq \Theta \subseteq \mathrm{Con}(\mathcal{A}) \subseteq \mathrm{Con}(\mathfrak{A})$, $\mathrm{Tr}(\bigcup \Theta)$, being well-known to be a congruence of $\mathfrak{A}$, is then easily seen to be a congruence of $\mathcal{A}$. Therefore, $\partial(\mathcal{A}) \triangleq (\bigcup \mathrm{Con}(\mathcal{A})) \in \mathrm{Con}(\mathcal{A})$, in which case this is the greatest congruence of $\mathcal{A}$ (it is this fact that justifies using the symbol ∂), while $\mathrm{Con}(\mathcal{A}) = \{\theta \in \mathrm{Con}(\mathfrak{A}) | \theta \subseteq \partial(\mathcal{A})\}$. Then, $\mathcal{A}$ is said to be *simple*, provided $\partial(\mathcal{A}) = \Delta_A$. Given any $\theta \in \mathrm{Con}(\mathfrak{A}[\mathcal{A}])$, we have the *quotient* Σ-matrix $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$, in which case $\nu_\theta \in \mathrm{hom}^{\mathrm{S}}_{[\mathrm{S}]}(\mathcal{A}, \mathcal{A}/\theta)$. The quotient $\Re(\mathcal{A}) \triangleq (\mathcal{A}/\partial(\mathcal{A}))$ is called the *reduction of* $\mathcal{A}$.

Corollary 2.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices and $h \in \mathrm{hom}^{(\mathrm{S})}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Then, for every $\vartheta \in \mathrm{Con}(\mathcal{B})$, $h^{-1}[\vartheta] \in \{\theta \in \mathrm{Con}(\mathcal{A}) \mid (\ker h) \subseteq \theta\}$(, whereas $h[h^{-1}[\vartheta]] = \vartheta$, while, conversely, for every $\theta \in \mathrm{Con}(\mathcal{A})$ such that $(\ker h) \subseteq \theta$, $h[\theta] \in \mathrm{Con}(\mathcal{B})$, whereas $h^{-1}[h[\theta]] = \theta$).*

Proof. With using Lemma 2.2. First, consider any $\vartheta \in \mathrm{Con}(\mathcal{B})$. Then, the fact that $h^{-1}[\vartheta][D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}$ is by the fact that $\vartheta[D^{\mathcal{B}}] \subseteq D^{\mathcal{B}}$, while $D^{\mathcal{A}} = h^{-1}[D^{\mathcal{B}}]$.(Conversely, consider any $\theta \in \mathrm{Con}(\mathcal{A})$ such that $\ker h \subseteq \theta$. Then, the fact that $(h[\theta])[D^{\mathcal{B}}] \subseteq D^{\mathcal{B}}$ is by the fact that $\theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}$, while $D^{\mathcal{A}} = h^{-1}[D^{\mathcal{B}}]$.) □

By Corollary 2.13, we immediately have:

Corollary 2.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices and $h \in \mathrm{hom}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Suppose $\mathcal{A}$ is simple. Then, h is injective.*

Proposition 2.15 (Matrix Homomorphism Theorem). *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Σ-matrices, $f \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$ and $g \in \mathrm{hom}^{(\mathrm{S})}_{[\mathrm{S}]}(\mathcal{A}, \mathcal{C})$. Suppose $(\ker f) \subseteq (\ker g)$. Then, $h \triangleq (g \circ f^{-1}) \in \mathrm{hom}^{(\mathrm{S})}_{[\mathrm{S}]}(\mathcal{B}, \mathcal{C})$.*

Proof. The fact that $h \in \mathrm{hom}(\mathfrak{B}, \mathfrak{C})$(and $h[B] = C$) is well-known due to the Algebra Homomorphism Theorem. Finally, we also have $h^{-1}[D^{\mathcal{C}}] = f[g^{-1}[D^{\mathcal{C}}]][=] \supseteq f[D^{\mathcal{A}}] = f[f^{-1}[D^{\mathcal{B}}]] = D^{\mathcal{B}}$, for $f[A] = B$, as required. □

Proposition 2.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices and $h \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Then, $\partial(\mathcal{A}) = h^{-1}[\partial(\mathcal{B})]$ and $\partial(\mathcal{B}) = h[\partial(\mathcal{A})]$.*

Proof. As $\Delta_B \in \mathrm{Con}(\mathcal{B})$, by Corollary 2.13, we have $\ker h = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, and so $\ker h \subseteq \partial(\mathcal{A})$, in which case, by Corollary 2.13, we get:

$$\begin{aligned} h^{-1}[\partial(\mathcal{B})] &\subseteq \partial(\mathcal{A}), \\ h[h^{-1}[\partial(\mathcal{B})]] &= \partial(\mathcal{B}), \\ h[\partial(\mathcal{A})] &\subseteq \partial(\mathcal{B}), \\ h^{-1}[h[\partial(\mathcal{A})]] &= \partial(\mathcal{A}). \end{aligned}$$

These collectively imply the equalities to be proved, as required. □

Since, for any equivalence θ on any set A, it holds that $\nu_\theta[\theta] = \Delta_{A/\theta}$, as an immediate consequence of Proposition 2.16, we also have:

Corollary 2.17. *Let $\mathcal{A}$ be a Σ-matrix. Then, $\Re(\mathcal{A})$ is simple.*

Proposition 2.18. *Let C be a Σ-logic and M a finite class of finite Σ-matrices. Suppose C if finitely-defined by M. Then, C is defined by M. In particular, C is inductive.*

Proof. In that case, $C' \triangleq \mathrm{Cn}^{\omega}_{\mathsf{M}} \subseteq C$, for C' is inductive, while $\equiv_C = \equiv_{C'}$. For proving the converse point-wise inclusion, it suffices to prove that $\mathsf{M} \subseteq \mathrm{Mod}(C)$. For consider any $\mathcal{A} \in \mathsf{M}$, any $\Gamma \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, any $\varphi \in C(\Gamma)$ and any $h \in \mathrm{hom}(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h[\Gamma] \subseteq D^{\mathcal{A}}$. Then, $\alpha \triangleq |A| \in (\wp_{\infty\setminus 1}(\omega) \cap \omega)$. Take any bijection $e : V_\alpha \to A$ to be extended to a $g \in \mathrm{hom}(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $e^{-1} \circ (h \restriction V_\omega)$ is extended to a Σ-substitution σ, in which case $\sigma(\varphi) \in C(\sigma[\Gamma])$, for C is structural, while $\sigma[\Gamma \cup \{\varphi\}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. For every $\mathcal{B} \in \mathsf{M}$, we have the equivalence relation $\theta^{\mathcal{B}} \triangleq \{\langle a, b\rangle \in B^2 \mid (a \in D^{\mathcal{B}}) \Leftrightarrow (b \in D^{\mathcal{B}})\}$ on B, in which case $B/\theta^{\mathcal{B}}$ is finite, for B is so. Moreover, as both α, M and all members of it are finite, we have the finite set $I \triangleq \{\langle h', \mathcal{B}\rangle \mid \mathcal{B} \in \mathsf{M}, h' \in \mathrm{hom}(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B})\}$, in which case, for each $i \in I$, we set $h_i \triangleq \pi_0(i)$, $\mathcal{B}_i \triangleq \pi_1(i)$ and $\theta_i \triangleq \theta^{\mathcal{B}_i}$. Then, by (2.2), we have $\theta \triangleq (\equiv_{C'} \cap \mathrm{Eq}^{\alpha}_{\Sigma}) = (\mathrm{Eq}^{\alpha}_{\Sigma} \cap \bigcap_{i \in I} h_i^{-1}[\theta_i])$, in which case, for every $i \in I$, $\theta \subseteq h_i^{-1}[\theta_i] = \ker(\nu_{\theta_i} \circ h_i)$, and so $g_i \triangleq (\nu_{\theta_i} \circ h_i \circ \nu_\theta^{-1}) : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to B_i$. In this way, $f \triangleq (\prod_{i \in I} g_i) : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to (\prod_{i \in I} B_i)$ is injective, for $(\ker f) = ((\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)^2 \cap \bigcap_{i \in I} (\ker g_i))$ is diagonal. Hence, $\mathrm{Fm}^{\alpha}_{\Sigma}/\theta$ is finite, for $\prod_{i \in I} B_i$ is so, and so is $(\sigma[\Gamma]/\theta) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)$. For each $c \in (\sigma[\Gamma]/\theta)$, choose any $\phi_c \in (\sigma[\Gamma] \cap \nu_\theta^{-1}[\{c\}]) \neq \varnothing$. Put $\Delta \triangleq \{\phi_c \mid c \in (\sigma[\Gamma]/\theta)\} \in \wp_\omega(\sigma[\Gamma])$. Consider any $\psi \in \sigma[\Gamma]$. Then, $\Delta \ni \phi_{[\psi]_\theta} \equiv_C \psi$, in which case $\psi \in C(\Delta)$, and so $\sigma[\Gamma] \subseteq C(\Delta)$. In this way, $\sigma(\varphi) \in C(\Delta) = C'(\Delta)$, for $\Delta \in \wp_\omega(\mathrm{Fm}^{\omega}_{\Sigma})$, so, by (2.2), $\sigma(\varphi) \in \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\Delta) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{A}}(\Delta)$. Moreover, $g[\Delta] \subseteq g[\sigma[\Gamma]] = h[\Gamma] \subseteq D^{\mathcal{A}}$, and so $h(\varphi) = g(\sigma(\varphi)) \in D^{\mathcal{A}}$, as required. □

Proposition 2.19. *Let M be a class of truth-non-empty Σ-matrices. Then, the logic of M is non-pseudo-axiomatic.*

Proof. Consider any $\varphi \in \bigcap_{k \in \omega} \mathrm{Cn}^{\omega}_{\mathsf{M}}(x_k)$, any $\mathcal{A} \in \mathsf{M}$ and any $h \in \mathrm{hom}(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, $\varphi \in \mathrm{Fm}^{k}_{\Sigma}$, for some $k \in \omega$. Choose any $a \in D^{\mathcal{A}} \neq \varnothing$. Let $g \in \mathrm{hom}(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h \restriction (V_\omega \setminus \{x_k\})) \cup \{\langle x_k, a\rangle\}$. Then, $g(x_k) = a \in D^{\mathcal{A}}$, and so $h(\varphi) = g(\varphi) \in D^{\mathcal{A}}$, as required. □

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, the Σ-matrix $(\prod_{i \in I} \mathcal{A}_i) \triangleq \langle \prod_{i \in I} \mathfrak{A}_i, A \cap \bigcap_{i \in I} \pi_i^{-1}[D^{\mathcal{A}_i}] \rangle$ is called the *direct product of* $\overline{\mathcal{A}}$. (As usual, when $I = 2$, $\mathcal{A}_0 \times \mathcal{A}_1$ stands for the direct product involved. Likewise, if $(\mathrm{img}\, \overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$[and $I = 2$], where $\mathcal{A}$ is a Σ-matrix, $\mathcal{A}^I \triangleq (\prod_{i \in I} \mathcal{A}_i)$ is called the *direct I-power [square] of* $\mathcal{A}$.) Finally, any submatrix $\mathcal{B}$ of $\prod_{i \in I} \mathcal{A}_i$ is referred to as a *subdirect product of* $\overline{\mathcal{A}}$, whenever, for each $i \in I$, $\pi_i[B] = A_i$.

Lemma 2.20 (Subdirect Product Lemma). *Let M be a[finite] class of[finite] Σ-matrices and $\mathcal{A}$ a {truth-non-empty }(simple) $([\omega\cap](\omega+1))$-generated model of the logic of M. Then, there is some strict surjective homomorphism from a subdirect product of a [finite]tuple constituted by consistent {truth-non-empty }submatrices of members of M onto $\Re(\mathcal{A})$ (resp., onto $\mathcal{A}$ itself).*

Proof. Take any $A' \in \wp_{[\omega\cap](\omega+1)}(A)$ generating $\mathfrak{A}$ and any $a \in A \neq \varnothing$, in which case $A'' \triangleq (A' \cup \{a\}) \in (\wp_{[\omega\cap](\omega+1)}(A) \setminus 1)$, and so $\alpha \triangleq |A''| \in (([\omega\cap](\omega+1)) \setminus 1) \subseteq \wp_{\infty\setminus 1}(\omega)$. Next, take any bijection from V_α onto A'' to be extended to a surjective $h \in \hom(\mathrm{Fm}^\alpha_\Sigma, \mathfrak{A})$, in which case it is a surjective strict homomorphism from $\mathcal{B} \triangleq \langle \mathrm{Fm}^\alpha_\Sigma, X\rangle$, where $X \triangleq h^{-1}[D^{\mathcal{A}}]$, onto $\mathcal{A}$, and so, by (2.5), $\mathcal{B}$ is a model of the logic of M. Then, applying (2.2) twice, we get $\mathrm{Cn}^\alpha_{\mathsf{M}}(X) \subseteq \mathrm{Cn}^\alpha_{\mathcal{B}}(X) \subseteq X \subseteq \mathrm{Cn}^\alpha_{\mathsf{M}}(X)$. Furthermore, we have the[finite] set $I \triangleq \{\langle h', \mathcal{D}\rangle \mid h' \in \hom(\mathcal{B}, \mathcal{D}), \mathcal{D} \in \mathsf{M}, (\mathrm{img}\, h') \nsubseteq D^{\mathcal{D}}\}$, in which case, for every $i \in I$, we set $h_i \triangleq \pi_0(i)$, and so $\mathcal{C}_i \triangleq (\pi_1(i){\restriction}(\mathrm{img}\, h_i))$ is a consistent {truth-non-empty }submatrix of $\pi_1(i) \in \mathsf{M}$. Clearly, $X = \mathrm{Cn}^\alpha_{\mathsf{M}}(X) = (\mathrm{Fm}^\alpha_\Sigma \cap \bigcap_{i\in I} h_i^{-1}[D^{\mathcal{C}_i}])$. Therefore, $g \triangleq (\prod_{i\in I} h_i) : \mathrm{Fm}^\alpha_\Sigma \to (\prod_{i\in I} C_i)$ is a strict homomorphism from $\mathcal{B}$ to $\prod_{i\in I} \mathcal{C}_i$ such that, for each $i \in I$, $(\pi_i \circ g) = h_i$, in which case $\pi_i[g[\mathrm{Fm}^\alpha_\Sigma]] = h_i[\mathrm{Fm}^\alpha_\Sigma] = C_i$, and so g is a surjective strict homomorphism from $\mathcal{B}$ onto the subdirect product $\mathcal{E} \triangleq ((\prod_{i\in I} \mathcal{C}_i){\restriction}(\mathrm{img}\, g))$ of $\overline{\mathcal{C}}$. Put $\theta \triangleq \eth(\mathcal{A})(= \Delta_A)$ and $\mathcal{F} \triangleq (\mathcal{A}/\theta)$. Then, $f \triangleq (\nu_\theta \circ h) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{F})$. Therefore, by Corollaries 2.13, 2.17 and Proposition 2.16, we have $(\ker g) = g^{-1}[\Delta_E] \subseteq \eth(\mathcal{B}) = f^{-1}[\Delta_F] = (\ker f)$, in which case, by Proposition 2.15, $e \triangleq (f \circ h^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{F})$,(and so $(\nu_\theta^{-1} \circ e) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{A})$,) as required. ☐

Given a class M of Σ-matrices, the class of all [sub]direct products of tuples(of cardinality $\in K \subseteq \infty$) constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$. Clearly, $\mathrm{Mod}(C)$, where C is a Σ-logic, is closed under $\mathbf{P}$.

Corollary 2.21. *Let K and M be classes of Σ-matrices, C the logic of M and C' an extension of C. Suppose(both M and all members of it are finite and) $[\Re](\mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))) \subseteq \mathsf{K}$ {in particular, $[\Re](\mathbf{S}(\mathbf{P}_{(\omega)}(\mathsf{M}))) \subseteq \mathsf{K}$ ⟨in particular, $\mathsf{K} \supseteq \mathsf{M}$ is closed under both $\mathbf{S}$ and $\mathbf{P}_{(\omega)}$[as well as $\Re$]⟩}. Then, C' is (finitely-)defined by $\mathsf{S} \triangleq (\mathrm{Mod}_{[*]}(C') \cap \mathsf{K})$.*

Proof. Clearly, $C' \subseteq \mathrm{Cn}^\omega_{\mathsf{S}}$, for $\mathsf{S} \subseteq \mathrm{Mod}(C')$. Conversely, consider any $(\Gamma \cup \{\varphi\}) \in \wp_{(\omega)}(\mathrm{Fm}^\omega_\Sigma)$, in which case(there is some $\alpha' \in (\omega \setminus 1)$ such that $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\alpha'}_\Sigma$, and so) $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^\alpha_\Sigma$, where $\alpha \triangleq ((\alpha' \cap)\omega) \in \wp_{\infty\setminus 1}(\omega)$, such that $\varphi \notin C'(\Gamma)$. Then, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, C'(\Gamma)\rangle$ is a model of C' {in particular, of C}, and so is its $(\alpha+1)$-generated submatrix $\mathcal{A} \triangleq \langle \mathfrak{Fm}^\alpha_\Sigma, C'(\Gamma) \cap \mathrm{Fm}^\alpha_\Sigma\rangle$, in view of (2.5), in which case $\varphi \notin \mathrm{Cn}^\alpha_{\mathcal{A}}(\Gamma)$, by the idempotencity of C', and so $\varphi \notin \mathrm{Cn}^\omega_{\mathcal{A}}(\Gamma)$, in view of (2.2). Therefore, by Lemma 2.20, there are some $\mathcal{B} \in \mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))$, in which case $\mathcal{D} \triangleq [\Re](\mathcal{B}) \in [\Re](\mathbf{P}^{\mathrm{SD}}_{(\omega)}(\mathbf{S}_*(\mathsf{M}))) \subseteq \mathsf{K}$, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{A}/\eth(\mathcal{A}))$. Then, by (2.5), $\mathrm{Cn}^\omega_{\mathcal{D}} = \mathrm{Cn}^\omega_{\mathcal{A}}$, in which case[, by Corollary 2.17,] $\mathcal{D} \in \mathsf{S}$, and so $\varphi \notin \mathrm{Cn}^\omega_{\mathsf{S}}(\Gamma)$, as required. ☐

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^\alpha_{\Sigma} \subseteq \mathrm{Fm}^\alpha_{\Sigma'}$ and $\hom(\mathfrak{Fm}^\alpha_{\Sigma'}, \mathfrak{Fm}^\alpha_{\Sigma'}) = \{h{\restriction}\mathrm{Fm}^\alpha_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{Fm}^\alpha_\Sigma), h[\mathrm{Fm}^\alpha_{\Sigma'}] \subseteq \mathrm{Fm}^\alpha_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^\omega_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^\omega_{\Sigma'}$, called the *$\Sigma'$-fragment of C*, in which case C is said to be an *expansion of C'*. In that case, given also any class M of Σ-matrices defining C, in its turn, C' is defined by $\mathsf{M}{\restriction}\Sigma'$.

As a matter of fact, the above purely-formal definition of paraconsitency appears to be too expansive, because any consistent non-pseudo-axiomatic/truth-non-empty Σ-logic/-matrix (including the classical one) occurs to be $\wr$-paraconsistent, whenever $(\wr x_0) \triangleq x_0$ (less trivial intuitively non-acceptable instances of such an odd "paraconsistency" are provided by modal logics with $(\wr x_0) \triangleq (\Box x_0)$ and the necessity rule). Intuitively, it is clear that, when dealing with $\wr$-paraconsistency of a logic, $\wr$ must be a kind of negation for the logic. In its turn, this raises the question: what is negation? Refraining from vain attempts to specify it *completely and irrevocably*, we just restrict our consideration by, so to say, *subclassical* negation. Intuitively, $\wr$ is viewed as a subclassical negation for a logic C, whenever the $\wr$-fragment of C is a sublogic of the negation fragment of the classical logic. And what is more, formally speaking, we require refutation of merely most unacceptable *oddity* rules. Thus, a (possibly, secondary) unary connective $\wr$ of Σ is referred to as a *subclassical negation for* a Σ-logic C, provided:

$$\wr^m x_0 \notin C(\wr^n x_0), \tag{2.7}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd. This declines the bizarre instances mentioned above, when $m = 1$ and $n = 0$. Nevertheless, we retain the above definition of paraconsistency to prove most strong versions of maximal paraconsistency results perfectly independent from specifying what is negation.

3. Preliminary key issues

3.1. **Congruence and equality determinants.** A *[binary]relational Σ-scheme* is any $\varepsilon \subseteq (\wp_\omega(\mathrm{Fm}^{[2\cap]\omega}_\Sigma) \times \mathrm{Fm}^{[2\cap]\omega}_\Sigma)$, in which case, given any Σ-matrix $\mathcal{A}$, we set $\theta^{\mathcal{A}}_\varepsilon \triangleq \{\langle a, b\rangle \in A^2 \mid \mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]\} \subseteq A^2$. Note that, given a one more Σ-matrix $\mathcal{B}$ and an $h \in \hom^{(\mathrm{S})}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, we have:

$$h^{-1}[\theta^{\mathcal{B}}_\varepsilon] \subseteq (=)[=]\theta^{\mathcal{A}}_\varepsilon. \tag{3.1}$$

A *[unary]unitary relational Σ-scheme* is any $\Upsilon \subseteq \mathrm{Fm}^{[1\cap]\omega}_\Sigma$, in which case we have the [binary]relational Σ-scheme $\varepsilon_\Upsilon \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in \sigma_{1:+1}[\Upsilon]\}$ such that $\theta^{\mathcal{A}}_{\varepsilon_\Upsilon}$, where $\mathcal{A}$ is any Σ-matrix, is an equivalence relation on A.

A *[binary]congruence/equality determinant for* a class of Σ-matrices M is any [binary]relational Σ-scheme ε such that, for each $\mathcal{A} \in \mathsf{M}$, $\theta^{\mathcal{A}}_\varepsilon \in \mathrm{Con}(\mathcal{A})/= \Delta_A$, respectively.

Then, according to [21]/[20], a *[unary]unitary congruence/equality determinant for* a class of Σ-matrices M is any [unary]unitary relational Σ-scheme Υ such that ε_Υ is a/an congruence/equality determinant for M. (It is unary unitary equality determinants that are equality determinants in the sense of [20].)

Example 3.1 (cf. [21]). Given any Σ-matrix $\mathcal{A}$, it is routine checking that the equivalence relation $\theta^{\mathcal{A}} \triangleq \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^\omega_\Sigma}} \in \mathrm{Con}(\mathfrak{A})$. Moreover, as $x_0 \in \mathrm{Fm}^\omega_\Sigma$, we clearly have $\theta^{\mathcal{A}}[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}$. Thus, $\mathrm{Fm}^\omega_\Sigma$ is a unitary congruence determinant for every Σ-matrix. ☐

Example 3.2 (cf. Example 1 of [20])**.** $\{x_0\}$ is a unary unitary equality determinant for any consistent truth-non-empty two-valued matrix. □

Example 3.3. [cf. Example 2 of [20]] Let $j \in 2$, $\vec{k} \in 2^2$, $\wr$ a (possibly, secondary) unary connective of Σ and $\mathcal{A}$ a Σ-matrix. Suppose $A \subseteq 2^2$, $D^{\mathcal{A}} = (A \cap \pi_j^{-1}[\{k_1\}])$ and $(\wr^{\mathfrak{A}})^{-1}[D^{\mathcal{A}}] = (A \cap \pi_{1-j}^{-1}[\{k_0\}])$. Then, $\Upsilon_\wr \triangleq \{x_0, \wr x_0\}$ is a unary unitary equality determinant for $\mathcal{A}$. □

Lemma 3.4. *Let $\mathcal{A}$ be a Σ-matrix and ε a congruence determinant for $\mathcal{A}$. Then, $\partial(\mathcal{A}) = \theta^{\mathcal{A}}_\varepsilon$. In particular, $\mathcal{A}$ is simple, whenever ε is an equality determinant for it.*

Proof. Consider any $\theta \in \mathrm{Con}(\mathcal{A})$ and any $\langle a, b\rangle \in \theta$. Then, as $\mathrm{Con}(\mathcal{A}) \ni \theta^{\mathcal{A}}_\varepsilon \supseteq \Delta_A \ni \langle a, a\rangle$, we have $\mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/a]$, in which case, by the reflexivity of θ, we get $\mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]$, and so $\langle a, b\rangle \in \theta^{\mathcal{A}}_\varepsilon$, as required. □

It is remarkable that Proposition 2.16 equally ensues from Lemmas 2.2, 3.4, (3.1) and Example 3.1.

Lemma 3.5. *Let M be a class of Σ-matrices, C the logic of M and $\mathcal{B} \in \mathrm{Mod}_*(C)$. Then, $\mathfrak{B} \in \mathbf{V}(\pi_0[\mathsf{M}])$.*

Proof. Consider any $(\phi \approx \psi) \in \mathrm{Eq}^\omega_\Sigma$ being true in $\pi_0[\mathsf{M}]$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{B})$. Take any $\varphi \in \mathrm{Fm}^\omega_\Sigma$ and any $v : V_{\omega\setminus 2} \to B$. Then, there is some $k \in (\omega \setminus 1)$ such that $(\phi \approx \psi) \in \mathrm{Eq}^k_\Sigma$. Put $\varphi' \triangleq \sigma_{1:+k}(\varphi)$. Then, for each $\mathcal{A} \in \mathsf{M}$ and every $g \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$, we have $g(\phi) = g(\psi)$, in which case $g(\varphi'[x_0/\phi]) = g(\varphi'[x_0/\psi])$, and so the rules $(\varphi'[x_0/\phi]) \vdash (\varphi'[x_0/\psi])$ and $(\varphi'[x_0/\psi]) \vdash (\varphi'[x_0/\phi])$ are true in M, and so in $\mathcal{B}$. Let $h' \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{B})$ extend $(h{\restriction}V_k) \cup [x_{i+k}/v(x_{i+1})]_{i\in(\omega\setminus 1)}$. Then, $(\sigma_{1:+1}(\varphi)[x_0/h(\phi); v]) = h'(\varphi'[x_0/\phi]) \in D^{\mathcal{B}}$ iff $D^{\mathcal{B}} \ni h'(\varphi'[x_0/\psi]) = (\sigma_{1:+1}(\varphi)[x_0/h(\psi); v])$. Thus, $\mathcal{B} \models (\forall_{\omega\setminus 2}((\sigma_{1:+1}(\varphi)[x_0/x_1]) \leftrightarrow \sigma_{1:+1}(\varphi)))[x_0/h(\phi), x_1/h(\psi)]$, for all $\varphi \in \mathrm{Fm}^\omega_\Sigma$. Hence, by Example 3.1, we eventually get $\langle h(\phi), h(\psi)\rangle \in \partial(\mathcal{B}) = \Delta_B$, as required. □

Lemma 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, ε a/an congruence/equality determinant for $\mathcal{B}$ and $h \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$/ injective. Suppose either ε is binary or $h[A] = B$. Then, ε is a/an congruence/equality determinant for $\mathcal{A}$.*

Proof. In that case, by (3.1), we have $\theta^{\mathcal{A}}_\varepsilon = h^{-1}[\theta^{\mathcal{B}}_\varepsilon]$. In this way, Corollary 2.13/injectivity of h completes the argument. □

Lemma 3.7. *Let $\mathcal{A}$ be a Σ-matrix with unary unitary equality determinant Υ, $\mathcal{B}$ a submatrix of $\mathcal{A}$ and $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$. Then, h is diagonal.*

Proof. Consider any $a \in B$ and any $v \in \Upsilon$. Then, $(v^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (v^{\mathfrak{B}}(a) \in D^{\mathcal{B}}) \Leftrightarrow (v^{\mathfrak{A}}(h(a)) \in D^{\mathcal{A}})$. Thus, $h(a) = a$, as required. □

3.2. **False-singular consistent weakly conjunctive matrices.** Given any consistent false-singular Σ-matrix $\mathcal{A}$, the unique element of $A \setminus D^{\mathcal{A}}$ is denoted by $\dashv^{\mathcal{A}}$.

Lemma 3.8. *Let $\diamond$ be a (possibly, secondary) binary connective of Σ, $\mathcal{A}$ a consistent false-singular weakly $\diamond$-conjunctive Σ-matrix, $n \in \omega$, $\overline{\mathcal{B}}$ an n-tuple constituted by consistent submatrices of $\mathcal{A}$ and $\mathcal{C}$ a subdirect product of $\overline{\mathcal{B}}$. Then, $(n \times \{\dashv^{\mathcal{A}}\}) \in C$.*

Proof. In case $n = 0$, we simply have $(n \times \{\dashv^{\mathcal{A}}\}) = \varnothing \in C$, for $C \neq \varnothing$.

Now, assume $n \neq \varnothing$. Define a $\bar{c} \in C^n$ as follows. Consider any $i \in n$. Then, as $\mathcal{B}_i$, being a submatrix of the false-singular matrix $\mathcal{A}$, is consistent, $\dashv^{\mathcal{A}} \in B_i$. Therefore, since $\pi_i[C] = B_i$, there is some $c_i \in C$ such that $\pi_i(c_i) = \dashv^{\mathcal{A}}$. Finally, put $b \triangleq (\diamond^{\mathfrak{C}}\bar{c}) \in C$. Then, for each $i \in I$, we have $\pi_i(b) = \dashv^{\mathcal{A}}$, for $\mathcal{A}$ is both weakly $\diamond$-conjunctive and false-singular, as required. □

3.2.1. *Classical matrices and logics.* Fix any (possibly, secondary) unary $\wr$ and binary $\diamond$ connectives of Σ.

A Σ-matrix $\mathcal{A}$ is said to be *$\wr$-classical*, provided $A = \{\mathsf{f}, \mathsf{t}\}$, $D^{\mathcal{A}} = \{\mathsf{t}\}$, $\wr^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$ and $\wr^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$, in which case it is two-valued, truth-non-empty and both consistent and false-singular with $\dashv^{\mathcal{A}} = \mathsf{f}$ but not $\wr$-paraconsistent.

A Σ-logic is said to be *$\wr$-[sub]classical*, whenever it is defined by [resp., has a model being] a $\wr$-classical matrix, in which case $\wr$ is a subclassical negation for it. Then, a Σ-logic is said to be *inferentially $\wr$-classical*, whenever it is either $\wr$-classical or the purely inferential version of a $\wr$-classical Σ-logic.

Lemma 3.9. *Let $\mathcal{A}$ be a $\wr$-classical weakly $\diamond$-conjunctive Σ-matrix and $\mathcal{B}$ a (simple)consistent finitely-generated model of the logic of $\mathcal{A}$. Then, $\mathcal{A}$ is embeddable into $\Re(\mathcal{B})$ (resp., into $\mathcal{B}$).*

Proof. Put $\mathcal{E} \triangleq \Re(\mathcal{B})$ (resp., $\mathcal{E} \triangleq \mathcal{B}$). Then, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some $n \in \omega$, some n-tuple $\overline{\mathcal{C}}$ constituted by consistent submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case, by (2.5), $\mathcal{D}$ is consistent, and so $n \neq 0$. Then, by Lemma 3.8, $D \ni a \triangleq (n \times \{\mathsf{f}\})$, and so $D \ni \wr^{\mathfrak{D}} a = (n \times \{\mathsf{t}\})$. Hence, as $n \neq 0$, $e \triangleq \{\langle b, n \times \{b\}\rangle \mid b \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{E})$, and so Corollary 2.14, Example 3.2 and Lemma 3.4 complete the argument. □

Corollary 3.10. *Any $\sim$-classical weakly $\diamond$-conjunctive logic is maximal.*

Proof. Let $\mathcal{A}$ be a $\sim$-classical weakly $\diamond$-conjunctive matrix. Consider any consistent extension C' of the logic C of $\mathcal{A}$, in which case $x_0 \notin C'(\varnothing)$. Then, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, C'(\varnothing)\rangle$ is a model of C' (in particular, of C), and so is its consistent finitely generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_\Sigma, \mathrm{Fm}^1_\Sigma \cap C'(\varnothing)\rangle$, in view of (2.5). In this way, (2.5) and Lemma 3.9 complete the argument. □

Corollary 3.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\sim$-classical weakly $\diamond$-conjunctive Σ-matrices. Suppose $\mathcal{B}$ is a model of the logic of $\mathcal{A}$. Then, $\mathcal{B} = \mathcal{A}$. In particular, $\mathcal{A}$ and $\mathcal{B}$ are equal, whenever they define same logic.*

Proof. In that case, $\mathcal{B}$ is both simple, by Example 3.2 and Lemma 3.4, and finite (in particular, finitely generated). Hence, by Lemma 3.9, there is an embedding e of $\mathcal{A}$ into $\mathcal{B}$, in which case $(\mathcal{A}{\restriction}\varnothing) = (\mathcal{B}{\restriction}\varnothing)$ and $e \in \hom_{\mathrm{S}}(\mathcal{A}{\restriction}\varnothing, \mathcal{B}{\restriction}\varnothing)$. Then, by Example 3.2 and Lemma 3.7, e is diagonal, and so $\mathcal{A} = \mathcal{B}$, for $A = B$, as required. □

3.3. **Disjunctivity.** Fix any set A and any $\delta : A^2 \to A$. Given any $X, Y \subseteq A$, set $\delta(X,Y) \triangleq \delta[X \times Y]$. Then, a $Z \subseteq A$ is said to be *[weakly]δ-disjunctive*, provided, for all $a, b \in A$, it holds that $((\{a,b\} \cap Z) \neq \varnothing) \Leftrightarrow [\Rightarrow](\delta(a,b) \in Z)$, in which case, for all $X, Y \subseteq A$, we have $((X \subseteq Z)|(Y \subseteq Z)) \Leftrightarrow [\Rightarrow](\delta(X,Y) \subseteq Z)$. Next, a closure operator C over A is said to be *[weakly]δ-disjunctive*, provided, for all $a, b \in A$ and every $Z \subseteq A$, it holds that

$$C(Z \cup \delta(a,b))[\subseteq] = (C(Z \cup \{a\}) \cap C(Z \cup \{b\})), \tag{3.2}$$

in which case the following [resp., (3.3) and (3.4) alone, being equivalent to the weak δ-disjunctivity of C] clearly hold, by (3.2) with $Z = \varnothing$:

$$\delta(a,b) \in C(a), \tag{3.3}$$
$$\delta(a,b) \in C(b), \tag{3.4}$$
$$a \in C(\delta(a,a)), \tag{3.5}$$
$$\delta(b,a) \in C(\delta(a,b)), \tag{3.6}$$
$$C(\delta(\delta(a,b),c)) = C(\delta(a,\delta(b,c))), \tag{3.7}$$

for all $a, b, c \in A$.

Lemma 3.12. *Let C be a closure operator over A and $\mathcal{B}$ a closure basis of $\operatorname{img} C$. Suppose each element of $\mathcal{B}$ is δ-disjunctive. Then,*

$$(C(Z \cup X) \cap C(Z \cup Y)) = C(Z \cup \delta(X,Y)), \tag{3.8}$$

for all $X, Y, Z \subseteq A$. In particular, C is δ-disjunctive and the following holds:

$$\delta(C(X), a) \subseteq C(\delta(X,a)), \tag{3.9}$$

for all $(X \cup \{a\}) \subseteq A$.

Proof. First, for all $a \in A$, we have:

$$\begin{aligned}
&(a \in C(Z \cup X) \cap C(Z \cup Y))\\
\Leftrightarrow\ &\forall W \in \mathcal{B} : ((((Z \subseteq W) \& (X \subseteq W)) \Rightarrow (a \in W))\\
&\&(((Z \subseteq W) \& (Y \subseteq W)) \Rightarrow (a \in W)))\\
\Leftrightarrow\ &\forall W \in \mathcal{B} : (((Z \subseteq W) \& (X \subseteq W | Y \subseteq W)) \Rightarrow (a \in W))\\
\Leftrightarrow\ &\forall W \in \mathcal{B} : (((Z \subseteq W) \& (\delta(X,Y) \subseteq W)) \Rightarrow (a \in W))\\
\Leftrightarrow\ &(a \in C(Z \cup \delta(X,Y))),
\end{aligned}$$

in which case (3.8) holds, and so immediately does its particular case (3.2). Finally, applying (3.8) with $Z = \varnothing$ twice, we also get $\delta(C(X), a) \subseteq C(\delta(C(X), a)) = (C(C(X)) \cap C(a)) = (C(X) \cap C(a)) = C(\delta(X,a))$, in which case (3.9) holds, as required. ☐

Lemma 3.13. *Let C be a δ-disjunctive closure operator over A and $X \in (\operatorname{img} C)$. Then, X is δ-disjunctive iff it is pair-wise-meet-irreducible in $\operatorname{img} C$, and so it is finitely-meet-irreducible in $\operatorname{img} C$ iff it is δ-disjunctive and proper.*

Proof. First, assume X is not δ-disjunctive. Then, in view of (3.3) and (3.4), X is weakly δ-disjunctive, so there is some $\vec{a} \in (A \setminus X)^2$, in which case, for each $i \in 2$, it holds that $X \neq C(X \cup \{a_i\}) \in (\operatorname{img} C)$, such that $\delta(\vec{a}) \in X$. Therefore, by (3.2), we have $X = (\bigcap_{i \in 2} C(X \cup \{a_i\}))$. Hence, X is not pair-wise-meet-irreducible in $\operatorname{img} C$.

Conversely, assume X is not pair-wise-meet-irreducible in $\operatorname{img} C$. Then, there is some $\vec{Y} \in ((\operatorname{img} C) \setminus \{X\})^2$ such that $X = (\bigcap_{i \in 2} Y_i)$, in which case, for each $i \in 2$, $X \subsetneq Y_i$, so there is some $a_i \in (Y_i \setminus X) \neq \varnothing$. In this way, by (3.2), we have $\delta(\vec{a}) \in C(X \cup \delta(\vec{a})) = (\bigcap_{i \in 2} C(X \cup \{a_i\})) \subseteq (\bigcap_{i \in 2} Y_i) = X$. Thus, X is not δ-disjunctive, as required. ☐

3.3.1. *Disjunctive logics and matrices.* Fix any (possibly, secondary) binary connective $\veebar$ of Σ.

Remark 3.14. In view of (2.3) and (2.5), given two Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ such that there is a [surjective]strict homomorphism from $\mathcal{A}$ [on]to $\mathcal{B}$, $\mathcal{A}$ is (weakly)$\veebar$-disjunctive if[f] $\mathcal{B}$ is so. ☐

Corollary 3.15. *Let I be a finite set, $\overline{\mathcal{A}}$ an I-tuple of $\veebar$-disjunctive Σ-matrices and $\mathcal{B}$ a consistent $\veebar$-disjunctive subdirect product of $\overline{\mathcal{A}}$. Then, $(\pi_i \restriction B) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{A}_i)$, for some $i \in I$.*

Proof. Then, by Remark 3.14, $\mathcal{B} \triangleq \{B \cap \pi_i^{-1}[D^{\mathcal{A}_i}] \mid i \in I\}$ is a finite set of $\veebar^{\mathfrak{B}}$-disjunctive subsets of B. Let C be the closure operator over B dual to the closure system with basis $\mathcal{B}$. Then, $D^{\mathcal{B}} = (B \cap \bigcap \mathcal{B}) \in (\operatorname{img} C)$ is both $\veebar^{\mathfrak{B}}$-disjunctive and proper. Hence, by Lemmas 3.12 and 3.13, $D^{\mathcal{B}} \in \mathcal{B}$, as required. ☐

Corollary 3.16. *Let $\alpha \in \wp_{\infty \setminus 1}(\omega)$ and M a class of [non-]weakly $\veebar$-disjunctive Σ-matrices. Then, $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ is [non-]weakly $\veebar$-disjunctive[and holds (3.9)].*

Proof. The "weak" case is evident.[Conversely, for each $\mathcal{A} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$, $h^{-1}[D^{\mathcal{A}}]$ is $\veebar$-disjunctive, by Remark 3.14. Then, Lemma 3.12 completes the argument.] ☐

Corollary 3.17. *Let $\mathcal{A}$ be a false-singular Σ-matrix and C the logic of $\mathcal{A}$. Then, the following are equivalent:*

(i) *C is [non-]weakly $\veebar$-disjunctive;*
(ii) *$\mathcal{A}$ is [non-]weakly $\veebar$-disjunctive;*
(iii) *C holds both (3.3) and (3.4) [as well as (3.5)].*

Proof. First, (ii)⇒(i) is by Corollary 3.16. Next, (iii) is a particular case of (i). Finally, assume (iii) holds. Consider any $a, b \in A$. In case $(a/b) \in D^{\mathcal{A}}$, by (3.3)/(3.4), we have $(a \veebar^{\mathfrak{A}} b) \in D^{\mathcal{A}}$.[Now, assume $(\{a, b\} \cap D^{\mathcal{A}}) = \varnothing$. Then, $D^{\mathcal{A}} \not\ni a = b$. Therefore, by (3.5), we get $D^{\mathcal{A}} \not\ni (a \veebar^{\mathfrak{A}} a) = (a \veebar^{\mathfrak{A}} b)$.] Thus, (ii) holds, as required. □

Corollary 3.18. *Let C be an inductive Σ-logic. Then, the following are equivalent:*

(i) *C is $\veebar$-disjunctive;*
(ii) *$\operatorname{img} C$ has a basis consisting of $\veebar$-disjunctive sets;*
(iii) *(3.3), (3.5), (3.6) and (3.9) hold;*
(iv) *(3.3), (3.5), (3.6) hold and, for any axiomatization $\mathcal{C}$ of C, every $(\Gamma \vdash \phi) \in \mathrm{SI}_\Sigma(\mathcal{C})$ and each $\psi \in \mathrm{Fm}^\omega_\Sigma$, it holds that $(\phi \veebar \psi) \in C(\Gamma \veebar \psi)$.*

Proof. First, (i)⇒(ii) is by Remark 2.1 and Lemma 3.13. Next, (ii)⇒(iii) is by Lemma 3.12. Further, (iv) is a particular case of (iii). Then, the converse is proved by induction on the length of $\mathcal{C}$-derivations. Finally, assume (iii) holds, in which case (3.4) holds by (3.3) and (3.6), and so does the inclusion from left to right in (3.2), by (3.3) and (3.4). Conversely, consider any $\varphi \in (C(Z \cup \{\phi\}) \cap C(Z \cup \{\psi\}))$. Then, by (3.3), (3.6) and (3.9), we have $(\psi \veebar \varphi) \in C(Z \cup \{\phi \veebar \psi\})$. Likewise, by (3.3), (3.5) and (3.9), we also have $\varphi \in C(Z \cup \{\psi \veebar \varphi\})$. Hence, we eventually get $\varphi \in C(Z \cup \{\phi \veebar \psi\})$, in which case (3.2) holds, and so does (i), as required. □

Corollary 3.19. *Any axiomatic extension of an inductive $\veebar$-disjunctive Σ-logic is $\veebar$-disjunctive.*

Proof. By Corollary 3.18(i)⇔(iv) and (3.3). □

3.3.1.1. Disjunctive extensions of logics defined by finite classes of finite disjunctive matrices. Given a Σ-rule $\Gamma \vdash \phi$ and a Σ-formula ψ, put $((\Gamma \vdash \phi) \veebar \psi) \triangleq ((\Gamma \veebar \psi) \vdash (\phi \veebar \psi))$. (This notation is naturally extended to Σ-calculi member-wise.)

Lemma 3.20. *Let $\Gamma \vdash \phi$ be a Σ-rule and $\mathcal{A}$ a $\veebar$-disjunctive Σ-matrix. Then, $\mathcal{A} \in \mathrm{Mod}(\sigma_{+1}(\Gamma \vdash \phi) \veebar x_0)$ iff $\mathcal{A} \in \mathrm{Mod}(\Gamma \vdash \phi)$.*

Proof. The "if" part is by the strucuturality of $\mathrm{Cn}^\omega_{\mathcal{A}}$ and Corollary 3.16(3.9). Conversely, assume $\mathcal{A} \in \mathrm{Mod}(\sigma_{+1}(\Gamma \vdash \phi) \veebar x_0)$. Consider any $h \in \hom(\mathfrak{Fm}^\omega, \mathfrak{A})$ such that $h(\phi) \notin D^{\mathcal{A}}$. Let $g \in \hom(\mathfrak{Fm}^\omega, \mathfrak{A})$ extend $[x_0/h(\phi); x_{i+1}/h(x_i)]_{i \in \omega}$, in which case $(g \circ \sigma_{+1}) = h$, and so, by the $\veebar$-disjunctivity of $\mathcal{A}$, we have $g(\sigma_{+1}(\phi) \veebar x_0) = (h(\phi) \veebar^{\mathfrak{A}} h(\phi)) \notin D^{\mathcal{A}}$. Hence, there is some $\psi \in \Gamma$ such that $(h(\psi) \veebar^{\mathfrak{A}} h(\phi)) = g(\sigma_{+1}(\psi) \veebar x_0) \notin D^{\mathcal{A}}$, in which case, by the $\veebar$-disjunctivity of $\mathcal{A}$, we eventually get $h(\psi) \notin D^{\mathcal{A}}$, and so $\mathcal{A} \in \mathrm{Mod}(\Gamma \vdash \phi)$, as required. □

Theorem 3.21. *Let M be a finite class of finite $\veebar$-disjunctive matrices, C the logic of M and $\mathsf{K}^{[*]} \triangleq \mathbf{S}^{[*]}_*(\mathsf{M})$. Then, the following hold:*

(i) *the mappings*

$$\begin{aligned} C' &\mapsto (\mathrm{Mod}(C') \cap \mathsf{K}^{[*]}), \\ \mathsf{S} &\mapsto \mathrm{Cn}^\omega_{\mathsf{S}}. \end{aligned}$$

are inverse to one another dual isomorphisms between the lattice of all $\veebar$-disjunctive [non-pseudo-axiomatic]extensions of C and that of all relative equality-free first-order universal Horn model subclasses of $\mathsf{K}^{[*]}$;

(ii) *for any Σ-calculus $\mathcal{C}$, the following hold:*
 a) *the extension of C relatively axiomatized by $\mathcal{C}$, being $\veebar$-disjunctive [and non-pseudo-axiomatic], corresponds to the relative equality-free first-order universal Horn model subclass of $\mathsf{K}^{[*]}$ relatively axiomatized by $\mathcal{C}$;*
 b) *[providing $(\mathcal{C} \cap \mathrm{Fm}^\omega_\Sigma) \neq \varnothing$,]the relative equality-free universal fitst-order Horn model subclass of $\mathsf{K}^{[*]}$ relatively axiomatized by $\mathcal{C}$ corresponds to the $\veebar$-disjunctive [non-pseudo-axiomatic]extension of C relatively axiomatized by $(\mathcal{C} \cap \mathrm{Fm}^\omega_\Sigma) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}^\omega_\Sigma] \veebar x_0)$;*

(iii) *[providing every member of M is truth-non-empty,]relative equality-free first-order universal positive Horn model subclasses of $\mathsf{K}^{[*]}$ correspond exactly to [non-pseudo-axiomatic]axiomatic extensions of C, corresponding objects having same axiomatic relative axiomatizations;*

(iv) *for any $\mathsf{C} \subseteq \mathsf{K}^{[*]}$, $\mathbf{S}^{[*]}_*(\mathsf{C})$, being a relative equality-free first-order universal Horn model subclass of $\mathsf{K}^{[*]}$, corresponds to the logic of C.*

In particular, $\veebar$-disjunctive extensions of C are inductive.

Proof. (i) First, the fact that $(\mathrm{Mod}(\mathrm{Cn}^\omega_{\mathsf{S}}) \cap \mathsf{K}^{[*]}) = \mathsf{S}$, where S is a relative equality-free first-order universal Horn model subclass of $\mathsf{K}^{[*]}$, is immediate, while the fact that $\mathrm{Cn}^\omega_{\mathsf{S}}$ is a $\veebar$-disjunctive[and non-pseudo-axiomatic] extension of C is by (2.5), Remark 3.14 and Corollary 3.16[as well as Proposition 2.19]. Now, consider any $\veebar$-disjunctive [non-pseudo-axiomatic]extension C' of C. Then, we have the inductive $\veebar$-disjunctive [non-pseudo-axiomatic]extension C'' of C (for C is inductive) defined as follows: for every $Z \subseteq \mathrm{Fm}^\omega_\Sigma$, put $C''(Z) \triangleq (\bigcup C'[\wp_\omega(Z)])$. Consider any Σ-rule $\Gamma \vdash \varphi$ such that $\varphi \notin C''(\Gamma)$[and $\Gamma \neq \varnothing$]. Then, by Corollary 3.18(i)⇒(ii), there is some $\veebar$-disjunctive $X \in (\operatorname{img} C'') \subseteq (\operatorname{img} C)$ such that $\Gamma \subseteq X \not\ni \varphi$. Moreover, as Γ is finite, there is some $\alpha \in (\omega \setminus 1) \subseteq \wp_{\infty \setminus 1}(\omega)$ such that $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^\alpha_\Sigma$, in which case, in view of (2.2), $\Gamma \subseteq Y \triangleq (X \cap \mathrm{Fm}^\alpha_\Sigma) \in (\operatorname{img} \mathrm{Cn}^\alpha_{\mathsf{M}})$ is both $\veebar$-disjunctive[, non-empty] and proper, for $\varphi \in (\mathrm{Fm}^\alpha_\Sigma \setminus Y)$. Furthermore, by the structurality of C'', $\langle \mathfrak{Fm}^\omega_\Sigma, X \rangle$ is a model of C'', and so is its consistent [truth-non-empty]submatrix $\mathcal{D} \triangleq \langle \mathfrak{Fm}^\alpha_\Sigma, Y \rangle$, in view of (2.5). On the other hand, by Corollary 3.16, $\mathrm{Cn}^\alpha_{\mathsf{M}}$ is $\veebar$-disjunctive. Hence, by Lemma 3.13, Y is finitely-meet-irreducible in $\operatorname{img} \mathrm{Cn}^\alpha_{\mathsf{M}}$. And what is more, since both α, M and all members of M are finite, $\mathcal{B} \triangleq \{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})\}$ is a finite basis of $\operatorname{img} \mathrm{Cn}^\alpha_{\mathsf{M}}$. Therefore, $Y \in \mathcal{B}$, in which case there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$ such that $Y = h^{-1}[D^{\mathcal{A}}]$, and so h is a surjective strict homomorphism from $\mathcal{D}$ onto $\mathcal{B} \triangleq (\mathcal{A} {\restriction} (\operatorname{img} h))$. In this way, by (2.5), $\mathcal{B}$ is a consistent [truth-non-empty]model of C''.

Finally, as $\Gamma \subseteq Y = h^{-1}[D^{\mathcal{B}}] \not\ni \varphi$, we conclude that $\Gamma \vdash \varphi$ is not true in $\mathcal{B} \in \mathsf{S} \triangleq (\mathrm{Mod}(C'') \cap \mathsf{K}^{[*]})$ under h. Thus, since both S and all members of it are finite, in which case $C''' \triangleq \mathrm{Cn}^{\omega}_{\mathsf{S}}$ is inductive[and non-pseudo-axiomatic, by Proposition 2.19], and so $C'' = C'''$, by Proposition 2.18, we eventually get $C' = C''' = C''$, as required, for, in that case, C', being inductive, is axiomatized by a Σ-calculus.

(ii) Consider any Σ-calculus $\mathcal{C}$. Then:

a) is immediate, in view of (2.5), due to which $\mathsf{K} \subseteq \mathrm{Mod}(C)$.

b) Let C' be the extension of C relatively axiomatized by $\mathcal{C}' \triangleq ((\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}^{\omega}_{\Sigma}] \veebar x_0))$. Then, C being inductive, is axiomatized by a Σ-calculus $\mathcal{C}''$, in which case C' is axiomatized by $\mathcal{C}'' \cup \mathcal{C}'$, and so is inductive. Moreover, by Corollary 3.16, C is $\veebar$-disjunctive, in which case C', being an extension of C, inherits (3.3), (3.5), (3.6) and (3.7) held by C. Then, we prove the $\veebar$-disjunctivity of C' with applying Corollary 3.18(i)⇔(iv) to both C and C'. For consider any Σ-substitution σ and any $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$. First, consider any $\phi \in (\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma})$. Then, by the structurality of C' and (3.3), we have $(\sigma(\phi) \veebar \psi) \in C'(\varnothing)$. Now, consider any $(\Gamma \vdash \phi) \in (\mathcal{C} \setminus \mathrm{Fm}^{\omega}_{\Sigma})$. Let ς be the Σ-substitution extending $(\sigma{\restriction}(V_\omega \setminus V_1)) \cup [x_0/(\sigma(x_0) \veebar \psi)]$, in which case $(\varsigma \circ \sigma_{+1}) = (\sigma \circ \sigma_{+1})$, and so, by (3.7) and the structurality of C', we eventually get $(\sigma[\sigma_{+1}[\Gamma] \veebar x_0] \veebar \psi) = ((\varsigma[\sigma_{+1}[\Gamma]] \veebar \sigma(x_0)) \veebar \psi) \vdash_{C'} (\varsigma[\sigma_{+1}[\Gamma]] \veebar (\sigma(x_0) \veebar \psi)) = \varsigma[\sigma_{+1}[\Gamma] \veebar x_0] \vdash_{C'} \varsigma(\sigma_{+1}(\varphi) \veebar x_0) = (\varsigma(\sigma_{+1}(\varphi)) \veebar (\sigma(x_0) \veebar \psi)) \vdash_{C'} ((\varsigma(\sigma_{+1}(\varphi)) \veebar \sigma(x_0)) \veebar \psi) = (\sigma(\sigma_{+1}(\varphi) \veebar x_0) \veebar \psi)$. In this way, C' is disjunctive.[And what is more, since $(\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \neq \varnothing$, C' is not purely inferential, and so is non-pseudo-axiomatic.] Then, **a)** and Lemma 3.20 complete the argument.

(iii) is by (ii) and Corollary 3.19[as well as Proposition 2.19, due to which C, being the axiomatic extension of C relatively axiomatized by $\varnothing$, is non-pseudo-axiomatic].

(iv) is by (2.5). ☐

As it is demonstrated by Theorem 4.44 below, $(\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}^{\omega}_{\Sigma}] \veebar x_0)$ cannot be replaced by $\mathcal{C}$ in the item (ii)**b)** of Theorem 3.21, and so the reservations "positive" and "axiomatic" cannot be omitted in its item (iii).

Lemma 3.22. *Let M be a finite class of finite $\veebar$-disjunctive matrices, C the logic of M and $\mathcal{A}$ a (simple)consistent{ truth-non-empty} finitely-generated $\veebar$-disjunctive model of C.([Suppose every member of $\mathbf{S}^{\{*\}}_{*}(\mathsf{M})$ is simple.]) Then, $\mathcal{A} \in \mathbf{H}^{-1}(\mathbf{H}(\mathbf{H}^{-1}(\mathbf{S}^{\{*\}}_{*}(\mathsf{M}))))$ (resp., $\mathcal{A} \in \mathbf{H}(\mathbf{H}^{-1}(\mathbf{S}^{\{*\}}_{*}(\mathsf{M})))$ [resp., $\mathcal{A} \in \mathbf{I}(\mathbf{S}^{\{*\}}_{*}(\mathsf{M}))$]). In particular, $\mathcal{A} \in \mathbf{S}^{\{*\}}_{*}(\mathsf{M})$, whenever $(\mathsf{M} \cup \{\mathcal{A}\}) \subseteq \mathbf{S}(\mathcal{B})$, where $\mathcal{B}$ is any Σ-matrix with unary unitary equality determinant.*

Proof. Set $\mathcal{D} \triangleq \Re(\mathcal{A})$ (resp., $\mathcal{D} \triangleq \mathcal{A}$). Then, by Lemma 2.20, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}^{\{*\}}_{*}(\mathsf{M})^I$, in which case, by Remark 3.14, every member of $\mathrm{img}\,\overline{\mathcal{C}}$ is $\veebar$-disjunctive, some subdirect product $\mathcal{E}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{D})$, in which case, by (2.5) and Remark 3.14, $\mathcal{E}$ is consistent and $\veebar$-disjunctive, and so, by Corollary 3.15, there is some $i \in I$ such that $h \triangleq (\pi_i{\restriction}D) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{C}_i)$.([Moreover, in that case, by Proposition 2.16, we have $(\ker g) = \partial(\mathcal{E}) = (\ker h)$. Therefore, by Proposition 2.15, $h \circ g^{-1}$ is an isomorphism from $\mathcal{D} = \mathcal{A}$ onto $\mathcal{C}_i$.]) Finally, Lemmas 3.4, 3.6 and 3.7 complete the argument. ☐

By (2.5), Remark 3.14 and Lemma 3.22, we immediately have:

Corollary 3.23. *Let M and $\mathsf{K}^{[*]}$ be as in Theorem 3.21, $\mathsf{S} \subseteq \mathsf{K}^{[*]}$ and C' the logic of S.(Suppose $\mathsf{M} = \{\mathcal{A}\}$, where $\mathcal{A}$ is a Σ-matrix with unary unitary equality determinant.) Then, $(\mathrm{Mod}(C') \cap \mathsf{K}^{[*]}) = (\mathbf{H}^{-1}(\mathbf{H}(\mathbf{H}^{-1}(\mathbf{S}^{[*]}_{*}(\mathsf{S})))) \cap \mathsf{K}^{[*]})(= (\mathbf{S}^{[*]}_{*}(\mathsf{S})))$.*

Theorem 3.24. *Let M, C and $\mathsf{K}^{[*]}$ be as in Theorem 3.21.(Suppose $\mathsf{M} = \{\mathcal{A}\}$, where $\mathcal{A}$ is a Σ-matrix with unary unitary equality determinant.) Then, the set of all relative first-order equality-free Horn model subclasses of $\mathsf{K}^{[*]}$ is a closure system over $\mathsf{K}^{[*]}$. Moreover, for any $\mathsf{S} \subseteq \mathsf{K}^{[*]}$, the logic of S is the $\veebar$-disjunctive [non-pseudo-axiomatic]extension of C corresponding to $(\mathbf{H}^{-1}(\mathbf{H}(\mathbf{H}^{-1}(\mathbf{S}^{[*]}_{*}(\mathsf{S})))) \cap \mathsf{K}^{[*]})(= \mathbf{S}^{[*]}_{*}(\mathsf{S}))$. In particular, the complete lattice joins of the closure system involved are exactly unions, in which case the lattice under consideration is distributive, and so is that of all $\veebar$-disjunctive [non-pseudo-axiomatic]extensions of C.(Moreover, relative first-order equality-free Horn model subclasses of $\mathsf{K}^{[*]}$ are exactly lower cones of it, under identification of its members with carriers of their underlying algebras.)*

Proof. We use (2.5), Remark 3.14, Corollaries 3.16, 3.23 and Theorem 3.21[as well as Proposition 2.19] tacitly. Consider any set I and any I-tuple $\overline{\mathcal{C}}$, constituted by Σ-calculi. For every $i \in I$, put $\mathsf{S}_i \triangleq (\mathrm{Mod}(\mathcal{C}_i) \cap \mathsf{K}^{[*]})$ and $C_i \triangleq \mathrm{Cn}^{\omega}_{\mathsf{S}_i}$. Then, we clearly have $(\mathrm{Mod}(\bigcup_{i \in I} \mathcal{C}_i) \cap \mathsf{K}^{[*]}) = (\mathsf{K}^{[*]} \cap \bigcap_{i \in I} \mathsf{S}_i)$. And what is more, the logic C' of $\mathsf{S} \triangleq (\bigcup_{i \in I} \mathsf{S}_i) \subseteq \mathsf{K}^{[*]}$ is a $\veebar$-disjunctive [non-pseudo-axiomatic]extension of C and is equal to $\bigcap_{i \in I} C_i$, in which case it is the meet of $\{C_i \mid i \in I\}$ in the lattice of all $\veebar$-disjunctive [non-pseudo-axiomatic]extensions of C, and so the join of $\{\mathsf{S}_i \mid i \in I\}$ is equal to $(\mathrm{Mod}(C') \cap \mathsf{K}^{[*]}) = (\mathbf{H}^{-1}(\mathbf{H}(\mathbf{H}^{-1}(\mathbf{S}^{[*]}_{*}(\mathsf{S})))) \cap \mathsf{K}^{[*]}) = (\mathsf{K}^{[*]} \cap \bigcup_{i \in I} \mathbf{H}^{-1}(\mathbf{H}(\mathbf{H}^{-1}(\mathbf{S}^{[*]}_{*}(\mathsf{S}_i))))) = (\bigcup_{i \in I}(\mathbf{H}^{-1}(\mathbf{H}(\mathbf{H}^{-1}(\mathbf{S}^{[*]}_{*}(\mathsf{S}_i)))) \cap \mathsf{K}^{[*]})) = (\bigcup_{i \in I}(\mathrm{Mod}(C_i) \cap \mathsf{K}^{[*]})) = (\bigcup_{i \in I} \mathsf{S}_i)$, as required. ☐

3.4. **Distributive and De Morgan lattices.** Let $\Sigma^{+}_{[01]} \triangleq (\{\wedge, \vee\}[\cup\{\bot, \top\}])$ be the [bounded]lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction)[and nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)].

Lemma 3.25. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be lattices, a a unit/zero of $\mathfrak{A}$, b a unit/zero of $\mathfrak{B}$ and $h \in \hom(\mathfrak{A}, \mathfrak{B})$. Suppose $h[A] = B$. Then, $h(a) = b$.*

Proof. Then, there is some $c \in A$ such that $h(c) = b$, in which case $(a(\vee/\wedge)^{\mathfrak{A}}c) = a$, and so $h(a) = (h(a)(\vee/\wedge)^{\mathfrak{B}}b) = b$, as required. ☐

Given any $\Sigma \supseteq \Sigma^{+}$, $\phi \lessapprox \psi$ is used as an abbreviation for $(\phi \wedge \psi) \approx \phi$, where $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$. Then, any Σ-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\Sigma^{+}$ is a lattice is well-known to be congruence-distributive (cf., e.g., Example 2 on p. 12 of [10]), the partial ordering of $\mathfrak{A}{\restriction}\Sigma^{+}$ being denoted by $\leqslant^{\mathfrak{A}}$.

Given any $n \in (\omega \setminus 1)$, by $\mathfrak{D}_{n[,01]}$ we denote the [bounded] distributive lattice given by the chain n, viz., the $\Sigma^+_{[01]}$-algebra with carrier n such that $(\wedge/\vee)^{\mathfrak{D}_n} \triangleq ((\min/\max)\upharpoonright n^2)[$ and $(\bot/\top)^{\mathfrak{D}_n} \triangleq (0/(n-1))]$.

Here, we deal with the signature $\Sigma_{0[1]} \triangleq (\Sigma^+_{[01]} \cup \{\sim\})$ with unary $\sim$ (weak negation).

A *[bounded]De Morgan lattice* (cf. [2], [16], [17]) is any $\Sigma_{0[1]}$-algebra $\mathfrak{A}$ such that $\mathfrak{A}\upharpoonright\Sigma^+_{[01]}$ is a [bounded]distributive lattice (cf. [2]) and the following Σ_0-identities are true in $\mathfrak{A}$:

(3.10) $$\sim\sim x_0 \approx x_0,$$

(3.11) $$\sim(x_0 \vee x_1) \approx \sim x_0 \wedge \sim x_1,$$

(3.12) $$\sim(x_0 \wedge x_1) \approx \sim x_0 \vee \sim x_1,$$

the variety of all them being denoted by [B]DML. Then, a *[bounded]Kleene lattice* is any [bounded]De Morgan lattice satisfying the Σ_0-identity:

(3.13) $$(x_0 \wedge \sim x_0) \lessapprox (x_1 \vee \sim x_1),$$

the variety of all them being denoted by [B]KL. Next, a *[bounded]Boolean lattice* is any [bounded]De Morgan lattice satisfying the Σ_0-identity:

(3.14) $$x_0 \lessapprox (x_1 \vee \sim x_1),$$

the variety of all them being denoted by [B]BL $\subseteq$ [B]KL.[3]

By $\mathfrak{DM}_{4[,01]}$ we denote the [bounded]De Morgan lattice such that $(\mathfrak{DM}_{4[,01]}\upharpoonright\Sigma^+_{[01]}) \triangleq \mathfrak{D}^2_{2[,01]}$ and $\sim^{\mathfrak{DM}_{4[,01]}}\vec{a} \triangleq \langle 1-a_{1-i}\rangle_{i\in 2}$, for all $\vec{a} \in 2^2$.

Remark 3.26. Since any non-empty proper prime filter of $\mathfrak{D}^2_{2[,01]}$ contains t but not f, and so contains b iff it does not contain n, $F_j \triangleq (2^2 \cap \pi_j^{-1}[\{1\}])$, where $j \in 2$, are exactly all non-empty proper prime filters of $\mathfrak{D}^2_{2[,01]}$, in which case $\langle \mathfrak{DM}_{4[,01]}, F_j\rangle$ is both $\wedge$-conjunctive and $\vee$-disjunctive, while, by Example 3.3 with $\vec{k} = \Delta_2$ and $\wr = \sim$, we see that $\Upsilon_\sim$ is a unary unitary equality determinant for it. □

Recall also the following rather well-known (within Universal Algebra) fact:

Lemma 3.27. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{DM}_4$. Then,* $\mathrm{Con}(\mathfrak{B}) \subseteq \{\Delta_B, B^2\}$. *In particular, $\mathfrak{B}$ is simple iff $|B| > 1$.*

Proof. Consider any $\theta \in (\mathrm{Con}(\mathfrak{B}) \setminus \{\Delta_B\})$. Take any $\vec{a} \in (\theta \setminus \Delta_B) \neq \varnothing$. Consider the following exhaustive cases:

1. $\mathrm{img}\,\vec{a} \subseteq \{\mathsf{f}, \mathsf{t}\}$.
   Then, $\mathrm{img}\,\vec{a} = \{\mathsf{f}, \mathsf{t}\}$, for $a_0 \neq a_1$, and so $\mathsf{f}\ \theta\ \mathsf{t}$.
2. $\mathrm{img}\,\vec{a} \subseteq \{\mathsf{n}, \mathsf{b}\}$.
   Then, $\mathrm{img}\,\vec{a} = \{\mathsf{n}, \mathsf{b}\}$, for $a_0 \neq a_1$, in which case $\mathsf{n}\ \theta\ \mathsf{b}$, and so $\mathsf{f} = (\mathsf{n} \wedge^{\mathfrak{B}} \mathsf{b})\ \theta\ (\mathsf{n} \wedge^{\mathfrak{B}} \mathsf{n}) = \mathsf{n} = (\mathsf{n} \vee^{\mathfrak{B}} \mathsf{n})\ \theta\ (\mathsf{n} \vee^{\mathfrak{B}} \mathsf{b}) = \mathsf{t}$.
3. $a_i \in \{\mathsf{f}, \mathsf{t}\}$, while $a_{1-i} \in \{\mathsf{b}, \mathsf{n}\}$, for some $i \in 2$.
   Then, $a_i\ \theta\ a_{1-i} = \sim^{\mathfrak{B}} a_{1-i}\ \theta \sim^{\mathfrak{B}} a_i$, and so $\mathsf{f}\ \theta\ \mathsf{t}$, because $\sim^{\mathfrak{B}}\langle j, j\rangle = \langle 1-j, 1-j\rangle$, for all $j \in 2$.

Thus, in any case, we have $\mathsf{f}\ \theta\ \mathsf{t}$. Therefore, for every $c \in B$, we get $c = (\mathsf{f} \vee^{\mathfrak{B}} c)\ \theta\ (\mathsf{t} \vee^{\mathfrak{B}} c) = \mathsf{t}$. Hence, $\theta = B^2$, as required. □

Given any $n \in (\omega \setminus 1)$, by $\mathfrak{K}_{n[,01]}$ we denote the chain [bounded]Kleene lattice such that $(\mathfrak{K}_{n[,01]}\upharpoonright\Sigma^+_{[01]}) \triangleq \mathfrak{D}_{n[,01]}$ and $\sim^{\mathfrak{K}_{n[,01]}} i \triangleq (n-1-i)$, for all $i \in n$, $\mathfrak{K}_{2[,01]}$ being a [bounded]Boolean lattice. Then, $e_n \triangleq \{\langle 0,0\rangle, \langle 1, n-1\rangle\} \in \hom(\mathfrak{K}_{2[,01]}, \mathfrak{K}_{n[,01]})$ is injective. Moreover, for any $n \in (\omega \setminus 3)$, $\hbar_n \triangleq (\{\langle 0,0\rangle, \langle n-1, 2\rangle\} \cup (((n-1) \setminus 1) \times \{1\})) \in \hom(\mathfrak{K}_{n[,01]}, \mathfrak{K}_{3[,01]})$ is surjective. Finally, for any $i \in 2$, $e_{3,i} \triangleq \{\langle 0, \mathsf{f}\rangle, \langle 2, \mathsf{t}\rangle, \langle 1, \langle i, 1-i\rangle\rangle\} \in \hom(\mathfrak{K}_{3[,01]}, \mathfrak{DM}_{4[,01]})$ is injective.

4. Four-valued expansions of Belnap's logic

Fix any language $\Sigma \supseteq \Sigma_{0[1]}$ such that either $\Sigma \supseteq \Sigma_{01}$ or $(\Sigma \cap \Sigma_{01}) = \Sigma_0$ and any Σ-algebra $\mathfrak{A}$ such that $(\mathfrak{A}\upharpoonright\Sigma_{0[1]}) = \mathfrak{DM}_{4[,01]}$. Put $\mathcal{A} \triangleq \langle \mathfrak{A}, 2^2 \cap \pi_0^{-1}[\{1\}]\rangle$, $\overleftarrow{\mathcal{A}} \triangleq \langle \mathfrak{A}, 2^2 \cap \pi_1^{-1}[\{1\}]\rangle$ and $\overrightarrow{\mathcal{A}} \triangleq \langle \mathfrak{A}, \{\mathsf{t}\}\rangle$. Since [bounded]Belnap's four-valued logic (cf. [3]), denoted by $C_{[\mathrm{B}]\mathrm{B}}$ from now on, is defined by $\mathcal{DM}_{4[,01]} \triangleq (\mathcal{A}\upharpoonright\Sigma_{0[1]})$ (cf. [13]),[4] the logic C of $\mathcal{A}$ is a four-valued expansion of $C_{[\mathrm{B}]\mathrm{B}}$. We start our study from marking its framework.

4.1. Characteristic matrix expansions.

Lemma 4.1. *Let Σ' be an algebraic signature, $\wr$ a (possibly, secondary) unary connective of Σ', $\mathcal{A}'$ a Σ'-matrix, I a set, $\overline{\mathcal{D}}$ an I-tuple constituted by submatrices of $\mathcal{A}'$, $\mathcal{E}$ a submatrix of $\prod_{i\in I}\mathcal{D}_i$ and $a \in D^{\mathcal{E}}$. Suppose $\wr^{\mathfrak{E}} a \in D^{\mathcal{E}}$. Then, $a \in (D^{\mathcal{A}'} \cap (\wr^{\mathfrak{A}'})^{-1}[D^{\mathcal{A}'}])^I$.*

Proof. Then, for each $i \in I$, both $\pi_i(a) \in D^{\mathcal{A}'}$ and $\wr^{\mathfrak{A}'}\pi_i(a) = \pi_i(\wr^{\mathfrak{E}} a) \in D^{\mathcal{A}'}$, as required. □

Next, a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be *regular*, provided, for every $\varsigma \in \Sigma$ of arity $m \in \omega$, $\varsigma^{\mathfrak{B}} : B^m \to B$ is *regular*, i.e., monotonic with respect to the *information* partial ordering $\sqsubseteq$ on A defined by $(\vec{a} \sqsubseteq \vec{b}) \overset{\text{def}}{\Longleftrightarrow} ((a_0 \leqslant b_0)\&(b_1 \leqslant a_1))$, for all $\vec{a}, \vec{b} \in A$, in the sense that, for all $\bar{a}, \bar{b} \in B^m$, $\varsigma^{\mathfrak{B}}(\bar{a}) \sqsubseteq \varsigma^{\mathfrak{B}}(\bar{b})$, whenever $a_i \sqsubseteq b_i$, for each $i \in m$. (Clearly, every subalgebra of $\mathfrak{DM}_{4[,01]}$ is regular.) Likewise, $\mathfrak{B}$ is said to be *b-idempotent*, where $b \in B$, provided, for every $\varsigma \in \Sigma$ of arity $m \in \omega$, $\varsigma^{\mathfrak{B}} : B^m \to B$ is *b-idempotent* in the sense that $\varsigma^{\mathfrak{B}}(m \times \{b\}) = b$. (Clearly, $\mathfrak{B}$ is b-idempotent iff $\{b\}$ forms a subalgebra of it.) Finally, $\mathfrak{B}$ is said to be *specular*, whenever $(\mu\upharpoonright B) \in \hom(\mathfrak{B}, \mathfrak{A})$. (Clearly, $\mathfrak{DM}_{4[,01]}$ is specular.)

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[3]According to [2], "Boolean/Kleene/De Morgan algebra" traditionally stands for "bounded Boolean/Kleene/De Morgan lattice".

[4]This equally ensues from Theorem 4.53(x)⇒(v) below, (2.5), the $\wedge$-conjuctivity (cf. Remark 3.26 with $j = 0$) and the finiteness (and so the inductivity of the logic) of $\mathcal{DM}_{4[,01]}$ as well as the fact that $\mathcal{DM}_4\upharpoonright\{\mathsf{n}\}$ is truth-empty, while $\mu \in \hom(\mathfrak{DM}_{4[,01]}, \mathfrak{DM}_{4[,01]})$.

Lemma 4.2. *Let I be a set, $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, $\mathcal{B}$ a Σ-matrix and e an embedding of $\mathcal{B}$ into $\prod_{i\in I} \mathcal{C}_i$. Suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, $\{I \times \{a\} \mid a \in \{\mathsf{f},\mathsf{t}\}\} \subseteq e[B]$ and, for each $i \in I$, $\{\mathsf{f},\mathsf{b},\mathsf{t}\} \cup C_i$ forms a regular subalgebra of $\mathfrak{A}$ and either $\mathsf{n} \notin C_i$ or $\mathfrak{A}{\upharpoonright}\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is specular. Then, $(B \dotplus 2) \triangleq ((B \times \{\mathsf{b}\}) \cup \{\langle e^{-1}(I \times \{\mathsf{f}\}),\mathsf{f}\rangle, \langle e^{-1}(I \times \{\mathsf{t}\}),\mathsf{t}\rangle\})$ forms a subalgebra of $\mathfrak{B} \times (\mathfrak{A}{\upharpoonright}\{\mathsf{f},\mathsf{b},\mathsf{t}\})$, in which case $\pi_0{\upharpoonright}(B \dotplus 2)$ is a surjective strict homomorphism from $(\mathcal{B} \dotplus 2) \triangleq ((\mathcal{B} \times (\mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{b},\mathsf{t}\})){\upharpoonright}(B \dotplus 2))$ onto $\mathcal{B}$.*

Proof. Consider any $\varsigma \in \Sigma$ of arity $n \in \omega$ and any $\bar{b} \in (B \dotplus 2)^n$. In case $\varsigma^{\mathfrak{A}}(\bar{a}) = \mathsf{b}$, where $\bar{a} \triangleq (\pi_1 \circ \bar{b})$, we clearly have $\varsigma^{\mathfrak{B}\times\mathfrak{A}}(\bar{b}) \in (B \times \{\mathsf{b}\}) \subseteq (B \dotplus 2)$. Otherwise, since $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, we have $\varsigma^{\mathfrak{A}}(\bar{a}) \in \{\mathsf{f},\mathsf{t}\}$. Put $N \triangleq \{k \in n \mid a_k = \mathsf{b}\}$. Consider any $i \in I$. Put $\bar{c} \triangleq (\pi_i \circ e \circ \pi_0 \circ \bar{b})$. Then, for every $j \in (n \setminus N)$, it holds that $C_i \ni c_j = a_j \in \{\mathsf{f},\mathsf{t}\}$. Hence, $c_j \sqsubseteq a_j$, for all $j \in n$. Therefore, by the regularity of $\mathfrak{A}{\upharpoonright}(\{\mathsf{f},\mathsf{b},\mathsf{t}\} \cup C_i)$, we have $\varsigma^{\mathfrak{A}}(\bar{c}) \sqsubseteq \varsigma^{\mathfrak{A}}(\bar{a})$. Consider the following complementary cases:

(1) $\mathsf{n} \in C_i$.
Then, $C_i \ni \mu(a_j) \sqsubseteq c_j$, for all $j \in n$. Therefore, as, in that case, $\mathfrak{A}{\upharpoonright}\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is specular, by the regularity of $\mathfrak{A}{\upharpoonright}(\{\mathsf{f},\mathsf{b},\mathsf{t}\} \cup C_i)$, we have $\varsigma^{\mathfrak{A}}(\bar{a}) = \mu(\varsigma^{\mathfrak{A}}(\bar{a})) = \varsigma^{\mathfrak{A}}(\mu \circ \bar{a}) \sqsubseteq \varsigma^{\mathfrak{A}}(\bar{c})$, and so we get $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$.

(2) $\mathsf{n} \notin C_i$.
Then, $\varsigma^{\mathfrak{A}}(\bar{c}) \in C_i \subseteq \{\mathsf{f},\mathsf{b},\mathsf{t}\}$. Therefore, since both f and t are minimal elements of the poset $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ ordered by $\sqsubseteq$, we get $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$.

Thus, in any case, we have $\varsigma^{\mathfrak{A}}(\bar{c}) = \varsigma^{\mathfrak{A}}(\bar{a})$. and so, by the injectivity of e, we get $\varsigma^{\mathfrak{B}\times\mathfrak{A}}(\bar{b}) \in \{\langle e^{-1}(I \times \{\mathsf{f}\}),\mathsf{f}\rangle, \langle e^{-1}(I \times \{\mathsf{t}\}),\mathsf{t}\rangle\}$ $\subseteq (B \dotplus 2)$, as required. □

Lemma 4.3. *Let $\mathcal{B}$ be a model of C. Suppose either $\mathfrak{A}$ is b-idempotent or both $\mathfrak{A}$ is regular and $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{0[1]}$), while $\mathcal{B}$ is not a model of the rule:*

$$\{x_0, {\sim}x_0\} \vdash (x_1 \vee {\sim}x_1). \tag{4.1}$$

Then, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to $\Re(\mathcal{D})$.

Proof. In that case, there are some $a, b \in B$ such that (4.1) is not true in $\mathcal{B}$ under $[x_0/a, x_1/b]$. Then, in view of (2.5), the submatrix $\mathcal{E}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a finitely-generated model of C, in which (4.1) is not true under $[x_0/a, x_1/b]$. Hence, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some set J, some J-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{F}$ of $\overline{\mathcal{C}}$, in which case $(\mathfrak{F}{\upharpoonright}\Sigma_0) \in \mathsf{DML}$, for $\mathsf{DML} \ni \mathfrak{DM}_4$ is a variety, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}, \Re(\mathcal{E}))$, in which case, by (2.5), $\mathcal{F}$ is a model of C, in which case it is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.26 with $j = 0$), but is not a model of (4.1), in which case there are some $c, d \in F$ such that $\{c, {\sim}^{\mathfrak{F}}c\} \subseteq D^{\mathcal{F}} \not\ni d \geqslant^{\mathfrak{F}} {\sim}^{\mathfrak{F}}d$. Then, by Lemma 4.1, $c = (I \times \{\mathsf{b}\})$, in which case ${\sim}^{\mathfrak{F}}c = c$, and so $(F \setminus D^{\mathcal{F}}) \ni e \triangleq ((c \wedge^{\mathfrak{F}} d) \vee^{\mathfrak{F}} {\sim}^{\mathfrak{F}}d) = {\sim}^{\mathfrak{F}}e \leqslant^{\mathfrak{F}} d$. Hence, $e \in \{\mathsf{b},\mathsf{n}\}^J$, while $K \triangleq \{i \in J \mid \pi_i(e) = \mathsf{n}\} \neq \varnothing$. Given any $\bar{a} \in A^2$, set $(a_0|a_1) \triangleq ((K \times \{a_0\}) \cup ((J \setminus K) \times \{a_1\}))$. In this way, we have:

$$F \ni c = (\mathsf{b}|\mathsf{b}), \tag{4.2}$$

$$F \ni e = (\mathsf{n}|\mathsf{b}), \tag{4.3}$$

$$F \ni (c \wedge^{\mathfrak{F}} e) = (\mathsf{f}|\mathsf{b}), \tag{4.4}$$

$$F \ni (c \vee^{\mathfrak{F}} e) = (\mathsf{t}|\mathsf{b}). \tag{4.5}$$

Consider the following complementary cases:

(1) either $\mathfrak{A}$ is b-idempotent or $K = J$.
Then, $f \triangleq \{\langle x, (x|\mathsf{b})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{F}$, in which case $g' \triangleq (g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A}, \Re(\mathcal{E}))$, and so, by Corollary 2.14, Lemma 3.4 and Remark 3.26 with $j = 0$, g' is injective. In this way, g' is an isomorphism from $\mathcal{A}$ onto the submatrix $\mathcal{G} \triangleq (\Re(\mathcal{E}){\upharpoonright}(\operatorname{img} g'))$ of $\Re(\mathcal{E})$, and so $h \triangleq g'^{-1} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$.

(2) $\mathfrak{A}$ is not b-idempotent and $K \neq J$.
Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\phi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{f}$ and $\psi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{t}$, where $\phi \triangleq (x_0 \wedge (\varphi \wedge {\sim}\varphi))$ and $\psi \triangleq (x_0 \vee (\varphi \vee {\sim}\varphi))$, and so, by (4.2), we get:

$$F \ni \phi^{\mathfrak{F}}(c) = (\mathsf{f}|\mathsf{f}), \tag{4.6}$$

$$F \ni \psi^{\mathfrak{F}}(c) = (\mathsf{t}|\mathsf{t}). \tag{4.7}$$

Moreover, in that case, both $\mathfrak{A}$ is regular and $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$. And what is more, $e' \triangleq \{\langle a', \langle a'\rangle\rangle\}$ is an embedding of $\mathcal{A}$ into $\mathcal{A}^1$ such that $\{1 \times \{x\} \mid x \in \{\mathsf{f},\mathsf{t}\}\} = e'[\{\mathsf{f},\mathsf{t}\}] \subseteq e'[A]$. In this way, Lemma 4.2 with $I = 1$ and both $\mathcal{A}$ and e' instead of $\mathcal{B}$ and e, respectively, used tacitly throughout the rest of the proof, is well-applicable to $\mathcal{A}$. Then, since $K \neq \varnothing \neq (J \setminus K)$, by (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7), we see that $f \triangleq \{\langle\langle x, y\rangle, (x|y)\rangle \mid \langle x, y\rangle \in (A \dotplus 2)\}$ is an embedding of $\mathcal{H} \triangleq (\mathcal{A} \dotplus 2)$ into $\mathcal{F}$, while $h' \triangleq (\pi_0{\upharpoonright}(A \dotplus 2)) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H}, \mathcal{A})$. Then, $g' \triangleq (g \circ f) \in \hom_{\mathrm{S}}(\mathcal{H}, \Re(\mathcal{E}))$, and so g' is a surjective strict homomorphism from $\mathcal{H}$ onto the submatrix $\mathcal{G} \triangleq (\Re(\mathcal{E}){\upharpoonright}(\operatorname{img} g'))$ of $\Re(\mathcal{E})$. And what is more, by Lemma 3.4 and Remark 3.26 with $j = 0$, $\mathcal{A}$ is simple. Hence, by Corollary 2.13 and Proposition 2.16, we get $(\ker g') \subseteq \partial(\mathcal{H}) = (\ker h')$. Therefore, by Proposition 2.15, $h \triangleq (h' \circ g'^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$.

Thus, in any case, there are some submatrix $\mathcal{G}$ of $\mathcal{E}/\theta$, where $\theta \triangleq \partial(\mathcal{E})$, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. Then, $\mathcal{D} \triangleq (\mathcal{E}{\upharpoonright}\nu_\theta^{-1}[G])$, being a submatrix of $\mathcal{E}$, is so of $\mathcal{B}$, in which case $h'' \triangleq (\nu_\theta{\upharpoonright}D) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{G})$ is surjective, and so is $h''' \triangleq (h \circ h'') \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{A})$. On the other hand, by Lemma 3.4 and Remark 3.26 with $j = 0$, $\mathcal{A}$ is simple. Hence, by Proposition 2.16, $\vartheta \triangleq \partial(\mathcal{D}) = (\ker h''')$. Therefore, by Proposition 2.15, $\nu_\vartheta \circ h'''^{-1}$ is an isomorphism from $\mathcal{A}$ onto $\Re(\mathcal{D})$, as required. □

Corollary 4.4. *Let C' be an extension of C. Suppose either $\mathfrak{A}$ is b-idempotent or both $\mathfrak{A}$ is regular and $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{0[1]}$), while the rule* (4.1) *is not satisfied in C'. Then, $C' = C$.*

Proof. In that case, $\sim(x_1 \vee \sim x_1) \notin T \triangleq C'(\{x_0, \sim x_0\})$, so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C) not being a model of (4.1). In this way, (2.5) and Lemma 4.3 complete the argument. □

Proposition 4.5. *Let M be a class of Σ-matrices. Suppose either $\mathfrak{A}$ is b-idempotent or both $\mathfrak{A}$ is regular and $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a specular subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{0[1]}$), while C is defined by M. Then, there are some $\mathcal{B} \in \mathsf{M}$ and some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to $\Re(\mathcal{D})$.*

Proof. Note that the rule (4.1) is not satisfied in C, because it is not true in $\mathcal{A}$ under $[x_0/\mathsf{b}, x_1/\mathsf{n}]$. Therefore, as C is defined by M, there is some $\mathcal{B} \in \mathsf{M} \subseteq \mathrm{Mod}(C)$ not being a model of (4.1), in which case Lemma 4.3 completes the argument. □

Now, we are in a position to argue several interesting corollaries of Proposition 4.5:

Corollary 4.6. *Let M be a class of Σ-matrices. Suppose the logic of M is an expansion of C_{B} (in particular, $\Sigma = \Sigma_0$ and the logic of M is C_{B} itself). Then, some $\mathcal{B} \in \mathsf{M}$ is not truth-/false-singular. In particular, any four-valued expansion of C_{B} (including C_{B} itself) is defined by no truth-/false-singular matrix.*

Proof. By contradiction. For suppose every member of M is truth-/false-singular. Then, $\mathsf{M}{\restriction}\Sigma_0$ is a class of truth-/false-singular Σ_0-matrices defining C_{B}. Then, by Proposition 4.5, there are some $\mathcal{B} \in (\mathsf{M}{\restriction}\Sigma_0)$ and some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{E} \triangleq (\mathcal{D}/\theta)$, where $\theta \triangleq \partial(\mathcal{D})$, in which case $\mathcal{E}$ is truth-/false-singular, for $\mathcal{D}$ is so, because $\mathcal{B}$ is so/, while $((D/\theta) \setminus (D^{\mathcal{D}}/\theta)) \subseteq ((D \setminus D^{\mathcal{D}})/\theta)$, and so is $\mathcal{DM}_4$. This contradiction completes the argument. □

Corollary 4.7. *Any four-valued $\Sigma_{0[1]}$-matrix $\mathcal{B}$ defining $C_{[\mathrm{B}]\mathrm{B}}$ is isomorphic to $\mathcal{DM}_{4[,01]}$.*

Proof. By Proposition 4.5, there are then some submatrix $\mathcal{D}$ of $\mathcal{B}$ and some isomorphism e from $\mathcal{DM}_{4[,01]}$ onto $\mathcal{D}/\theta$, where $\theta \triangleq \partial(\mathcal{D})$, in which case $4 = |DM_{4[,01]}| = |D/\theta| \leqslant |D| \leqslant |B| = 4$, in which case $4 = |D/\theta| = |D| = |B|$, and so ν_θ is injective, whereas $D = B$. In this way, $e^{-1} \circ \nu_\theta$ is an isomorphism from $\mathcal{B}$ onto $\mathcal{DM}_{4[,01]}$, as required. □

This, in its turn, enables us to prove:

Theorem 4.8. *Any four-valued expansion of $C_{[\mathrm{B}]\mathrm{B}}$ is defined by an expansion of $\mathcal{DM}_{4[,01]}$.*

Proof. Let $\mathcal{B}$ be a four-valued Σ-matrix defining an expansion of $C_{[\mathrm{B}]\mathrm{B}}$. Then, $\mathcal{B}{\restriction}\Sigma_{0[1]}$ is a four-valued $\Sigma_{0[1]}$-matrix defining $C_{[\mathrm{B}]\mathrm{B}}$ itself. Hence, by Corollary 4.7, there is an isomorphism e from $\mathcal{B}{\restriction}\Sigma_{0[1]}$ onto $\mathcal{DM}_{4[,01]}$. In that case, e is an isomorphism from $\mathcal{B}$ onto the expansion $\langle e[\mathfrak{B}], e[D^{\mathcal{B}}]\rangle$ of $\mathcal{DM}_{4[,01]}$. In this way, (2.5) completes the argument. □

Thus, the way of construction of four-valued expansions chosen in the beginning of this section does exhaust *all* of them. And what is more, any of them is defined by a unique expansion of $\mathcal{DM}_4$, as it follows from the theorem immediately ensuing from the following key lemma " killing several (more precisely, $|\mathbf{S}_*(\mathcal{DM}_4)| = 5$; cf. Subsubsection 6.1.4) birds with one stone":

Lemma 4.9 (Four-Valued Key Lemma)**.** *Let $\mathcal{B}$ be a Σ-matrix. Suppose $(\mathcal{B}{\restriction}\Sigma_0) \in \mathbf{S}_*(\mathcal{DM}_4)$ and $\mathcal{B}$ is a model of C. Then, $\mathcal{B}$ is a submatrix of $\mathcal{A}$.*

Proof. In that case, $\mathcal{B}$ is consistent and, being finite, is finitely-generated. In addition, by Lemmas 3.4, 3.6 and Remark 3.26 with $j = 0$, it is simple. And what is more, by Remarks 3.14 and 3.26 with $j = 0$, $\mathcal{B}$ is $\vee$-disjunctive. Therefore, as $\mathcal{A}$ is finite, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$, in which case, by Remark 3.14 and (2.5), $\mathcal{D}$ is consistent and $\vee$-disjunctive, and so, by Corollary 3.15, there is some $i \in I$ such that $h \triangleq (\pi_i{\restriction}D) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$. Moreover, by Lemmas 3.4, 3.6 and Remark 3.26 with $j = 0$, $\mathcal{C}_i$ is simple. Therefore, by Proposition 2.16, $(\ker h) = \partial(\mathcal{D}) = (\ker g)$. Hence, by Proposition 2.15, $e \triangleq (h \circ g^{-1}) \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{C}_i) \subseteq \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A}) \subseteq \hom_{\mathrm{S}}(\mathcal{B}{\restriction}\Sigma_0, \mathcal{DM}_4)$, in which case, by Lemma 3.7 and Remark 3.26 with $j = 0$, e is diagonal, as required. □

By (2.5) and Lemma 4.9, we immediately get the following universal characterization:

Corollary 4.10. *Let $\mathcal{B} \in \mathbf{S}_*(\mathcal{DM}_4)$. Then, the logic of an expansion of $\mathcal{B}$ is an extension of C iff B forms a subalgebra of $\mathfrak{A}$.*

Theorem 4.11. *Let $\mathcal{B}$ be a Σ-matrix. Suppose $(\mathcal{B}{\restriction}\Sigma_0) = \mathcal{DM}_4$ and $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$). Then, $\mathcal{B} = \mathcal{A}$.*

Proof. Then, by Lemma 4.9, $\mathcal{B}$ is a submatrix of $\mathcal{A}$, in which case $\mathcal{B} = \mathcal{A}$, for $B = A$, as required. □

In view of Theorem 4.11, $\mathcal{A}$ is said to be *characteristic for C*. Subsections 4.2, 4.3, 4.4, 4.5 and 4.6 provide characterizations of certain properties of four-valued expansions of C_{B} via respective properties of their characteristic matrices. And what is more, $\mathcal{A}$ is $\vee$-disjunctive and has a unary unitary equality determinant (cf. Remark 3.26 with $j = 0$), so Theorems 3.21 and 3.24 are well applicable to C immediately yielding the item (1k) of the Abstract (cf. Subsubsection 6.1.4 for more detail).

4.1.1. *Minimal four-valuedness.* As a one more interesting consequence of Proposition 4.5, we have:

Theorem 4.12. *Let M be a class of Σ-matrices. Suppose the logic of M is an expansion of C_{B} (in particular, $\Sigma = \Sigma_0$ and the logic of M is C_{B} itself). Then, $4 \leqslant |B|$, for some $\mathcal{B} \in \mathsf{M}$. In particular, any four-valued expansion of C_{B} (including C_{B} itself) is minimally four-valued.*

Proof. In that case, C_{B} is defined by $\mathsf{M}{\restriction}\Sigma_0$, and so, by Proposition 4.5, there are some $\mathcal{B} \in \mathsf{M}$ and some submatrix $\mathcal{D}$ of $\mathcal{B}{\restriction}\Sigma_0$ such that $\mathcal{DM}_4$ is isomorphic to $\mathcal{D}/\theta$, where $\theta \triangleq \partial(\mathcal{D})$. In this way, $4 = |DM_4| = |D/\theta| \leqslant |D| \leqslant |B|$, as required. □

4.2. **Relevance Principle.**

Lemma 4.13. *C is purely-inferential iff $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. First, assume $\{\mathsf{n}\}$ forms a subalgebra of $\mathcal{A}$, in which case $\mathcal{A}\restriction\{\mathsf{n}\}$ is a truth-empty submatrix of $\mathcal{A}$, and so C is purely inferential, in view of (2.5).

Conversely, assume $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{n}) \neq \mathsf{n}$, in which case $(\varphi^{\mathfrak{A}}(\mathsf{n}) \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} \varphi^{\mathfrak{A}}(\mathsf{n})) \in D^{\mathcal{A}}$, and so $((x_0 \vee \sim x_0) \vee (\varphi \vee \sim\varphi)) \in C(\varnothing)$, as required. □

Lemma 4.14. *C has no inconsistent formula iff $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. First, assume $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $(\varphi^{\mathfrak{A}}(\mathsf{b}) \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}} \varphi^{\mathfrak{A}}(\mathsf{b})) \notin D^{\mathcal{A}}$, and so $((x_0 \wedge \sim x_0) \wedge (\varphi \wedge \sim\varphi))$ is an inconsistent formula of C.

Conversely, assume $\{\mathsf{b}\}$ forms a subalgebra of $\mathcal{A}$. Let us prove, by contradiction, that C has no inconsistent formula. For suppose some $\varphi \in \mathrm{Fm}^\omega_\Sigma$ is an inconsistent formula of C, in which case $\varphi \in \mathrm{Fm}^\alpha_\Sigma$, for some $\alpha \in (\omega \setminus 1)$, while $x_\alpha \in C(\varphi)$. Let $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $(V_\alpha \times \{\mathsf{b}\}) \cup (V_{\omega\setminus\alpha} \times \{\mathsf{f}\})$. Then, $h(\varphi) = \mathsf{b} \in D^{\mathcal{A}}$, whereas $h(x_\alpha) = \mathsf{f} \notin D^{\mathcal{A}}$. This contradiction completes the argument. □

Theorem 4.15. *The following are equivalent:*

(i) *C holds Relevance Principle;*
(ii) *C is purely inferential and has no inconsistent formula;*
(iii) *both $\{\mathsf{n}\}$ and $\{\mathsf{b}\}$ form subalgebras of $\mathfrak{A}$.*

Proof. First, (ii) is a particular case of (i). Next, (ii)⇒(iii) is by Lemmas 4.13 and 4.14.

Finally, assume (iii) holds. Consider any $\alpha \in (\omega \setminus 1)$, any $\phi \in \mathrm{Fm}^\alpha_\Sigma$ and any $\psi \in \mathrm{Fm}^{\omega\setminus\alpha}_\Sigma$. Let $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $(V_\alpha \times \{\mathsf{b}\}) \cup (V_{\omega\setminus\alpha} \times \{\mathsf{n}\})$. Then, $h(\phi) = \mathsf{b} \in D^{\mathcal{A}}$, whereas $h(\psi) = \mathsf{n} \notin D^{\mathcal{A}}$. Thus, $\psi \notin C(\phi)$, and so (i) holds, as required. □

Corollary 4.16 (cf. Theorem 4.2 of [13] for the case $\Sigma = \Sigma_0$)**.** *C has no proper extension holding Relevance Principle.*

Proof. Consider any extension C' of C holding Relevance Principle, in which case C, being a sublogic of C', does so as well, and so, by Theorem 4.15(i)⇒(iii), $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$. Moreover, as C' is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.26 with $j = 0$), (4.1) is not satisfied in C', for $1 \in (\omega \setminus 1)$, while $(x_0 \wedge \sim x_0) \in \mathrm{Fm}^1_\Sigma$, whereas $(x_1 \vee \sim x_1) \in \mathrm{Fm}^{\omega\setminus 1}_\Sigma$. In this way, Corollary 4.4 completes the argument. □

Perhaps, this is the principal maximality of C.

4.3. **Maximality.** Clearly, $\mathcal{A}$ is consistent and truth-non-empty, and so C is inferentially consistent. In this connection, we have:

Theorem 4.17. *C is [inferentially]maximal iff $\mathcal{A}$ has no proper consistent submatrix[other than that with carrier $\{\mathsf{n}\}$].*

Proof. First, consider any proper consistent submatrix $\mathcal{B}$ of $\mathcal{A}$[such that $B \neq \{\mathsf{n}\}$, in which case $(\{\mathsf{n}, \mathsf{f}\} \cap B) \neq \varnothing$, and so $\mathsf{t} \in B$, in which case $\mathcal{B}$ is truth-non-empty]. Then, by (2.5), the logic C' of $\mathcal{B}$ is a[n inferentially] consistent extension of C. For proving that $C' \neq C$, consider the following complementary cases:

(1) $\mathsf{b} \in B$.
Then, $\mathsf{n} \notin B$, for $B \neq A$, while $(\mathsf{n} \wedge^{\mathfrak{B}} \mathsf{b}) = \mathsf{f}$, whereas $(\mathsf{n} \vee^{\mathfrak{B}} \mathsf{b}) = \mathsf{t}$. In that case, $(x_0 \vee \sim x_0) \in (C'(\varnothing) \setminus C(\varnothing))$.
(2) $\mathsf{b} \notin B$.
Then, $\mathcal{B}$ is not $\sim$-paraconsistent, as opposed to $\mathcal{A}$, and so is C', as opposed to C.

Thus, in any case, $C' \neq C$, and so C is not [inferentially]maximal.

Conversely, assume $\mathcal{A}$ has no proper consistent submatrix[other than that with carrier $\{\mathsf{n}\}$]. Consider any [inferentially]consistent extension C' of C. Then, $x_0 \notin T \triangleq C'(\varnothing[\cup\{x_1\}])[\ni x_1]$, while, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its consistent [truth-non-empty]finitely-generated submatrix $\mathcal{B} = \langle \mathfrak{Fm}^2_\Sigma, \mathrm{Fm}^2_\Sigma \cap T\rangle$, in view of (2.5). Hence, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent [truth-non-empty]submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B}/\partial(\mathcal{B}))$, in which case, by (2.5), $\mathcal{D}$ is a consistent model of C', and so, in particular, $I \neq \varnothing$. Moreover, for any $i \in I$,[as $\mathcal{C}_i$ is truth-non-empty, in which case $C_i \neq \{\mathsf{n}\}$, and so] $\mathcal{C}_i = \mathcal{A}$ is truth-non-empty anyway. Hence, by the following claim, both $D \ni a \triangleq (I \times \{\mathsf{f}\})$ and $D \ni b \triangleq (I \times \{\mathsf{t}\})$:

Claim 4.18. *Let I be a finite set, $\overline{\mathcal{C}}$ an I-tuple constituted by consistent truth-non-empty submatrices of $\mathcal{A}$ and $\mathcal{B}$ a subdirect product of $\overline{\mathcal{C}}$. Then, $\{I \times \{\mathsf{f}\}, I \times \{\mathsf{t}\}\} \subseteq B$.*

Proof. In that case, $\mathfrak{B}\restriction\Sigma^+$ is a finite lattice, so it has both a zero a and a unit b. Consider any $i \in I$. Then, as $\mathcal{C}_i$ is both consistent and truth-non-empty, we have both $(\{\mathsf{f}, \mathsf{n}\} \cap C_i) \neq \varnothing$ and $(\{\mathsf{b}, \mathsf{t}\} \cap C_i) \neq \varnothing$, in which case we get $\{\mathsf{f}, \mathsf{t}\} \subseteq C_i$, for $(\mathsf{n} \wedge^{\mathfrak{A}} \mathsf{b}) = \mathsf{f}$, while $\sim^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$, whereas $\sim^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$. Therefore, since $\pi_i[B] = C_i$ and $(\pi_i\restriction B) \in \hom(\mathfrak{B}\restriction\Sigma^+, \mathfrak{C}_i\restriction\Sigma^+)$ is surjective, by Lemma 3.25, we get $\pi_i(a) = \mathsf{f}$ and $\pi_i(b) = \mathsf{t}$. Thus, $B \ni a = (I \times \{\mathsf{f}\})$ and $B \ni b = (I \times \{\mathsf{t}\})$, as required. □

Next, if $\{\mathsf{f}, \mathsf{t}\} \subsetneq A$[distinct from $\{\mathsf{n}\}$] did form a subalgebra of $\mathfrak{A}$, $\mathcal{A}\restriction\{\mathsf{f}, \mathsf{t}\}$ would be a proper consistent submatrix of $\mathcal{A}$[other than that with carrier $\{\mathsf{n}\}$]. Therefore, there are some $\phi \in \mathrm{Fm}^2_\Sigma$ and $j \in 2$ such that $\phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}) = \langle j, 1-j\rangle$. Likewise, if $\{\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}\} \subsetneq A$[distinct from $\{\mathsf{n}\}$] did form a subalgebra of $\mathfrak{A}$, $\mathcal{A}\restriction\{\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}\}$ would be a proper consistent submatrix of $\mathcal{A}$[other than that with carrier $\{\mathsf{n}\}$]. Therefore, there is some $\psi \in \mathrm{Fm}^3_\Sigma$ such that $\psi^{\mathfrak{A}}(\mathsf{f}, \langle j, 1-j\rangle, \mathsf{t}) = \langle 1-j, j\rangle$. In this way, $\{\phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}), \psi^{\mathfrak{A}}(\mathsf{f}, \phi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}), \mathsf{t})\} = \{\mathsf{n}, \mathsf{b}\}$. Then, $D \supseteq \{\phi^{\mathfrak{D}}(a, b), \psi^{\mathfrak{D}}(a, \phi^{\mathfrak{D}}(a, b), b)\} = \{I \times \{\mathsf{n}\}, I \times \{\mathsf{b}\}\}$. Thus, $\{I \times \{c\} \mid c \in A\} \subseteq D$. Hence, as $I \neq \varnothing$, $\{\langle c, I \times \{c\}\rangle \mid c \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.5), C is an extension of C', and so $C' = C$, as required. □

4.4. **Subclassical expansions.**

Lemma 4.19. *Let $\mathcal{B}$ be a (simple)finitely generated consistent truth-non-empty model of C. Then, the following hold:*

(i) *$\mathcal{B}$ is $\sim$-paraconsistent, if $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$ and $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*
(ii) *$\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is embeddable into $\mathcal{B}/\eth(\mathcal{B})$ (resp., into $\mathcal{B}$ itself), if $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. Put $\mathcal{E} \triangleq (\mathcal{B}/\eth(\mathcal{B}))$ (resp., $\mathcal{E} \triangleq \mathcal{B}$). Then, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent truth-non-empty submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{E})$, in which case, by (2.5), $\mathcal{D}$ is consistent, and so, in particular, $I \neq \varnothing$. Hence, by Claim 4.18, both $D \ni a \triangleq (I \times \{\mathsf{f}\})$ and $D \ni b \triangleq (I \times \{\mathsf{t}\})$. Consider the following respective cases:

(i) $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$ and $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) \in \{\mathsf{n},\mathsf{b}\}$. Take any $i \in I \neq \varnothing$. Then, $\{\mathsf{f},\mathsf{t}\} = \pi_i[\{a,b\}] \subseteq C_i$. Moreover, $(\pi_i\restriction D) \in \hom^{\mathrm{S}}(\mathcal{D},\mathcal{C}_i)$, in which case, by (2.5) and (2.6), $\mathcal{C}_i$ is a model of $\sim(x_0 \wedge \sim x_0)$, and so $\mathsf{n} \notin C_i$, for $\sim^{\mathfrak{A}}(\mathsf{n} \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}}\mathsf{n}) = \mathsf{n} \notin D^{\mathcal{A}}$. And what is more, $\mathfrak{C}_i$ is a subalgebra of $\mathfrak{A}$. Hence, $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) \in C_i$, and so $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{b}$, for $\mathsf{n} \notin C_i$. Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a,b) = (I \times \{\mathsf{b}\})$, in which case $\sim^{\mathfrak{D}}c = c \in D^{\mathcal{D}}$, and so $\mathcal{D}$, being consistent, is $\sim$-paraconsistent, and so is $\mathcal{B}$, in view of (2.5), as required.
(ii) $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, $\mathcal{F} \triangleq (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\})$ is $\sim$-classical, in which case it is consistent, truth-non-empty, and two-valued, and so simple, in view of Example 3.2 and Lemma 3.4. Finally, as $\{I \times \{d\} \mid d \in F\} \subseteq D$ and $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in F\}$ is an embedding of $\mathcal{F}$ into $\mathcal{D}$, in which case, $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{F},\mathcal{E})$, and so Corollary 2.14 completes the argument. ☐

Theorem 4.20. *C is $\sim$-subclassical iff $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case the logic of $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is the only $\sim$-classical extension of C.*

Proof. Let $\mathcal{B}$ be a $\sim$-classical model of C, in which case it is consistent, truth-non-empty and two-valued, and so simple (cf. Example 3.2 and Lemma 3.4) and finite (in particular, finitely generated), but not $\sim$-paraconsistent.

First, consider any $a \in B$. Then, $\{a, \sim^{\mathfrak{B}}a\} \not\subseteq D^{\mathcal{B}} = \{\mathsf{t}\}$, for $\mathcal{B}$ is $\sim$-classical, in which case $(a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}}a) = \mathsf{f}$, for $\mathcal{B}$ is $\wedge$-conjunctive, because C is so, since $\mathcal{A}$ is so (cf. Remark 3.26 with $j = 0$), and so $\sim^{\mathfrak{B}}(a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}}a) = \mathsf{t}$. Thus, $\sim(x_0 \wedge \sim x_0)$ is true in $\mathcal{B}$. Hence, by Lemma 4.19(i), $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

Conversely, assume $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case, by (2.5), $\mathcal{D} \triangleq (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\})$ is a $\sim$-classical model of C, and so, by (2.5), Corollary 3.11 and Lemma 4.19(ii), we eventually get $\mathcal{D} = \mathcal{B}$, as required. ☐

In view of Theorem 4.20, the unique $\sim$-classical extension of a $\sim$-subclassical four-valued expansion $\boldsymbol{C}$ of C_{B} is said to be *characteristic for $\boldsymbol{C}$* and denoted by $\boldsymbol{C}^{\mathrm{PC}}$.

Theorem 4.21. *Let C' be an inferentially consistent (in particular, consistent non-pseudo-axiomatic) extension of C. Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is a model of C'.*

Proof. Then, $x_1 \notin C'(x_0) \ni x_0$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C'(x_0)\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely generated submatrix $\langle \mathfrak{Fm}^2_\Sigma, \mathrm{Fm}^2_\Sigma \cap C'(x_0)\rangle$, in view of (2.5). In this way, (2.5) and Lemma 4.19(ii) complete the argument. ☐

Example 4.22. When $\Sigma = \Sigma_0$, $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{B} \triangleq (\mathcal{A}\restriction\{\mathsf{n}\})$ is a consistent truth-empty submatrix of $\mathcal{A}$, and so, by (2.5), the logic C' of $\mathcal{B}$ is a consistent but inferentially inconsistent extension of C. Then, C' is not subclassical, because any classical logic is inferentially consistent, for any classical matrix is both consistent and truth-non-empty. In this way, the reservation "inferentially" cannot be omitted in the formulation of Theorem 4.21. ☐

4.5. **Paraconsistent and paracomplete extensions.** The axiomatic extension of C relatively axiomatized by the *Excluded Middle law* axiom $x_0 \vee \sim x_0$ is denoted by C^{EM}.

An extension C' of C is said to be *(maximally)[inferentially]paracomplete*, provided $(x_0 \vee \sim x_0) \notin C'(\varnothing[\cup\{x_1\}])$, that is, C' is not an extension of $C^{\mathrm{EM}}_{[+0]}$,(and C' has no proper [inferentially]paracomplete extension). Then, a model of C is said to be *[inferentially]paracomplete*, whenever the logic of it is so.

Clearly, a submatrix $\mathcal{B}$ of $\mathcal{A}$ is paracomplete/$\sim$-paraconsistent iff $\mathsf{n} \in B$/both $\mathsf{b} \in B$ and $(B \cap \{\mathsf{n},\mathsf{f}\}) \neq \varnothing$. In particular, $\mathcal{A}$ is both $\sim$-paraconsistent and paracomplete, and so is C.

By $\mathcal{A}_{\not{\mathrm{n}}}$ we denote the $\sim$-paraconsistent submatrix of $\mathcal{A}$ generated by $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, the logic of it being denoted by $C^{\not{\mathrm{n}}}$.

Lemma 4.23. *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of C. Then, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}_{\not{\mathrm{n}}}$ is embeddable into $\mathcal{D}/\eth(\mathcal{D})$.*

Proof. In that case, there are some $a \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}}a \in D^{\mathcal{B}}$ and some $b \in (B \setminus D^{\mathcal{B}})$. Then, in view of (2.5), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a,b\}$ is a $\sim$-paraconsistent finitely-generated model of C. Hence, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ constituted by consistent submatrices of $\mathcal{A}$, some subdirect product $\mathcal{E}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E},\mathcal{D}/\eth(\mathcal{D}))$. Hence, by (2.5), $\mathcal{E}$ is $\sim$-paraconsistent, in which case it is consistent, and so $I \neq \varnothing$. Take any $a \in D^{\mathcal{E}}$ such that $\sim^{\mathfrak{E}}a \in D^{\mathcal{E}}$. Then, by Lemma 4.1, $E \ni a = (I \times \{\mathsf{b}\})$, in which case, for each $i \in I$, $D^{\mathcal{C}_i} \ni \pi_i(a)$, and so $\mathcal{C}_i$ is truth-non-empty. Therefore, by Claim 4.18, we also have both $E \ni b \triangleq (I \times \{\mathsf{f}\})$ and $E \ni c \triangleq (I \times \{\mathsf{t}\})$. Consider the following complementary cases:

(1) $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, $A_{\not{\mathrm{n}}} = A$ and there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{b},\mathsf{t}) = \mathsf{n}$, in which case $E \ni \varphi^{\mathfrak{E}}(b,a,c) = (I \times \{\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{b},\mathsf{t})\}) = (I \times \{\mathsf{n}\})$, and so $\{I \times \{d\} \mid d \in A_{\not{\mathrm{n}}}\} \subseteq E$.

(2) $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, $A_{\not\eta} = \{\mathsf{f},\mathsf{b},\mathsf{t}\}$, and so $\{I \times \{d\} \mid d \in A_{\not\eta}\} \subseteq E$.

Thus, in any case, $\{I \times \{d\} \mid d \in A_{\not\eta}\} \subseteq E$. Then, as $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in A_{\not\eta}\}$ is an embedding of $\mathcal{A}_{\not\eta}$ into $\mathcal{E}$, in which case $(g \circ e) \in \hom_{\mathsf{S}}(\mathcal{A}_{\not\eta}, \mathcal{D}/\eth(\mathcal{D}))$, and so Corollary 2.14, Lemmas 3.4, 3.6 and Remark 3.26 with $j = 0$ complete the argument. □

Corollary 4.24. *$\mathcal{A}_{\not\eta}$ is a model of any $\sim$-paraconsistent extension of C. In particular, $C^{\not\eta}$ is the greatest $\sim$-paraconsistent extension of C, and so maximally $\sim$-paraconsistent, in which case an extension of C is $\sim$-paraconsistent iff it is a sublogic of $C^{\not\eta}$.*

Proof. Consider any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\})$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a $\sim$-paraconsistent model of C', and so of C. Then, (2.5) and Lemma 4.23 complete the argument. □

The logic of $\mathcal{DM}_{4[,01]}\upharpoonright\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is known as the *[bounded]logic of paradox* $LP_{[01]}$ [11] (cf. [14]).

Theorem 4.25. *The following are equivalent:*

(i) *C is maximally $\sim$-paraconsistent;*
(ii) *$C = C^{\not\eta}$;*
(iii) *$C^{\mathrm{EM}} \neq C^{\not\eta}$;*
(iv) *$\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*
(v) *C^{EM} is not $\sim$-paraconsistent;*
(vi) *C^{EM} is not maximally $\sim$-paraconsistent;*
(vii) *C^{EM} is either $\sim$-classical, if C is $\sim$-subclassical, or inconsistent, otherwise;*
(viii) *any consistent non-$\sim$-classical extension of C is paracomplete;*
(ix) *any $\sim$-paraconsistent extension of C is paracomplete;*
(x) *no expansion of LP is an extension of C;*
(xi) *C^{EM} is not an expansion of LP.*

Proof. First, (i)⇒(ii) is by (2.5). The converse is by Corollary 4.24. Thus, (i)⇔(ii) holds. Next, (ii)⇒(iii) is by the paracompleteness of C. In addition, (iv)⇒(ii) is immediate.

Further, assume $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $A_{\not\eta} = \{\mathsf{f},\mathsf{b},\mathsf{t}\}$, and so non-paracomplete submatrices of $\mathcal{A}$ are exactly submatrices of $\mathcal{A}_{\not\eta}$. Hence, by (2.5) and Proposition 2.12, $C^{\mathrm{EM}} = C^{\not\eta}$ is an expansion of LP. Thus, both (iii)⇒(iv) and (xi)⇒(iv) hold.

Furthermore, (vi) is a particular case of (v). Likewise, (v) is a particular case of (ix), while (ix) is a particular case of (viii). Moreover, (vi)⇒(iii) is by Corollary 4.24. And what is more, (vii)⇒(viii) is by Corollary 3.10.

Finally, assume (iv) holds. Let S be the set of all non-paracomplete consistent submatrices of $\mathcal{A}$, in which case, by Proposition 2.12, C^{EM} is defined by S. Consider any $\mathcal{B} \in \mathsf{S}$. Since it is not paracomplete, we have $\mathsf{n} \notin B$, in which case $\mathsf{f} \in B$, for it is consistent, and so $\mathsf{t} = \sim^{\mathfrak{A}}\mathsf{f} \in B$. Therefore, by (iv), $\mathsf{b} \notin B$, for $\{\mathsf{f},\mathsf{t}\} \subseteq B \not\ni \mathsf{n}$. Thus, $B = \{\mathsf{f},\mathsf{t}\}$. In this way, by Theorem 4.20, either $\mathsf{S} = \{\mathcal{B}\}$, in which case C^{EM} is $\sim$-classical, if C is $\sim$-subclassical, or $\mathsf{S} = \varnothing$, in which case C^{EM} is inconsistent, otherwise. Thus, (vii) holds.

After all, (xi/x) is a particular case of (x/ix), as required. □

4.5.1. *The resolutional extension.* By $C^{[\mathrm{EM}+]\mathrm{R}}$ we denote the *resolutional* extension of $C^{[\mathrm{EM}]}$, viz., the one relatively axiomatized by the *Resolution* rule:

$$\{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash x_0. \tag{4.8}$$

Put $\mathsf{S}_{[*,]\not b} \triangleq \{\mathcal{B} \in \mathbf{S}_{[*]}(\mathcal{A}) \mid \mathsf{b} \notin B\}$.

Lemma 4.26. *Let $\wr$ and $\veebar$ be (possibly, secondary) unary and binary connectives of Σ, C' a $\veebar$-disjunctive Σ-logic and C'' an extension of C'. Then,*

$$\{x_1 \veebar x_0, \wr x_1 \veebar x_0\} \vdash (x_2 \veebar x_0) \tag{4.9}$$

is satisfied in C'' iff

$$\{x_1 \veebar x_0, \wr x_1 \veebar x_0\} \vdash x_0 \tag{4.10}$$

is so.

Proof. In that case, (3.4) and (3.5), being valid for C', remain so for C''. First, assume (4.9) is satisfied in C'', in which case (4.9)$[x_2/x_0]$ is so, in view of the structurality of C'', and so is (4.10), in view of (3.5) and the transitivity of C''. Conversely, the fact that (4.10) and (3.4) are satisfied in C'' implies the fact that (4.9) is so, in view of the transitivity of C'', as required. □

By Lemmas 3.20, 4.26, Corollary 3.16 and Remark 3.26 with $j = 0$, we first have:

Corollary 4.27. *C^{R} is a proper extension of C.*

Theorem 4.28. *$C^{\mathrm{EM+R}}$ is equal to C^{PC}, if C is $\sim$-subclassical, and inconsistent, otherwise.*

Proof. With using Remark 3.26 with $j = 0$, Theorems 3.21, 4.20 and Lemma 4.26. Then, $C^{\mathrm{EM+R}}$ is defined by the set S of all non-paracomplete members of $\mathsf{S}_{*,\not b}$. In that case, $\mathsf{S} = \{\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}\}$, if $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and $\mathsf{S} = \varnothing$, otherwise, as required. □

By Remark 3.26 with $j = 0$, Theorem 3.21 and Lemma 4.26, we also have:

Lemma 4.29. *C^{R} is defined by $\mathsf{S}_{[*,]\not{\mathsf{b}}}$.*

By Lemmas 4.13 and 4.29, we first have:

Corollary 4.30. *C^{R} is purely inferential iff C is so. In particular, C^{R} is paracomplete, whenever C is purely inferential.*

In addition, we also get:

Corollary 4.31. *Suppose $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, C^{R} is defined by $\mathcal{A}_{\not{\mathsf{b}}} \triangleq \mathcal{A}\restriction\{\mathsf{f},\mathsf{n},\mathsf{t}\}$,*

Proof. In that case, $\mathsf{S}_{\not{\mathsf{b}}} = \mathbf{S}(\mathcal{A}_{\not{\mathsf{b}}})$, and so (2.5) and Lemma 4.29 complete the argument. □

Theorem 4.32. *The following are equivalent:*

(i) *C^{R} is paracomplete;*
(ii) *there is some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathsf{b} \notin B \ni \mathsf{n}$;*
(iii) *the carrier of the subalgebra of $\mathfrak{A}$ generated by $\{\mathsf{n}\}$ does not contain b;*
(iv) *there is no $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{n}) = \mathsf{b}$.*

Proof. In view of Lemma 4.29, C^{R} is paracomplete iff $\mathsf{S}_{\not{\mathsf{b}}}$ contains a paracomplete matrix. Thus, (i)⇔(ii) holds. Finally, (ii)⇔(iii)⇔(iv) are immediate. □

Lemma 4.33. *Let $a \in \{\mathsf{b},\mathsf{n}\}$. Suppose $\{\mathsf{f},[a,]\mathsf{t}\}$ forms a [regular]subalgebra of $\mathfrak{A}$. Then, $K^a_4 \triangleq \{\langle\mathsf{f},\mathsf{f}\rangle,\langle a,\mathsf{f}\rangle,\langle a,\mathsf{t}\rangle,\langle\mathsf{t},\mathsf{t}\rangle\}$ forms a subalgebra of $(\mathfrak{A}\restriction\{\mathsf{f},a,\mathsf{t}\}) \times (\mathfrak{A}\restriction\{\mathsf{f},\mathsf{t}\})$.*

Proof. Let $\mathfrak{B}$ be the subalgebra of $(\mathfrak{A}\restriction\{\mathsf{f},a,\mathsf{t}\}) \times (\mathfrak{A}\restriction\{\mathsf{f},\mathsf{t}\})$ generated by K^a_4. If $\langle\mathsf{t},\mathsf{f}\rangle$ was in B, there would be some $\varphi \in \mathrm{Fm}^4_\Sigma$ such that both $\varphi^{\mathfrak{A}}(\mathsf{f},a,a,\mathsf{t}) = \mathsf{t}$ and $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{f},\mathsf{t},\mathsf{t}) = \mathsf{f}$, in which case, since $(\mathsf{n}/\mathsf{b}) \sqsubseteq / \sqsupseteq b$, for every $b \in \{\mathsf{f},\mathsf{t}\}$, by the regularity of $\mathfrak{A}\restriction\{\mathsf{f},a,\mathsf{t}\}$, we would get $\mathsf{t} \sqsubseteq / \sqsupseteq \mathsf{f}$. Therefore, as $\sim^{\mathfrak{A}}(\mathsf{f}/\mathsf{t}) = (\mathsf{t}/\mathsf{f})$, we conclude that $B = K^a_4$, as required. □

Lemma 4.34. *Let $B \subseteq \{\mathsf{b},\mathsf{n}\}$. Suppose $\{\mathsf{f},\mathsf{t}\} \cup B$ forms a specular subalgebra of $\mathfrak{A}$. Then, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. By contradiction. For suppose $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. In that case, there are some $\varsigma \in \Sigma$ of some arity $n \in \omega$ and some $\bar{a} \in \{\mathsf{f},\mathsf{t}\}^n$ such that $\varsigma^{\mathfrak{A}}(\bar{a}) \in B$. Then, $(\mu \circ \bar{a}) = \bar{a}$, while $\mu(\varsigma^{\mathfrak{A}}(\bar{a})) \neq \varsigma^{\mathfrak{A}}(\bar{a})$, in which case $\mu \notin \hom(\mathfrak{A}\restriction(\{\mathsf{f},\mathsf{t}\} \cup B),\mathfrak{A})$, and so this contradiction completes the argument. □

Theorem 4.35. *Suppose $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ forms a regular specular subalgebra of $\mathfrak{A}$, in which case $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}_{\not{\mathsf{b}}}$ (cf. Lemma 4.34)[and $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{A}_{\not{\mathsf{b}}}$] (in particular, $\Sigma = \Sigma_{0[1]}$). Then, an extension of C is [non-]inferentially paracomplete iff it is a sublogic of C^{R}. In particular, C^{R} is maximally [non-]inferentially paracomplete.*

Proof. In that case, by Corollary 4.31, C^{R} is defined by the truth-non-empty paracomplete (and so inferentially paracomplete) Σ-matrix $\mathcal{A}_{\not{\mathsf{b}}}$, in which case, in particular, any extension of C, being a sublogic of C^{R}, is inferentially paracomplete, and so paracomplete.

Conversely, consider any [non-]inferentially paracomplete extension C' of C, in which case[, since $C'(\varnothing) \supseteq C(\varnothing) \neq \varnothing$, in view of Lemma 4.13,] $(x_0 \vee \sim x_0) \notin T \triangleq C'(x_1)$, while, by the structurality of C', $\langle\mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated inferentially paracomplete submatrix $\mathcal{B} \triangleq \langle\mathfrak{Fm}^2_\Sigma, T \cap \mathrm{Fm}^2_\Sigma\rangle$, in view of (2.5). Hence, by Lemma 2.20, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$, in which case $(\mathfrak{D}\restriction\Sigma_0) \in \mathsf{DML}$, for $\mathsf{DML} \ni \mathfrak{DM}_4$ is a variety, and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\Re(\mathcal{B}))$, in which case, by (2.5), $\mathcal{D}$ is an inferentially paracomplete model of C', and so there are some $a \in D^{\mathcal{D}} \subseteq \{\mathsf{b},\mathsf{t}\}^I$ and $b \in (D \setminus D^{\mathcal{D}})$ such that $\sim^{\mathfrak{D}} b \leqslant^{\mathfrak{D}} b$, in which case $b \lneqq^{\mathfrak{D}} c \triangleq (a \vee^{\mathfrak{D}} b) \in D^{\mathcal{D}}$. Put $J \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$, $K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{n}\} \neq \varnothing$, for $b \notin D^{\mathcal{D}}$, and $L \triangleq \{i \in I \mid \pi_i(b) = \mathsf{b} \neq \pi_i(c)\}$. Given any $\bar{a} \in A^4$, put $(a_0|a_1|a_2|a_3) \triangleq ((J \times \{a_0\}) \cup (K \times \{a_1\}) \cup (L \times \{a_2\}) \cup ((I \setminus (J \cup K \cup L)) \times \{a_3\}))$. Then, we have:

$$D \ni b = (\mathsf{t}|\mathsf{n}|\mathsf{b}|\mathsf{b}), \tag{4.11}$$

$$D \ni \sim^{\mathfrak{D}} b = (\mathsf{f}|\mathsf{n}|\mathsf{b}|\mathsf{b}), \tag{4.12}$$

$$D \ni c = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{b}), \tag{4.13}$$

$$D \ni \sim^{\mathfrak{D}} c = (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{b}) \tag{4.14}$$

Consider the following complementary cases:

(1) $\mathfrak{A}$ is b-idempotent.
Then, we have the following complementary subcases:
  (a) $J = \varnothing$,
  Then, since $K \neq \varnothing = J$, $\mathfrak{A}_{\not{\mathsf{b}}}$ is specular and $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, by (4.11), (4.13) and (4.14), we see that $\{\langle x,(x|x|\mu(x)|\mathsf{b})\rangle \mid x \in A_{\not{\mathsf{b}}}\}$ is an embedding of $\mathcal{A}_{\not{\mathsf{b}}}$ into $\mathcal{D}$. Hence, by (2.5), $\mathcal{A}_{\not{\mathsf{b}}}$ is a model of C', for $\mathcal{D}$ is so.
  (b) $J \neq \varnothing$.
  Then, taking Lemma 4.33 into account, since $K \neq \varnothing \neq J$, $\mathfrak{A}_{\not{\mathsf{b}}}$ is specular and $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, by (4.11), (4.12), (4.13) and (4.14), we see that $\{\langle\langle x,y\rangle,(y|x|\mu(x)|\mathsf{b})\rangle \mid \langle x,y\rangle \in K^{\mathsf{n}}_4\}$ is an embedding of $\mathcal{B} \triangleq ((\mathcal{A}_{\not{\mathsf{b}}} \times (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}))\restriction K^{\mathsf{n}}_4)$ into $\mathcal{D}$. Moreover, $(\pi_0\restriction K^{\mathsf{n}}_4) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B},\mathcal{A}_{\not{\mathsf{b}}})$. Hence, by (2.5), $\mathcal{A}_{\not{\mathsf{b}}}$ is a model of C', for $\mathcal{D}$ is so.

(2) $\mathfrak{A}$ is not b-idempotent.
Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\phi^{\mathfrak{A}}[\{\mathsf{b},\mathsf{t}\}] = \{\mathsf{t}\}$ and $\psi^{\mathfrak{A}}[\{\mathsf{b},\mathsf{t}\}] = \{\mathsf{f}\}$, where $\phi \triangleq (x_0 \vee (\varphi \vee \sim\varphi))$ and $\psi \triangleq \sim\phi$, and so, by (4.13), we get:

$$D \ni \psi^{\mathfrak{D}}(c) = (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}), \tag{4.15}$$

$$D \ni \phi^{\mathfrak{D}}(c) = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}). \tag{4.16}$$

Consider the following complementary subcases:

(a) $J = \varnothing$,
Then, since $K \neq \varnothing = J$ and $\mathfrak{A}_{\not\flat}$ is specular, by (4.11), (4.15) and (4.16), we see that $\{\langle x, (x|x|\mu(x)|\mu(x))\rangle \mid x \in A_{\not\flat}\}$ is an embedding of $\mathcal{A}_{\not\flat}$ into $\mathcal{D}$. Hence, by (2.5), $\mathcal{A}_{\not\flat}$ is a model of C', for $\mathcal{D}$ is so.

(b) $J \neq \varnothing$.
Then, taking Lemma 4.33 into account, since $K \neq \varnothing \neq J$ and $\mathfrak{A}_{\not\flat}$ is specular, by (4.11), (4.12), (4.15) and (4.16), we see that $\{\langle\langle x, y\rangle, (y|x|\mu(x)|\mu(x))\rangle \mid \langle x, y\rangle \in K_4^{\mathsf{n}}\}$ is an embedding of $\mathcal{B} \triangleq ((\mathcal{A}_{\not\flat} \times (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}))\restriction K_4^{\mathsf{n}})$ into $\mathcal{D}$. Moreover, $(\pi_0\restriction K_4^{\mathsf{n}}) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{A}_{\not\flat})$. Hence, by (2.5), $\mathcal{A}_{\not\flat}$ is a model of C', for $\mathcal{D}$ is so.

Thus, in any case, $\mathcal{A}_{\not\flat}$ is a model of C', and so $C' \subseteq C^{\mathrm{R}}$, as required. □

The logic of $\mathcal{DM}_{4[,01]}\restriction\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ is known as *Kleene's [bounded]three-valued logic* $K_{3[,01]}$ (cf. [6]).

Theorem 4.36. *The following are equivalent:*

(i) $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ *does not form a subalgebra of* $\mathfrak{A}$*;*
(ii) *[providing C is not purely inferential,]C^{R} is [non-]inferentially either $\sim$-classical, if C is $\sim$-subclassical, or inconsistent, otherwise;*
(iii) *[providing C is not purely inferential,]C^{R} is not [non-]inferentially paracomplete;*
(iv) *the Σ_0-fragment of C^{R} is not inferentially paracomplete;*
(v) *no expansion of K_3 is an extension of C;*
(vi) *C^{R} is not an expansion of K_3.*

Proof. First, (vi)⇒(i) is by Corollary 4.31.

Moreover, (vi) is a particular case of (v).

Next, assume (i) holds. We use Remark 2.10, Theorem 4.20 and Lemmas 4.13 and 4.29 tacitly. Consider the following four exhaustive cases:

(1) C is both $\sim$-subclassical and not purely inferential.
Then, $\mathsf{S}_{*,\not\flat} = \{\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}\}$, in which case C^{R} is $\sim$-classical, and so inferentially so.
(2) C is both purely-inferential and $\sim$-subclassical.
Then, $\mathsf{S}_{*,\not\flat} = \{\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}, \mathcal{A}\restriction\{\mathsf{n}\}\}$, in which case C^{R} is inferentially $\sim$-classical.
(3) C is both not $\sim$-subclassical and not purely inferential.
Then, $\mathsf{S}_{*,\not\flat} = \varnothing$, in which case C^{R} is inconsistent, and so inferentially so.
(4) C is both purely-inferential and not $\sim$-subclassical.
Then, $\mathsf{S}_{*,\not\flat} = \{\mathcal{A}\restriction\{\mathsf{n}\}\}$, in which case C^{R} is inferentially inconsistent.

Thus, (ii) holds.

Further, in view of Theorem 4.20, any [inferentially]$\sim$-classical extension of C is not [inferentially]paracomplete. And what is more, any [inferentially]paracomplete extension of C is clearly [inferentially]consistent. Hence, (ii)⇒(iii) holds.

Furthermore, (iii)⇒(iv) is by the fact that $x_0 \vee \sim x_0$ is a Σ_0-formula.

Finally, by Proposition 2.19, K_3 is non-pseudo-axiomatic. Moreover, it is paracomplete, and so inferentially so. And what is more, (4.8), being satisfied in K_3, is so in any expansion of it. In this way, (iv)⇒(v) holds, as required. □

In this connection, it is remarkable that paracomplete analogue of the "maximality" items (i) and (vi) of Theorem 4.25 do not hold, generally speaking, as it ensues from the following generic counterexamples collectively with Subsubsections 6.1.1 and 6.1.3:

Example 4.37. Suppose C is $\sim$-subclassical, i.e., $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (cf. Theorem 4.20). Then, $\mathcal{B} \triangleq (\mathcal{A}\times(\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}))$ is truth-non-empty, non-$\sim$-paraconsistent and, by (2.6), paracomplete, for $\mathcal{A}$ is so, in which case the logic of $\mathcal{B}$ is a proper (inferentially)paracomplete extension of C. □

Example 4.38. Let $\sqsupset$ be a (possibly, secondary) binary connective of Σ. Suppose both $\{\mathsf{f},\mathsf{t}\}$ and $\{\mathsf{f},\mathsf{n}[/\mathsf{b}],\mathsf{t}\}$ form subalgebras of $\mathfrak{A}$, in which case $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is a submatrix of $\mathcal{A}_{\not\flat}$, $\{\mathcal{A}_{\not\flat}[, \mathcal{A}_{\not\eta}]\}$ defining $C^{\mathrm{R}}[\cap C^{\mathrm{EM}}]$, in view of Corollary 4.31[and Theorem 4.25(iii)⇒(iv)], while $C^{\mathrm{R}}[\cap C^{\mathrm{EM}}]$ satisfies $x_0 \sqsupset x_0$, whereas $\{x_0, x_0 \sqsupset x_1\} \vdash x_1$ is true in $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$, in which case $\mathcal{B} \triangleq (\mathcal{A}_{\not\flat} \times (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}))$ is truth-non-empty, paracomplete, in view of (2.6), for $\mathcal{A}_{\not\flat}$ is so, and a model of the rule $\{\sim^i x_0 \sqsupset \sim^{1-i} x_0 \mid i \in 2\} \vdash (x_0 \vee \sim x_0)$, in its turn,[being also true in $\mathcal{A}_{\not\eta}$ but] not being true in $\mathcal{A}_{\not\flat}$ under $[x_0/\mathsf{n}]$, and so, by (2.5), the logic of $\{\mathcal{B}[, \mathcal{A}_{\not\eta}]\}$ is a proper [both $\sim$-paraconsistent and](inferentially)paracomplete extension of $C^{\mathrm{R}}[\cap C^{\mathrm{EM}}]$. □

Example 4.38 and Subsubsection 6.1.3 show that the preconditions in the formulation of Theorem 4.35 cannot be omitted. And what is more, as it follows from Theorem 4.35 [resp., Corollary 4.57(ii) below], the condition of existence of implication $\sqsupset$ holding both the Reflexivity axiom in $\{\mathcal{A}_{\not\flat}[, \mathcal{A}_{\not\eta}]\}$ and the Modus Ponens rule in $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is essential within Example 4.38.

4.5.2. *Miscellaneous extensions.* By $C^{[\mathrm{EM}+]\mathrm{NP}}$ we denote the least non-$\sim$-paraconsistent extension of $C^{[\mathrm{EM}]}$, viz., that which is relatively axiomatized by the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \sim x_0\} \vdash x_1. \tag{4.17}$$

Likewise, by $C^{[\mathrm{EM}+]\mathrm{MP}}$ we denote the extension of $C^{[\mathrm{EM}]}$ relatively axiomatized by the rule:

$$\{x_0, \sim x_0 \vee x_1\} \vdash x_1, \tag{4.18}$$

being nothing but *Modus Ponens* for the *material* implication $\sim x_0 \vee x_1$. (Clearly, it is a/an sublogic/extension of $C^{[\mathrm{EM}+](\mathrm{R/NP})}$, in view of (3.3) held in C by its $\vee$-disjunctivity (cf. Remark (3.26) with $j = 0$).) An extension of C is said to be *Kleene*, whenever it satisfies the rule:

$$\{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash ((x_2 \vee \sim x_2) \vee x_0) \tag{4.19}$$

Lemma 4.39. *Let I be a finite set, $\overline{\mathcal{C}} \in \{\mathcal{A}, \overleftarrow{\mathcal{A}}, \overrightarrow{\mathcal{A}}\}^I$, and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent submatrix of $\prod_{i \in I} \mathcal{C}_i$. Then,* $\hom(\mathcal{B}, \overrightarrow{\mathcal{A}}) \neq \varnothing$.

Proof. Consider the following complementary cases:

(1) $\mathcal{B}$ is truth-empty.
Take any $i \in I \neq \varnothing$, for $\mathcal{B}$ is consistent. Then, $h \triangleq (\pi_i \upharpoonright B) \in \hom(\mathfrak{B}, \mathfrak{A})$. Moreover, $D^{\mathcal{B}} = \varnothing \subseteq h^{-1}[\{\mathsf{t}\}]$. Hence, $h \in \hom(\mathcal{B}, \overrightarrow{\mathcal{A}})$.

(2) $\mathcal{B}$ is not truth-empty.
Then, $B \subseteq A^I$ is finite, for both I and A are so, and so is $D^{\mathcal{B}} \subseteq B$. Hence, $n \triangleq |D^{\mathcal{B}}| \in (\omega \setminus 1)$. Take any bijection $\bar{b} : n \to D^{\mathcal{B}}$. Then, by the $\wedge$-conjunctivity of $\mathcal{B}$, $a \triangleq (\wedge^{\mathfrak{B}} \bar{b}) \in D^{\mathcal{B}}$. Therefore, as $\mathcal{B}$ is consistent but not $\sim$-paraconsistent, $\sim^{\mathfrak{B}} a \notin D^{\mathcal{B}}$. Then, there is some $i \in I$, in which case $h \triangleq (\pi_i \upharpoonright B) \in \hom(\mathcal{B}, \mathcal{C}_i)$, such that $h(\sim^{\mathfrak{B}} a) \notin D^{\mathcal{C}_i}$. If there was some $j \in n$ such that $h(b_j) \neq \mathsf{t}$, we would have $\mathcal{C}_i \in \{\mathcal{A}, \overleftarrow{\mathcal{A}}\}$ and $(\{\mathsf{b}, \mathsf{n}\} \cap D^{\mathcal{C}_i}) \ni h(b_j) \leqslant^{\mathfrak{A}} h(a) \leqslant^{\mathfrak{A}} h(b_j)$, in which case we would get $h(a) = h(b_j)$, and so $h(\sim^{\mathfrak{B}} a) = \sim^{\mathfrak{A}} h(a) = \sim^{\mathfrak{A}} h(b_j) = h(b_j) \in D^{\mathcal{C}_i}$. Thus, $h \in \hom(\mathcal{B}, \overrightarrow{\mathcal{A}})$. □

Corollary 4.40. *Let I be a finite set , $\overline{\mathcal{C}} \in \{\mathcal{A}, \overleftarrow{\mathcal{A}}, \overrightarrow{\mathcal{A}}\}^I$, and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent non-paracomplete submatrix of $\prod_{i \in I} \mathcal{C}_i$. Then, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and* $\hom(\mathcal{B}, \mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\}) \neq \varnothing$.

Proof. Then, by Lemma 4.39, there is some $h \in \hom(\mathcal{B}, \overrightarrow{\mathcal{A}}) \neq \varnothing$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, and so $h \in \hom^{\mathrm{S}}(\mathcal{B}, \mathcal{D})$, where $\mathcal{D} \triangleq (\overrightarrow{\mathcal{A}} \upharpoonright D)$. Hence, by (2.6), $\mathcal{D}$ is not paracomplete. Therefore, as $x_0 \vee \sim x_0$ is not true in $\overrightarrow{\mathcal{A}}$ under $[x_0/(\mathsf{b}/\mathsf{n})]$, we have $(D \cap \{\mathsf{b}, \mathsf{n}\}) = \varnothing$. On the other hand, $\mathcal{B}$, being non-paracomplete, is truth-non-empty, for $B \neq \varnothing$. Therefore, $\mathsf{t} \in D$, in which case $\mathsf{f} = \sim^{\mathfrak{A}} \mathsf{t} \in D$, and so $D = \{\mathsf{f}, \mathsf{t}\}$, in which case $\mathcal{D} = (\mathcal{A} \upharpoonright D)$, as required. □

Theorem 4.41 (cf. [18] for the case $\Sigma = \Sigma_0$). *Suppose C is not maximally $\sim$-paraconsistent. Then, the following are equivalent:*

(i) *C is $\sim$-subclassical;*
(ii) *$C^{\mathrm{EM+NP}}$ is consistent;*
(iii) *C is $\sim$-subclassical and $C^{\mathrm{EM+NP}}$ is defined by $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})$.*

Proof. In that case, by Theorem 4.25(iii)⇒(i), C^{EM} is defined by $\mathcal{A}_{\not\mathsf{n}}$.

First, (iii)⇒(ii) is by the consistency of $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})$.

Next, assume (ii) holds, in which case $x_0 \notin T \triangleq C^{\mathrm{EM+NP}}(\varnothing)$, while, by the structurality of $C^{\mathrm{EM+NP}}$, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T \rangle$ is a model of $C^{\mathrm{EM+NP}}$ (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, T \cap \mathrm{Fm}^{1}_{\Sigma} \rangle$, in view of (2.5). Hence, by Lemma 2.20, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, in which case this is a submatrix of $\mathcal{A}^I$, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \Re(\mathcal{B}))$, in which case, by (2.5), $\mathcal{D}$ is a consistent model of $C^{\mathrm{EM+NP}}$, so it is neither $\sim$-paraconsistent nor paracomplete. Thus, by Corollary 4.40 and Theorem 4.20, (i) holds.

Finally, assume (i) holds. Then, by Theorem 4.20, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so of $\mathfrak{A}_{\not\mathsf{n}}$, in which case $\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\}$ is a submatrix of $\mathcal{A}_{\not\mathsf{n}}$, and so, by (2.5), $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})$ is a model of C^{EM}. Moreover, $\{a, \sim^{\mathfrak{A}} a\} \subseteq \{\mathsf{t}\}$, for no $a \in \{\mathsf{f}, \mathsf{t}\}$. Therefore, $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})$ is not $\sim$-paraconsistent, so it is a model of $C^{\mathrm{EM+NP}}$. Conversely, consider any finite set I, any $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A}_{\not\mathsf{n}})^I$ and any subdirect product $\mathcal{D} \in \mathrm{Mod}(C^{\mathrm{EM+NP}})$ of $\overline{\mathcal{C}}$, in which case $\mathcal{D}$ is a non-$\sim$-paraconsistent non-paracomplete submatrix of $\mathcal{A}^I$. Put $J \triangleq \hom(\mathcal{D}, \mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\}))$. Consider any $a \in (D \setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent, and so, by Lemma 4.40, there is some $g \in \hom(\mathcal{D}, \mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\}) \neq \varnothing$. Moreover, there is some $i \in I$, in which case $f \triangleq (\pi_i \upharpoonright D) \in \hom(\mathcal{D}, \mathcal{A}_{\not\mathsf{n}})$, such that $f(a) \notin D^{\mathcal{A}_{\not\mathsf{n}}}$. Then, $h \triangleq (f \times g) \in J$ and $h(a) \notin D^{\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})}$. In this way, $(\prod \Delta_J) \in \hom_{\mathrm{S}}(\mathcal{D}, (\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\}))^J)$. Thus, by (2.5) and Corollary 2.21, $C^{\mathrm{EM+NP}}$ is finitely-defined by $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})$. Then, the finiteness of A completes the argument. □

Remark 4.42. Let C' be a Kleene extension of C (in particular, a non-paracomplete one, in view of (3.3)). Then, we have $\{x_0 \vee x_1, \sim x_0 \vee x_1\} \vdash_{C'} (\sim(x_0 \vee x_1) \vee x_1)$. Therefore, in view of (3.3), C' satisfies (4.8) iff it satisfies (4.18). □

Lemma 4.43. *Let C' be an extension of C. Suppose C is not maximally $\sim$-paraconsistent,* (4.19) *is satisfied in C' (in particular, C' is not paracomplete, in view of* (3.3)*),* (4.18) *is not satisfied in C' and, for every $\varsigma \in \Sigma$, $\varsigma^{\mathfrak{A}_{\not\mathsf{n}}}$ is either regular or both b-idempotent and no more than binary. Then, C' is a sublogic of $C^{\mathrm{EM+NP}}$.*

Proof. The case, when $C^{\mathrm{EM+NP}}$ is inconsistent, is evident. Otherwise, by Theorems 4.20, 4.25(iv)⇒(i) and 4.41(ii)⇒(iii), $A_{\not\mathsf{n}} = \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ and $\{\mathsf{f}, \mathsf{t}\}$ form subalgebras of $\mathfrak{A}$, $C^{\mathrm{EM+NP}}$ being defined by the submatrix $\mathcal{B} \triangleq (\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\}))$ of $\mathcal{A}^2$, and so it suffices to prove that $\mathcal{B} \in \mathrm{Mod}(C')$. Then, by Corollary 2.21, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C') \subseteq \mathrm{Mod}(C)$ of it not being a model of (4.18), in which case it is $\wedge$-conjunctive, for $\mathcal{A}$ is so (cf. Remark 3.26 with $j = 0$), while $(\mathfrak{D} \upharpoonright \Sigma_0) \in \mathsf{DML}$, for $\mathsf{DML} \ni \mathfrak{DM}_4$ is a variety. Therefore, there are some $a \in D^{\mathcal{D}} \subseteq \{\mathsf{b}, \mathsf{t}\}^I$, in which case $\sim^{\mathfrak{D}} a \leqslant^{\mathfrak{D}} a$, and some $b \in (D \setminus D^{\mathcal{A}})$ such that $(\sim^{\mathfrak{D}} a \vee^{\mathfrak{D}} b) \in D^{\mathcal{A}}$. Hence, by (4.19), $(b \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} b) = ((b \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} b) \vee^{\mathfrak{D}} b) \in D^{\mathcal{A}}$, in which case $b \in \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}^I$. Put $J \triangleq \{i \in I \mid \pi_i(a) = \mathsf{b}\} \supseteq K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{f}\} \neq \varnothing$, for $(\sim^{\mathfrak{D}} a \vee^{\mathfrak{D}} b) \in D^{\mathcal{A}}$ and $b \notin D^{\mathcal{A}}$, and $L \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$, Then, given any $\vec{a} \in A^4$, set $(a_0|a_1|a_2|a_3) \triangleq (((I \setminus (L \cup K)) \times \{a_0\}) \cup ((L \setminus J) \times \{a_1\}) \cup ((L \cap J) \times \{a_2\}) \cup (K \times \{a_3\})) \in A^I$. In this way, $a = (\mathsf{t}|\mathsf{t}|\mathsf{b}|\mathsf{b})$ and $b = (\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{f})$. Therefore, we have:

$$D \ni e \triangleq (a \wedge^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{f}), \tag{4.20}$$

(4.21) $$D \ni {\sim}^{\mathfrak{D}} e = (\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{t}),$$

(4.22) $$D \ni c \triangleq (e \vee^{\mathfrak{D}} {\sim}^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{t}),$$

(4.23) $$D \ni {\sim}^{\mathfrak{D}} c = (\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{f}),$$

(4.24) $$D \ni d \triangleq (e \vee^{\mathfrak{D}} {\sim}^{\mathfrak{D}} a) = (\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{b}),$$

(4.25) $$D \ni {\sim}^{\mathfrak{D}} d = (\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{b}).$$

Consider the following complementary cases:

(1) $L \subseteq J$.
Then, given any $\vec{a} \in A^3$, set $(a_0|a_1|a_2) \triangleq (((I \setminus (L \cup K)) \times \{a_0\}) \cup ((L \cap J) \times \{a_1\}) \cup (K \times \{a_2\})) \in A^I$. In this way, by (4.20), (4.22) and (4.24), we have $e = (\mathsf{b}|\mathsf{b}|\mathsf{f}) \in D$, $c = (\mathsf{b}|\mathsf{b}|\mathsf{t}) \in D$ and $d = (\mathsf{b}|\mathsf{b}|\mathsf{b}) \in D$, respectively. Consider the following complementary subcases:
  (a) $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}_{\not\eta}$.
  Then, as $K \neq \varnothing$, $\{\langle x, (\mathsf{b}|\mathsf{b}|x)\rangle \mid x \in A_{\not\eta}\}$ is an embedding of $\mathcal{A}_{\not\eta}$ into $\mathcal{D}$.
  (b) $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}_{\not\eta}$.
  Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \in \{\mathsf{f},\mathsf{t}\}$, in which case $\phi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{f}$ and $\psi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{t}$, where $\phi \triangleq (\varphi \wedge {\sim}\varphi)$ and $\psi \triangleq (\varphi \vee {\sim}\varphi)$, and so both $D \ni \phi^{\mathfrak{D}}(d) = (\mathsf{f}|\mathsf{f}|\mathsf{f})$ and $D \ni \psi^{\mathfrak{D}}(d) = (\mathsf{t}|\mathsf{t}|\mathsf{t})$. Hence, as $I \supseteq K \neq \varnothing$, $\{\langle x, (x|x|x)\rangle \mid x \in A_{\not\eta}\}$ is an embedding of $\mathcal{A}_{\not\eta}$ into $\mathcal{D}$.
Thus, anyway, $\mathcal{A}_{\not\eta}$ is embeddable into $\mathcal{D}$, in which case, by (2.5), it is a model of C', and so is $\mathcal{B}$, for $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}_{\not\eta}$.

(2) $L \nsubseteq J$.
Consider the following complementary subcases:
  (a) either $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}_{\not\eta}$ or $((I \setminus (L \cup K)) \cup (L \cap J)) = \varnothing$.
  Then, taking (4.20), (4.22) and (4.24) into account, as $K \neq \varnothing \neq (L \setminus J)$, $\{\langle\langle x, y\rangle, (\mathsf{b}|y|\mathsf{b}|x)\rangle \mid \langle x, y\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$, and so, by (2.5), $\mathcal{B}$ is a model of C'.
  (b) $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}_{\not\eta}$ and $((I \setminus (L \cup K)) \cup (L \cap J)) \neq \varnothing$.
  Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \in \{\mathsf{f},\mathsf{t}\}$, in which case $\varphi^{\mathfrak{A}}[A_{\not\eta}] \subseteq \{\mathsf{f},\mathsf{t}\}$, for $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so $\phi^{\mathfrak{A}}[A_{\not\eta}] = \{\mathsf{f}\}$ and $\psi^{\mathfrak{A}}[A_{\not\eta}] = \{\mathsf{t}\}$, where $\phi \triangleq (\varphi \wedge {\sim}\varphi)$ and $\psi \triangleq (\varphi \vee {\sim}\varphi)$. In this way,

(4.26) $$D \ni \phi^{\mathfrak{D}}(a) = (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}),$$

(4.27) $$D \ni \psi^{\mathfrak{D}}(a) = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}).$$

  Consider the following complementary subsubcases:
    (i) $\mathfrak{A}_{\not\eta}$ is not regular.
    Then, there are some $\varsigma \in \Sigma$ of arity $n \in \omega$, some $\vec{g} \in (A^n_{\not\eta})^2$ and some $i \in 2$ such that $g^i_j \sqsubseteq g^{1-i}_j$, for all $j \in n$, but $\varsigma^{\mathfrak{A}}(\bar{g}^i) \not\sqsubseteq \varsigma^{\mathfrak{A}}(\bar{g}^{1-i})$, in which case $w \triangleq \varsigma^{\mathfrak{A}}(\bar{g}^i) \neq x \triangleq \varsigma^{\mathfrak{A}}(\bar{g}^{1-i}) \in \{\mathsf{f},\mathsf{t}\}$, and so $\bar{g}^i \neq \bar{g}^{1-i}$, in which case $y \triangleq g^i_j \in \{\mathsf{f},\mathsf{t}\}$ and $g^{1-i}_j = \mathsf{b}$, for some $j \in n$. Moreover, as $\varsigma^{\mathfrak{A}}$ is not regular, it is b-idempotent, in which case $\bar{g}^{1-i} \neq (n \times \{\mathsf{b}\})$, while $n \leqslant 2$, and so $n = 2$ and $z \triangleq g^{1-i}_{1-j} \neq \mathsf{b}$. Therefore, $g^i_{1-j} = z \in \{\mathsf{f},\mathsf{t}\}$, in which case $(z|z|z|z) \in D$, in view of (4.26) and (4.27). Moreover, by (4.24) and (4.25), we also have $(\mathsf{b}|y|\mathsf{b}|\mathsf{b}) \in D$. In this way, $D \ni f \triangleq \varsigma^{\mathfrak{D}}(\{\langle j, (\mathsf{b}|y|\mathsf{b}|\mathsf{b})\rangle, \langle 1-j, (z|z|z|z)\rangle\}) = (x|w|x|x)$. Consider the following complementary subsubsubcases:
      (A) $w = \mathsf{b}$.
      Then, taking (4.25) into account, we have $D \ni ((f \wedge^{\mathfrak{D}} {\sim}^{\mathfrak{D}} f) \vee^{\mathfrak{D}} {\sim}^{\mathfrak{D}} d) = (\mathsf{b}|\mathsf{b}|\mathsf{b}|\mathsf{b})$. Hence, as $I \supseteq K \neq \varnothing$, by (4.26) and (4.27), we see that $\{\langle u, (u|u|u|u)\rangle \mid u \in A_{\not\eta}\}$ is an embedding of $\mathcal{A}_{\not\eta}$ into $\mathcal{D}$. Therefore, by (2.5), $\mathcal{A}_{\not\eta}$ is a model of C', and so is $\mathcal{B}$, for $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}_{\not\eta}$.
      (B) $w \neq \mathsf{b}$.
      Then, $w \in \{\mathsf{f},\mathsf{t}\} \ni x$, so $D \supseteq \{f, {\sim}^{\mathfrak{D}} f\} = \{(\mathsf{f}|\mathsf{t}|\mathsf{f}|\mathsf{f}), (\mathsf{t}|\mathsf{f}|\mathsf{t}|\mathsf{t})\}$. Hence, as $K \neq \varnothing \neq (L \setminus J)$, by (4.24), (4.25), (4.26) and (4.27), we see that $\{\langle\langle u, v\rangle, (u|v|u|u)\rangle \mid \langle u, v\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$. Therefore, by (2.5), $\mathcal{B}$ is a model of C'.
    (ii) $\mathfrak{A}_{\not\eta}$ is regular.
    Then, Lemma 4.2, used tacitly throughout the rest of the proof, is well-applicable to $\mathcal{B}$. In this way, as $((I \setminus (L \cup K)) \cup (L \cap J)) \neq \varnothing \notin \{K, L \setminus J\}$, by (4.20), (4.21), (4.22), (4.23), (4.24), (4.25), (4.26) and (4.27), we see that $\{\langle\langle t, u, v\rangle, (v|u|v|t)\rangle \mid \langle t, u, v\rangle \in (B \dotplus 2)\}$ is an embedding of $\mathcal{B} \dotplus 2$ into $\mathcal{D}$, in which case, by (2.5), it is a model of C', and so is its strict surjective homomorphic image $\mathcal{B}$.

This completes the argument. □

Ii is remarkable that it is the gentle operation-wise condition that makes Lemma 4.43 well-applicable to the purely-implicative expansion of C_{BB} despite of the fact that, in that case, $\mathfrak{A}$ is neither regular nor b-idempotent. This equally concerns the following quite important result:

Theorem 4.44 (cf. [18] for the case $\Sigma = \Sigma_0$). *Suppose C is both ${\sim}$-subclassical and not maximally ${\sim}$-paraconsistent, while, for every $\varsigma \in \Sigma$, $\varsigma^{\mathfrak{A}_{\not\eta}}$ is either regular or both b-idempotent and no more than binary (in particular, $\Sigma = \Sigma_{0[1]}$). Then, proper consistent extensions of $C^{\mathrm{EM}} = C^{\not\eta}$ form the two-valued chain $C^{\mathrm{EM+NP}} \subsetneq C^{\mathrm{PC}} = C^{\mathrm{EM+(R/MP)}}$. Moreover, in case $\mathfrak{A}_{\not\eta}$ is regular (in particular, $\Sigma = \Sigma_{0[1]}$), both proper consistent extensions satisfy same axioms as C^{EM} do, and so are not axiomatic.*

Proof. With using Theorems 4.20, 4.21, 4.25(iii|iv|vi)⇒(i), 4.28, 4.41(i)⇒(iii), Corollary 3.10, Lemma 4.43 and Remark 4.42. First of all, (4.18) is not true in the consistent truth-non-empty Σ-matrix $\mathcal{B} \triangleq (\mathcal{A}_{\not\!\!/} \times (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}))$ under $[x_0/\langle\mathsf{b},\mathsf{t}\rangle, x_1/\langle\mathsf{f},\mathsf{t}\rangle]$.

Finally, assume $\mathfrak{A}_{\not\!\!/}$ is regular. Then, by Lemma 4.33, we have $\mathcal{D} \triangleq \langle\mathfrak{B}\restriction K_4^{\mathsf{b}}, K_4^{\mathsf{b}} \cap \pi_1^{-1}[\{\mathsf{t}\}]\rangle$, in which case both $(\pi_1\restriction K_4^{\mathsf{b}}) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\})$ and $(\pi_0\restriction K_4^{\mathsf{b}}) \in \hom^{\mathrm{S}}(\mathcal{D}, \mathcal{A}_{\not\!\!/})$, and so (2.5) and (2.6) complete the argument. □

In view of Lemma 4.26, Theorem 4.44 shows that $(\mathcal{C} \cap \mathrm{Fm}_\Sigma^\omega) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}_\Sigma^\omega] \veebar x_0)$ cannot be replaced by $\mathcal{C}$ in the item (ii)**b)** of Theorem 3.21, when taking $\mathsf{M} = \{\mathcal{A}_{\not\!\!/}\}$ and $\mathcal{C} = \{(4.17)\}$, and so the reservations "positive" and "axiomatic" cannot be omitted in its item (iii). In addition, the particular case of Theorem 4.44 with $\Sigma = \Sigma_{01}$ provides the " bounded" extension of [22] void of the rather unnatural restriction by merely non-empty sequents. This point, being essentially beyond the scopes of the present study, is going to be discussed in detail elsewhere.

4.5.2.1. Modus ponens versus truth-singularity.

Lemma 4.45. *Let $\mathcal{B}$ be a truth-singular $\wedge$-conjunctive Σ-matrix. Suppose $(\mathfrak{B}\restriction\Sigma_0) \in \mathsf{DML}$. Then, any $b \in D^{\mathcal{B}}$ is a unit of $\mathfrak{B}\restriction\Sigma^+$, in which case $\sim^{\mathfrak{B}} b$ is a zero of it, and so $\mathcal{B}$ is a model of* (4.18)*.*

Proof. In that case, $\mathfrak{B}\restriction\Sigma^+$ is a distributive lattice and $D^{\mathcal{B}}$ is a filter of it. Then, for any $a \in B$, we have $b \leqslant^{\mathfrak{B}} (a \vee^{\mathfrak{B}} b)$, in which case we get $(a \vee^{\mathfrak{B}} b) \in D^{\mathcal{B}}$, and so $(a \vee^{\mathfrak{B}} b) = b$, as required. □

As the truth-singularity is preserved under $\Re$, by the $\wedge$-conjunctivity of $\mathcal{A}$ (cf. Remark 3.26 with $j = 0$), (2.5), Lemmas 4.45, 3.5 and Corollary 2.17, we immediately get:

Corollary 4.46. *Any truth-singular model of C is a model of C^{MP}.*

Lemma 4.47. *$\Upsilon_0 \triangleq \{\sim^i x_0 \vee x_1 \mid i \in 2\}$ is a unitary congruence determinant for any $\wedge$-conjunctive Σ_0-matrix $\mathcal{B}$ such that $\mathfrak{B} \in \mathsf{DML}$.*

Proof. Using the distributivity of $\mathfrak{B}\restriction\Sigma^+$, the $\wedge$-conjunctivity of $\mathcal{B}$ as well as the identities (3.10), (3.11) and (3.12), it is routine checking that $\theta \triangleq \theta^{\mathcal{B}}_{\varepsilon\Upsilon_0} \in \mathrm{Con}(\mathfrak{B})$. Finally, consider any $\langle a, b\rangle \in \theta$. Then, $\mathcal{B} \models (\bigwedge \varepsilon\Upsilon_0)[x_0/a, x_1/b, x_2/(a \wedge^{\mathfrak{B}} b)]$, being a consequence of $\mathcal{B} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon\Upsilon_0)[x_0/a, x_1/b]$, implies $(a \in D^{\mathcal{B}}) \Leftrightarrow (b \in D^{\mathcal{B}})$, as required. □

By the congruence-distributivity of lattice expansions, Lemma 3.27 and Corollary 2.5, we also have:

Lemma 4.48. $\mathrm{Si}(\mathbf{V}(\mathfrak{A})) = \mathbf{IS}_{>1}\mathfrak{A}$.

Then, combining Lemmas 2.2, 3.27, 4.48, Remark 2.3 and Corollary 2.7, by the congruence-distributivity of lattice expansions, we get the following quite important non-trivial algebraic inheritance result:

Corollary 4.49. *Let $\mathfrak{B} \in \mathbf{V}(\mathfrak{A})$. Then,* $\mathrm{Con}(\mathfrak{B}) = \mathrm{Con}(\mathfrak{B}\restriction\Sigma_0)$.

In particular, by (2.5), Lemmas 3.6, 3.5, 4.47, Corollaries 2.17, 4.49 and the $\wedge$-conjunctivity of $\mathcal{A}$ (cf. Remark 3.26 with $j = 0$), we also have:

Corollary 4.50. *Υ_0 is a unitary congruence[equality] determinant for* $\mathrm{Mod}_{[*]}(C)$.

Note that the following rules are satisfied in C^{MP}, in view of (3.3) and (3.4) held in C by its weak $\vee$-disjunctivity (cf. Remark 3.26 with $j = 0$):

$$\{x_0, x_1, \sim^i x_0 \vee x_2\} \vdash (\sim^i x_1 \vee x_2), \tag{4.28}$$

where $i \in 2$. In this way, by Corollary 4.50, we get:

Corollary 4.51. *Any $\mathcal{B} \in \mathrm{Mod}_*(C^{\mathrm{MP}})$ is truth-singular.*

Theorem 4.52. *C^{MP} is defined by $\mathsf{S} \triangleq (\mathrm{Mod}(C) \cap \mathbf{P}^{\mathrm{SD}}(\mathbf{S}^*_*(\overrightarrow{\mathcal{A}})))$, and so by the class of all truth-singular models of C.*

Proof. As $\overrightarrow{\mathcal{A}}$ is truth-singular, while the truth-singularity is preserved under both $\mathbf{P}$ and $\mathbf{S}$, by Corollary 4.46, we have $\mathsf{S} \subseteq \mathrm{Mod}(C^{\mathrm{MP}})$. Conversely, consider any $\mathcal{B} \in (\mathrm{Mod}_*(C^{\mathrm{MP}}) \cap \Re(\mathbf{P}^{\mathrm{SD}}(\mathbf{S}_*(\mathcal{A}))))$, in which case $\mathcal{B} \in \mathrm{Mod}(C)$, while, by Corollary 4.51, $\mathcal{B}$ is truth-singular, whereas $(\mathfrak{B}\restriction\Sigma_0) \in \mathsf{DML}$, and so, by the $\wedge$-conjunctivity of $\mathcal{A}$ (cf. Remark 3.26 with $j = 0$) and Lemma 4.45, $D^{\mathcal{B}} = \{b\}$, whereas b is a unit of $\mathfrak{B}\restriction\Sigma^+$. Moreover, $\mathfrak{B} \in \mathbf{V}(\mathfrak{A})$, in which case, by Remark 2.3 and Lemma 4.48, $\mathfrak{B}$ is isomorphic to a subdirect product of some $\bar{\mathfrak{C}} \in (\mathbf{S}_{>1}\mathfrak{A})^I$, where I is a set, and so there is some embedding e of $\mathfrak{B}$ into $\prod_{i\in I} \mathfrak{C}_i$ such that, for each $i \in I$, $h_i \triangleq (\pi_i \circ e) \in \hom(\mathfrak{B}, \mathfrak{C}_i)$ is surjective, in which case C_i, being non-one-element, contains both t and f, and so, by Lemma 3.25, $h_i(b) = \mathsf{t}$. And what is more, for every $a \in B$ distinct from b, by the injectivity of e, there is some $i \in I$ such that $h_i(a) \neq h_i(b) = \mathsf{t}$. In this way, e is an isomorphism from $\mathcal{B}$ onto the subdirect product $(\prod_{i\in I}\langle\mathfrak{C}_i, \{\mathsf{t}\}\rangle)\restriction(\mathrm{img}\, e)$ of $\langle\langle\mathfrak{C}_i, \{\mathsf{t}\}\rangle\rangle_{i\in I} \in \mathbf{S}^*_*(\overrightarrow{\mathcal{A}})^I$. Hence, by (2.5), we get $\mathcal{B} \in \mathbf{I}(\mathsf{S})$. Then, Corollaries 2.21, 4.46 and (2.5) complete the argument. □

4.6. **Self-extensionality.**

Theorem 4.53. *The following are equivalent:*

(i) *C is self-extensional;*
(ii) *C has the property of Weak Contraposition with respect to $\sim$;*
(iii) *$\overleftarrow{\mathcal{A}}$ is a model of C;*
(iv) *C is defined by $\{\mathcal{A}, \overleftarrow{\mathcal{A}}\}$;*
(v) *$(\psi \in C(\phi)) \Leftrightarrow (\mathfrak{A} \models (\phi \lessapprox \psi))$, for all $\phi, \psi \in \mathrm{Fm}_\Sigma^\omega$;*

(vi) *there is some class* K *of* Σ*-algebras satisfying semilattice identities for* $\wedge$ *such that* $(\psi \in C(\phi)) \Leftrightarrow (\mathsf{K} \models (\phi \lessapprox \psi))$*, for all* $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*;*
(vii) $(\psi \equiv_C \phi) \Leftrightarrow (\mathfrak{A} \models (\phi \approx \psi))$*, for all* $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*;*
(viii) *there is some class* K *of* Σ*-algebras such that* $(\psi \equiv_C \phi) \Leftrightarrow (\mathsf{K} \models (\phi \approx \psi))$*, for all* $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*;*
(ix) *there is an injective homomorphism from* $\overleftarrow{\mathcal{A}}$ *to* $\mathcal{A}$*;*
(x) $\mathfrak{A}$ *is specular;*
(xi) μ *is an isomorphism from* $\overleftarrow{\mathcal{A}}$ *onto* $\mathcal{A}$*;*
(xii) $\overleftarrow{\mathcal{A}}$ *is isomorphic to* $\mathcal{A}$*;*
(xiii) C *is defined by* $\overleftarrow{\mathcal{A}}$*;*
(xiv) $\overrightarrow{\mathcal{A}}$ *is a model of* C*;*
(xv) *any* $\wedge$*-conjunctive truth-non-empty* Σ*-matrix* $\mathcal{B}$ *such that* $\mathfrak{B} \in \mathbf{V}(\mathfrak{A})$ *is a model of* C*;*

in which case $\mathrm{IV}(C) = \mathbf{V}(\mathfrak{A})$*.*

Proof. First, assume (i) holds. Consider any $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$ such that $\psi \in C(\phi)$. Then, since $\mathcal{A}$ is both $\wedge$-conjunctive and $\vee$-disjunctive (cf. Remark 3.26 with $j = 0$), we have $C(\phi \wedge \psi) = C(\{\phi, \psi\}) = C(\phi)$, in which case, by the validity of (3.12) in $\mathfrak{A}$ and (i), we get $C(\sim\psi) \supseteq C(\sim\phi \vee \sim\psi) = C(\sim(\phi \wedge \psi)) = C(\sim\phi) \ni \sim\phi$, and so (ii) holds.

In general, note that, since $\mathcal{A}$ is both finite and $\wedge$-conjunctive (cf. Remark 3.26 with $j = 0$), in which case C is inductive and $\wedge$-conjunctive, in view of Proposition 2.19, any $\wedge$-conjunctive truth-non-empty Σ-matrix $\mathcal{B}$ is a model of C iff $C(\phi) \subseteq \mathrm{Cn}^{\omega}_{\mathcal{B}}(\phi)$, for all $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$. In particular, (v)⇒(xv) is immediate, while (ii)⇒(iii) is by Remark 3.26 with $j = 1$ and the following claim:

Claim 4.54. *Suppose* C *has the Property of Weak Contraposition with respect to* $\sim$*. Then,* $C(\phi) \subseteq \mathrm{Cn}^{\omega}_{\overleftarrow{\mathcal{A}}}(\phi)$*, for all* $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$*.*

Proof. Consider any $\psi \in C(\phi)$, in which case $\sim\phi \in C(\sim\psi)$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Suppose $h(\phi) \in D^{\overleftarrow{\mathcal{A}}}$, in which case $h(\sim\phi) = \sim^{\mathfrak{A}} h(\phi) \notin D^{\mathcal{A}}$, and so $\sim^{\mathfrak{A}} h(\psi) = h(\sim\psi) \notin D^{\mathcal{A}}$, in which case $h(\psi) \in D^{\overleftarrow{\mathcal{A}}}$, as required. □

Likewise, assume (xiv) holds. Consider any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi \in C(\phi)$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h(\phi) \in D^{\overleftarrow{\mathcal{A}}}$. Then, by the structurality of C, Corollary 3.16(3.9) and Remark 3.26 with $j = 0$, $(\sigma_{+1}(\psi) \vee x_0) \in C(\sigma_{+1}(\phi) \vee x_0)$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $[x_0/\mathsf{b}; x_{i+1}/h(x_i)]_{i \in \omega}$, in which case $(g \circ \sigma_{+1}) = h$, and so we have $g(\sigma_{+1}(\phi) \vee x_0) = (h(\phi) \vee^{\mathfrak{A}} \mathsf{b}) = \mathsf{t}$. Hence, by (xiv), we get $(h(\psi) \vee^{\mathfrak{A}} \mathsf{b}) = g(\sigma_{+1}(\psi) \vee x_0) = \mathsf{t}$. Therefore, we eventually get $h(\psi) \in D^{\overleftarrow{\mathcal{A}}}$. Thus, (iii) holds.

On the other hand, $D^{\mathcal{A}}$ and $D^{\overleftarrow{\mathcal{A}}}$ are exactly all non-empty proper prime filters of $\mathfrak{A}{\restriction}\Sigma^+$ (cf. Remark 3.26). Therefore, (iv)⇒(v) is by the Prime Ideal Theorem for distributive lattices (in particular, for $(\mathfrak{A}{\restriction}\Sigma^+) = \mathfrak{D}^2_2$). In addition, both (v)⇒(vii) and (vi)⇒(viii) are by the semilattice identities for $\wedge$ that are true in $\mathfrak{A}$, while (vi/viii) is a particular case of (v/vii), respectively, whereas (viii)⇒(i) is immediate.

Now, assume (iii) holds.

In that case, (iv) is evident.

Moreover, $\overleftarrow{\mathcal{A}}$ is consistent and, being finite, is finitely generated. In addition, by Lemma 3.4 and Remark 3.26 with $j = 1$, $\overleftarrow{\mathcal{A}}$ is simple and $\vee$-disjunctive. Then, by Lemma 2.20, there is some finite set I, some I-tuple $\overline{\mathcal{C}}$ of submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \overleftarrow{\mathcal{A}})$, in which case, by Remark 3.14 and (2.5), $\mathcal{D}$ is consistent and $\vee$-disjunctive, and so, by Corollary 3.15, there is some $i \in I$ such that $h \triangleq (\pi_i{\restriction}D) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$. Moreover, by Lemmas 3.4, 3.6 and Remark 3.26 with $j = 0$, $\mathcal{C}_i$ is simple. Hence, by Proposition 2.16, $(\ker h) = \partial(\mathcal{D}) = (\ker g)$. Therefore, by Proposition 2.15, $e \triangleq (h \circ g^{-1}) \in \hom_{\mathrm{S}}(\overleftarrow{\mathcal{A}}, \mathcal{C}_i) \subseteq \hom(\overleftarrow{\mathcal{A}}, \mathcal{A})$ is injective, and so (ix) holds.

Furthermore, (ix)⇒(x) is by the following claim:

Claim 4.55. *Any injective homomorphism from* $\overleftarrow{\mathcal{A}}$ *to* $\mathcal{A}$ *is specular.*

Proof. Consider any injective $e \in \hom(\overleftarrow{\mathcal{A}}, \mathcal{A})$. Then, since $(\sim^{\mathfrak{A}} a = a) \Leftrightarrow (a \in \{\mathsf{n}, \mathsf{b}\})$, for all $a \in A$, we have both $e[\{\mathsf{n}, \mathsf{b}\}] \subseteq \{\mathsf{n}, \mathsf{b}\}$ and, by the injectivity of e, $e[\{f, \mathsf{t}\}] \subseteq \{\mathsf{f}, \mathsf{t}\}$. Moreover, as $\mathsf{n}, \mathsf{t} \in D^{\overleftarrow{\mathcal{A}}}$, while $(\{\mathsf{n}, \mathsf{b}\} \cap D^{\mathcal{A}}) = \{\mathsf{b}\}$, whereas $(\{\mathsf{f}, \mathsf{t}\} \cap D^{\mathcal{A}}) = \{\mathsf{t}\}$, we then get $e(\mathsf{n}) = \mathsf{b}$ and $e(\mathsf{t}) = \mathsf{t}$, respectively. Hence, by the injectivity of e, we eventually get $e(\mathsf{b}) = \mathsf{n}$ and $e(\mathsf{f}) = \mathsf{f}$, as required. □

Finally, (x)⇒(xi) is immediate, while (xii/iii/xiv) is a particular case of (xi/xiii/xv), respectively, whereas (xii)⇒(xiii) is by (2.5). After all, (vii) implies $\mathrm{IV}(C) = \mathbf{V}(\mathfrak{A})$, as required. □

As a first immediate *generic* consequence of Theorems 4.20, 4.53(i)⇒(x) and Lemma 4.34 with $B = \{\mathsf{b}, \mathsf{n}\}$ applicable to all bilattice expansions *at once* (cf. Subsubsection 6.1.2), we have:

Corollary 4.56. *Suppose* $\{\mathsf{f}, \mathsf{t}\}$ *does not form a subalgebra of* $\mathfrak{A}$*. Then,* C *is not self-extensional. In particular,* C *is* $\sim$*-subclassical, whenever it is self-extensional.*

Corollary 4.57. *Suppose* C *is self-extensional. Then,* $\{\mathsf{n}(, \mathsf{f}, \mathsf{t})\}$ *forms a subalgebra of* $\mathfrak{A}$ *iff* $\{\mathsf{b}(, \mathsf{f}, \mathsf{t})\}$ *does so. In particular, the following hold:*

(i) *the following are equivalent:*
   a) C *holds Relevance Principle;*
   b) C *is purely inferential;*
   c) C *has no inconsistent formula;*
   d) $\{\mathsf{n}\}$ *forms a subalgebra of* $\mathfrak{A}$*;*

e) $\{\mathsf{b}\}$ *forms a subalgebra of* $\mathfrak{A}$*;*
f) *there is no* $\psi \in \mathrm{Fm}^1_\Sigma$ *such that* $\psi^{\mathfrak{A}}[A] = \{\mathsf{t}\}$*;*
g) *there is no* $\phi \in \mathrm{Fm}^1_\Sigma$ *such that* $\phi^{\mathfrak{A}}[A] = \{\mathsf{f}\}$*.*

(ii) *[providing* C *is not purely inferential,]* C^{EM} *is{ maximally}* $\sim$*-paraconsistent iff* C^{R} *is [non-]inferentially paracomplete, in which case, when* $\mathfrak{A}_{\not{b}}$ *is regular (in particular,* $\Sigma = \Sigma_{0[1]}$*),* C^{R} *is maximally [non-]inferentially paracomplete, while any extension of* C *is both* $\sim$*-paraconsistent and [non-]inferentially paracomplete iff it is a sublogic of* $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$*, in its turn, being an axpansion of* $LP \cap K_3$*.*

Proof. Since $\mu[\{\mathsf{n}(,\mathsf{f},\mathsf{t})\}] = \{\mathsf{b}(,\mathsf{f},\mathsf{t})\}$, in view of Theorems 4.15(i)⇔(iii), 4.25, 4.35, 4.36, 4.53(i)⇒(x), Lemmas 4.13, 4.14 and Corollary 4.24, it only remains to prove the equivalence of the subitems **f)** and **g)** to others within (i).

First, **f)** is a particular case of **b)**. Next, **f)**⇔**g)** is by the fact $\sim^{\mathfrak{A}}(\mathsf{f}/\mathsf{t}) = (\mathsf{t}/\mathsf{f})$.

Finally, assume $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\varphi^{\mathfrak{A}}(\mathsf{n}) = \mu(\varphi^{\mathfrak{A}}(\mathsf{b}))$ and $\varphi^{\mathfrak{A}}[\{\mathsf{f},\mathsf{t}\}] \subseteq \{\mathsf{f},\mathsf{t}\}$, by Lemma 4.34 and Theorem 4.53(i)⇒(x), and so $\psi^{\mathfrak{A}}[A] = \{\mathsf{t}\}$, where $\psi \triangleq (x_0 \vee (\varphi \vee \sim\varphi)) \in \mathrm{Fm}^1_\Sigma$. Thus, **f)**⇒**e)** holds, as required. □

Corollary 4.57(i)**b)**⇐**f)**⇔**g)** collectively with Theorem 4.8 imply:

Corollary 4.58. *Any self-extensional four-valued expansion of* C_{B} *is not purely inferential iff it is definitionally equivalent to an expansion of* C_{BB}*.*

This clarifies the meaning of the bounded version C_{BB} of C_{B}. Subsubsection 6.1.3 shows that the condition of self-extensionality cannot be omitted in the formulations of Corollaries 4.57 and 4.58. As for Corollary 4.57(ii) (in case $\mathfrak{A}_{\not{b}}$ is regular), it clarifies the meaning of the self-extensional (in view of Lemma 4.60 below) meet $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ to be studied far more in Paragraph 6.1.4.1.

4.6.1. *Self-extensional extensions.* Note that $(\mathfrak{DM}_4{\restriction}\{\mathsf{f},\mathsf{t}\}) \in \mathsf{BL} \not\ni (\mathfrak{DM}_4{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}) \in \mathsf{KL} \not\ni \mathfrak{DM}_4$. In this way, by Remark 2.3 and Lemma 4.48, we immediately get:

Corollary 4.59. *Suppose* $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}$ *and* $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ *does not form [resp., forms] a subalgebra of* $\mathfrak{A}$*. Then, there is no non-trivial proper subvariety of* $\mathbf{V}(\mathfrak{A})$ *other than* $\mathbf{V}(\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{t}\})$ *relatively axiomatized by* (3.14)*[and* $\mathbf{V}(\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\})$ *relatively axiomatized by* (3.13)*].*

Lemma 4.60. *Suppose* $\mu \in \hom(\mathfrak{A},\mathfrak{A})$ *and (both)*$\{\mathsf{f},\mathsf{b},\mathsf{t}\}$*(and* $\{\mathsf{n}\}$*) forms a subalgebra of* $\mathfrak{A}$*. Then, the logic* C' *of* $\mathsf{S} \triangleq \{\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}, \overleftarrow{\mathcal{A}}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}\}$ *is a proper inferentially consistent inductive self-extensional* $\vee$*-disjunctive non-pseudo-axiomatic(purely inferential) both paracomplete and* $\sim$*-paraconsistent extension of* C*,* $\mathbf{V}(\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\})$ *being its intrinsic variety, and is also defined by* $\{\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}, \mathcal{A}{\restriction}\{\mathsf{f},\mathsf{n},\mathsf{t}\}\}$*, and so is equal to* $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ *and is relatively axiomatized by* (4.19)*.*

Proof. As both S and all members of it are finite, by (2.5) and Theorem 4.53(x)⇒(iii), C' is an inductive extension of C. And what is more, $\mu{\restriction}\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ is an isomorphism from $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ onto $\overleftarrow{\mathcal{A}}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, in which case, by (2.5), C' is equally defined by $\{\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}, \mathcal{A}{\restriction}\{\mathsf{f},\mathsf{n},\mathsf{t}\}\}$, and so is equal to $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$, in view of Theorem 4.25(iii)→(iv) and Corollary 4.31. In particular, it is both paracomplete and $\sim$-paraconsistent, for $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is $\sim$-paraconsistent, while $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{n},\mathsf{t}\}\}$ is paracomplete. Moreover, (4.1), being true in S, is not so in $\mathcal{A}$ under $[x_0/\mathsf{b}, x_1/\mathsf{n}]$. Hence, $C' \neq C$. On the other hand, $\{\mathsf{t}\}$ and $\{\mathsf{b},\mathsf{t}\}$ are exactly all proper non-empty prime filters of the three-element chain distributive lattice $(\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}){\restriction}\Sigma^+$. In particular, by Corollary 3.16, C' is $\vee$-disjunctive, while, by Theorems 3.21 and 3.24, it is relatively axiomatized by (4.19), for $\{\{\mathsf{f},\mathsf{n},\mathsf{t}\},\{\mathsf{f},\mathsf{b},\mathsf{t}\}\}^{\triangledown}$ is the lower cone of $\mathbf{S}_*(\mathcal{A})$ relatively axiomatized by (4.1). And what is more, by the Prime Ideal Theorem for distributive lattices, $\equiv_{C'}$ is the set of all Σ-identities true in $\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, i.e., in $\mathbf{V}(\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\})$. Thus, C' is self-extensional, $\mathbf{V}(\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\})$ being its intrinsic variety. Furthermore, as all members of S are consistent and truth-non-empty, C' is inferentially consistent and, by Proposition 2.19, is non-pseudo-axiomatic.(Finally, $\mathcal{A}{\restriction}\{\mathsf{n}\}$ is a truth-empty submatrix of $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{n},\mathsf{t}\}$. Thus, by (2.5), C' is purely inferential, as required.) □

After all, combining Propositions 2.18, 2.19, Remarks 2.8, 2.9, Theorems 4.20, 4.21, 4.25, 4.53, Lemmas 4.13, 4.60, Corollaries 4.56, 4.59 and Example 2.11, we eventually get:

Theorem 4.61. *Suppose* C *is self-extensional and[not] maximally* $\sim$*-paraconsistent(as well as purely inferential). Then, there is no inferentially consistent proper self-extensional (non-pseudo-axiomatic/purely-inferential)extension of* C *other than* $C^{\mathrm{PC}}_{(/+0)}$*[and* $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$*], being, in its[their] turn, both so and inductive[, while the former being a proper extension of the latter].*

On the other hand, any logic is either purely-inferential or, otherwise, non-pseudo-axiomatic. Therefore, by Remarks 2.8, 2.10, 3.26, 3.14, Corollaries 3.16, 4.56, 4.57, Theorems 3.21, 4.20, 4.61 and Lemma 4.60, we also get the following interesting non-trivial consequence:

Corollary 4.62. *Suppose* C *is self-extensional[and maximally* $\sim$*-paraconsistent]. Then, any extension of* C *is* $\vee$*-disjunctive if[f] it is self-extensional.*

4.6.2. *Semantics of miscellaneous extensions versus self-extensionality.* By Theorems 4.52, 4.53(i)⇔(xiv) and (2.5), we first get:

Corollary 4.63. C *is self-extensional iff* C^{MP} *is defined by* $\overrightarrow{\mathcal{A}}$*.*

Likewise, we also have the following one more characterization of the self-extensionality of C:

Theorem 4.64. C *is self-extensional iff* C^{NP} *is defined by* $\mathcal{A} \times \overrightarrow{\mathcal{A}}$*.*

Proof. We use Theorem 4.53(i)⇔(xiv) tacitly. First, $\Delta_A \times \Delta_A$ is an embedding of $\overrightarrow{\mathcal{A}}$ into $\mathcal{A} \times \overrightarrow{\mathcal{A}}$. In this way, (2.5) yields the "if" part. Conversely, assume C is self-extensional. Then, $\mathcal{A} \times \overrightarrow{\mathcal{A}}$ is a model of C. Moreover, $\{a, \sim^{\mathfrak{A}} a\} \subseteq \{\mathsf{t}\}$, for no $a \in A$. Therefore, $\mathcal{A} \times \overrightarrow{\mathcal{A}}$ is not $\sim$-paraconsistent, so it is a model of C^{NP}. Finally, consider any finite set I, any $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and any subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of $\overline{\mathcal{C}}$, in which case $\mathcal{D}$ is a non-$\sim$-paraconsistent submatrix of $\mathcal{A}^I$. Put $J \triangleq \hom(\mathcal{D}, \mathcal{A} \times \overrightarrow{\mathcal{A}})$. Consider any $a \in (D \setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent, and so, by Lemma 4.39, there is some $g \in \hom(\mathcal{D}, \overrightarrow{\mathcal{A}}) \neq \varnothing$. Moreover, there is some $i \in I$, in which case $f \triangleq (\pi_i \upharpoonright D) \in \hom(\mathcal{D}, \mathcal{A})$, such that $f(a) \notin D^{\mathcal{A}}$. Then, $h \triangleq (f \times g) \in J$ and $h(a) \notin D^{\mathcal{A} \times \overrightarrow{\mathcal{A}}}$. In this way, $(\prod \Delta_J) \in \hom_{\mathrm{S}}(\mathcal{D}, (\mathcal{A} \times \overrightarrow{\mathcal{A}})^J)$. Thus, by (2.5) and Corollary 2.21, C^{NP} is finitely-defined by $\mathcal{A} \times \overrightarrow{\mathcal{A}}$. Then, the finiteness of A completes the argument. □

5. Paraconsistent finitely-many-valued logics

The present section collectively with Subsection 6.2 exemplifying the former incorporates the material prepared by and announced in 1995 (cf. the paragraph after Theorem 2.1 in [14] and the reference [Pyn 95b] therein).

5.1. **Three-valued paraconsistent logics with subclassical negation.** Fix any (possibly, secondary) unary connective $\wr$ of Σ.

A Σ-matrix $\mathcal{A}$ is said to be *$\wr$-superclassical*, provided $A = \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, $D^{\mathcal{A}} = \{\mathsf{b}, \mathsf{t}\}$, $\wr^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$, $\wr^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$ and $\wr^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}}$, in which case it is three-valued, both consistent and false-singular with $\dashv^{\mathcal{A}} = \mathsf{f}$ as well as $\wr$-paraconsistent, while $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A} \upharpoonright \{\wr\}$, in which case $\wr$ is clearly a subclassical negation for the logic of $(\mathcal{A} \upharpoonright \{\wr\}) \upharpoonright \{\mathsf{f}, \mathsf{t}\}$, and so for that of $\mathcal{A}$, in view of (2.5). In this way, we have argued the routine part (viz., (ii)⇒(i)) of the following preliminary marking the framework of the present subsection:

Proposition 5.1. *Let C be a Σ-logic. Then, the following are equivalent:*
(i) *C is both three-valued and $\wr$-paraconsistent, while $\wr$ is a subclassical negation for C;*
(ii) *C is defined by a $\wr$-superclassical Σ-matrix.*

Proof. Assume (i) holds. Let $\mathcal{B}$ be any three-valued Σ-matrix defining C. Define an $e : \{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \to B$ as follows. In that case, $\mathcal{B}$ is $\sim$-paraconsistent, so there are some $e(\mathsf{b}) \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} e(\mathsf{b}) \in D^{\mathcal{B}}$ and some $e(\mathsf{f}) \in (B \setminus D^{\mathcal{B}})$, in which case $e(\mathsf{f}) \neq e(\mathsf{b})$. Next, by (2.7) with $m = 1$ and $n = 0$, there is some $e(\mathsf{t}) \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} e(\mathsf{t}) \notin D^{\mathcal{B}}$, in which case $e(\mathsf{f}) \neq e(\mathsf{t}) \neq e(\mathsf{b})$. In this way, $e : \{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \to B$ is injective, and so bijective, for $|B| = 3$. Hence, it is an isomorphism from $\mathcal{A} \triangleq \langle e^{-1}[\mathfrak{B}], \{\mathsf{b}, \mathsf{t}\}\rangle$ onto $\mathcal{B}$. Therefore, by (2.5), C is defined by $\mathcal{A}$. Furthermore, $\sim^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}}$, while $\sim^{\mathfrak{A}}\mathsf{t} \notin D^{\mathcal{A}}$, in which case $\sim^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$, and so it only remains to show that $\sim^{\mathfrak{A}}\mathsf{f} = \mathsf{t}$. We do it by contradiction. For suppose $\sim^{\mathfrak{A}}\mathsf{f} \neq \mathsf{t}$, in which case we have the following two exhaustive cases:

(1) $\sim^{\mathfrak{A}}\mathsf{f} = \mathsf{f}$.
This contradicts to (2.7) with $m = 0$ and $n = 1$.
(2) $\sim^{\mathfrak{A}}\mathsf{f} = \mathsf{b}$.
As $\sim^{\mathfrak{A}}\mathsf{b} \in \{\mathsf{b}, \mathsf{t}\}$, we then have the following two exhaustive subcases:
(a) $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$.
Then, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a = \mathsf{b} \in D^{\mathcal{A}}$, for each $a \in D^{\mathcal{A}}$. This contradicts to (2.7) with $m = 3$ and $n = 0$.
(b) $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$.
Then, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\mathsf{f} = \mathsf{f}$. This contradicts to (2.7) with $m = 0$ and $n = 3$.

Thus, in any case, we come to a contradiction, as required. □

Proposition 5.2. *Any three-valued $\wr$-paraconsistent Σ-logic C with subclassical negation $\wr$ is minimally three-valued.*

Proof. By contradiction. For supposed C is defined by a Σ-matrix $\mathcal{A}$ such that $|A| < 3$, in which case it is $\wr$-paraconsistent, and so both consistent and truth-non-empty. Therefore, there is some $a \in A$ such that $D^{\mathcal{A}} = \{a\}$. Hence, $\sim^{\mathfrak{A}} a = a$. This contradicts to (2.7) with $m = 1$ and $n = 0$, as required. □

Remark 5.3. By Example 3.3 with $j = 0$ and $\vec{k} = \Delta_2$, $\Upsilon_\wr$ is a unary unitary equality determinant for any $\wr$-superclassical Σ-matrix $\mathcal{A}$. □

5.1.1. *Maximal paraconsistency of three-valued paraconsistent logics with subclassical negation.* Fix any $\wr$-superclassical Σ-matrix $\mathcal{A}$. Let C be the logic of $\mathcal{A}$.

Then, a *ternary* b*-relative (weak classical)conjunction for* $\mathfrak{A}$ is any $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{f}, \mathsf{t}) = \mathsf{f} = \varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{t}, \mathsf{f})$ or, equivalently, the rules of the form $\{x_0, \wr x_0, \varphi[x_{2-i}/(\wr x_{i+1})]\} \vdash x_{i+1}$, where $i \in 2$, are satisfied in C.

We start from proving the following key lemma "killing two birds (both the sufficiency part of the characterization of the maximal $\wr$-paraconsistency of three-valued $\wr$-paraconsistent logics with subclassical negation $\wr$ and the uniqueness of a $\wr$-superclassical matrix defining any given maximally $\wr$-paraconsistent three-valued logic with subclassical negation $\wr$) with one stone":

Lemma 5.4 (Three-Valued Key Lemma)**.** *Let $\mathcal{B}$ a(simple) finitely-generated $\wr$-paraconsistent model of C. Suppose either $\mathfrak{A}$ has a ternary b-relative conjunction or $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A}$ is embeddable into $\mathcal{B}/\eth(\mathcal{B})$ (resp., into $\mathcal{B}$).*

Proof. Put $\mathcal{E} \triangleq (\mathcal{B}/\eth(\mathcal{B})$ (resp., $\mathcal{E} \triangleq \mathcal{B}$). Then, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case, by (2.5), $\mathcal{D}$ is $\wr$-paraconsistent, and so there are some $a \in D^{\mathcal{D}}$ such that $\sim^{\mathfrak{D}} a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$. Then, by Lemma 4.1, $D \ni a = (I \times \{\mathsf{b}\})$. Consider the following complementary cases:

(1) $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, $\mathfrak{A}$ has a ternary b-relative conjunction $\varphi \in \mathrm{Fm}^3_\Sigma$. Put $c \triangleq \varphi^{\mathfrak{D}}(a, b, \wr^{\mathfrak{D}} b) \in D$, $d \triangleq \wr^{\mathfrak{D}} c \in D$ and $J \triangleq \{i \in I \mid \pi_i(b) \neq \mathsf{b}\} \neq \varnothing$, for $b \notin D^{\mathcal{D}}$. Given any $\vec{a} \in A^2$, set $(a_0|a_1) \triangleq ((\{a_0\} \times J) \cup (\{a_1\} \times (I \setminus J))) \in D$. Then, we have $c = (\mathsf{f}|\mathsf{b})$, $a = (\mathsf{b}|\mathsf{b})$ and $d = (\mathsf{t}|\mathsf{b})$. In this way, since $J \neq \varnothing$, while $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle a', (a'|\mathsf{b})\rangle \mid a' \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

(2) $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\{\varphi^{\mathfrak{A}}(\mathsf{b}), \wr^{\mathfrak{A}} \varphi^{\mathfrak{A}}(\mathsf{b})\} = \{\mathsf{f}, \mathsf{t}\}$, and so $D \supseteq \{a, \varphi^{\mathfrak{D}}(a), \wr^{\mathfrak{D}} \varphi^{\mathfrak{D}}(a)\} = \{I \times \{a'\} \mid a' \in A\}$. Therefore, as $I \neq \varnothing$, for $b \notin D^{\mathcal{D}}$, $\{\langle a', I \times \{a'\}\rangle \mid a' \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Thus, anyway, there is an injective $e \in \mathrm{hom_S}(\mathcal{A}, \mathcal{D})$, in which case $(g \circ e) \in \mathrm{hom_S}(\mathcal{A}, \mathcal{E})$, and so Corollary 2.14, Lemma 3.4 and Remark 5.3 complete the argument. ☐

Lemma 5.5. *Suppose[C is $\sim$-subclassical,] $\mathfrak{A}$ has no ternary b-relative conjunction and $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$. Then, C has a proper $\wr$-paraconsistent[$\wr$-subclassical] extension[, in which case $\wr$ is a subclassical negation for it].*

Proof. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}^3$ generated by $\{\langle \mathsf{b},\mathsf{b},\mathsf{b}\rangle, \langle \mathsf{b},\mathsf{f},\mathsf{t}\rangle, \langle \mathsf{b},\mathsf{t},\mathsf{f}\rangle\}$. If $\langle \mathsf{f}, a, b\rangle$ was in B, for any $a, b \in A$, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{b},\mathsf{b}) = \mathsf{f}$, in which case $\{\mathsf{b}\}$ would not form a subalgebra of $\mathfrak{A}$. Therefore, as $\wr^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$, we conclude that $(((\{\mathsf{f},\mathsf{t}\} \times A) \times A) \cap B) = \varnothing$. Likewise, if $\langle \mathsf{b},\mathsf{f},\mathsf{f}\rangle$ was in B then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f},\mathsf{t}) = \mathsf{f} = \varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{t},\mathsf{f})$, in which case it would be a ternary b-relative conjunction for $\mathfrak{A}$. Therefore, as $\wr^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$ and $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, we conclude that $(\{\langle \mathsf{b},\mathsf{f},\mathsf{f}\rangle, \langle \mathsf{b},\mathsf{t},\mathsf{t}\rangle\} \cap B) = \varnothing$. Thus, $B = \{\langle \mathsf{b},\mathsf{b},\mathsf{b}\rangle, \langle \mathsf{b},\mathsf{f},\mathsf{t}\rangle, \langle \mathsf{b},\mathsf{t},\mathsf{f}\rangle\}$, in which case $D^{\mathcal{B}} = \{\langle \mathsf{b},\mathsf{b},\mathsf{b}\rangle\} \neq B$, and so, as $\wr^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, $\mathcal{B}$ is $\sim$-paraconsistent, while the rule $x_0 \vdash \wr x_0$ is true in $\mathcal{B}$, and so is its logical consequence

$$\{x_0, x_1, \wr x_1\} \vdash \wr x_0, \tag{5.1}$$

not being true in $\mathcal{A}$ under $[x_0/\mathsf{t}, x_1/\mathsf{b}]$.[Moreover, (5.1) is true in any $\wr$-classical model $\mathcal{C}'$ of C, for $\mathcal{C}'$ is not $\wr$-paraconsistent]. In this way, taking (2.5) into account, the logic of $\{\mathcal{B}[, \mathcal{C}']\}$ is a proper $\wr$-paraconsistent[$\wr$-subclassical] extension of C, as required. ☐

Theorem 5.6. *[Suppose C is $\wr$-subclassical. Then,]C has no proper $\wr$-paraconsistent[$\wr$-subclassical] extension iff either $\mathfrak{A}$ has a ternary b-relative conjunction or $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$.*

Proof. Assume either $\mathfrak{A}$ has a ternary b-relative conjunction or $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Consider any $\wr$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \wr x_0\}) \supseteq \{x_0, \wr x_0\}$, while, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated $\wr$-paraconsistent submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_\Sigma, T \cap \mathrm{Fm}^2_\Sigma\rangle$, in view of (2.5). Then, by Lemma 5.4, $\mathcal{A}$ is embeddable into $\mathcal{B}/\partial(\mathcal{B})$, in which case, by (2.5), it is a model of C', and so $C' = C$. Thus, C is maximally $\wr$-paraconsistent. In this way, Lemma 5.5 completes the argument. ☐

On the other hand, Subsubsections 6.1.1 and 6.1.2 definitely show that the maximal paraconsistency is not at all a prerogative of merely three-valued logics. And what is more, as it is shown in the next subsection, there is no limit of the number of truth values, for which minimally many-valued maximally paraconsistent logics exist.

Lemma 5.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\wr$-superclassical Σ-matrices and $e \in \mathrm{hom_S}(\mathcal{A}, \mathcal{B})$. Then, e is diagonal. In particular, $\mathcal{A} = \mathcal{B}$.*

Proof. In that case, $(\mathcal{A}{\restriction}\{\wr\}) = (\mathcal{B}{\restriction}\{\wr\})$ is $\wr$-superclassical and $e \in \mathrm{hom_S}(\mathcal{A}{\restriction}\{\wr\}, \mathcal{B}{\restriction}\{\wr\})$. Therefore, by Lemma 3.7 and Remark 5.3, e is diagonal, and so $\mathcal{A} = \mathcal{B}$, for $A = B$, as required. ☐

After all, the second "bird" is as follows:

Theorem 5.8. *Let $\mathcal{B}$ be a $\wr$-superclassical Σ-matrix. Suppose $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$) and C is maximally $\wr$-paraconsistent. Then, $\mathcal{B} = \mathcal{A}$.*

Proof. Then, by Lemma 3.4 and Remark 5.3, $\mathcal{B}$ is a simple finite (and so finitely-generated) $\wr$-paraconsistent model of C. Hence, by Lemma 5.4 and Theorem 5.6, $\mathcal{A}$ is embeddable into $\mathcal{B}$. In this way, Lemma 5.7 completes the argument. ☐

In view of Proposition 5.1 and Theorem 5.8, the unique $\wr$-superclassical Σ-matrix defining a given three-valued maximally $\wr$-paraconsistent Σ-logic $\boldsymbol{C}$ with subclassical negation $\wr$ is said to be *characteristic for $\boldsymbol{C}$*.

5.1.2. *Weakly conjunctive three-valued paraconsistent logics with subclassical negation.* Fix (in addition to $\wr$) any (possibly, secondary) binary connective $\diamond$ of Σ.

Remark 5.9. Given any weakly $\diamond$-conjunctive $\wr$-superclassical Σ-matrix $\mathcal{A}$, $(x_1 \wedge x_2)$ is clearly a ternary b-relative conjunction for $\mathfrak{A}$. ☐

By Proposition 5.1, Theorems 5.6, 5.8 and Remark 5.9, we immediately get:

Corollary 5.10. *Any three-valued $\wr$-paraconsistent weakly $\diamond$-conjunctive Σ-logic C with subclassical negation $\wr$ is maximally $\wr$-paraconsistent.*

Corollary 5.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\wr$-superclassical Σ-matrices and C the logic of $\mathcal{A}$. Suppose $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$) and C is weakly $\diamond$-conjunctive. Then, $\mathcal{B} = \mathcal{A}$.*

Since the three-valued submatrix arising in the formulation of the following corollary is both $\wedge$-conjunctive and $\sim$-superclassical, Proposition 5.1 and Corollary 5.10 yield a supplementary generic insight into the following particular case of Corollary 4.24:

Corollary 5.12. *Let $\mathcal{A}$ be as in Section 4. Suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, the logic of $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is maximally $\sim$-paraconsistent.*

5.1.2.1. Subclassical three-valued paraconsistent weakly conjunctive logics. Fix any $\sim$-superclassical weakly $\diamond$-conjunctive Σ-matrix $\mathcal{A}$. Let C be the logic of it.

Lemma 5.13. *Let $\mathcal{B}$ a (simple)finitely generated consistent model of C. Then, the following hold:*

(i) *$\mathcal{B}$ is $\wr$-paraconsistent, if $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*
(ii) *$\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is embeddable into $\mathcal{B}/\partial(\mathcal{B})$ (resp., into $\mathcal{B}$ itself), if $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. Put $\mathcal{E} \triangleq (\mathcal{B}/\partial(\mathcal{B}))$ (resp., $\mathcal{E} \triangleq \mathcal{B}$). Then, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some $n \in \omega$, some n-tuple $\overline{\mathcal{C}}$ constituted by consistent submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{E})$, in which case, by (2.5), $\mathcal{D}$ is consistent, and so, in particular, $n \neq 0$. Hence, by Lemma 3.8, $D \ni a \triangleq (n\times\{\mathsf{f}\})$, in which case $D \ni b \triangleq {\sim}^{\mathfrak{D}} a = (n\times\{\mathsf{t}\})$. Consider the following respective cases:

(i) $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{b}$. Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a, b) = (I \times \{\mathsf{b}\})$, in which case $\wr^{\mathfrak{D}} c \in D^{\mathcal{D}}$, and so $\mathcal{D}$, being consistent, is $\wr$-paraconsistent, and so is $\mathcal{B}$, in view of (2.5), as required.
(ii) $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, $\mathcal{F} \triangleq (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})$ is $\wr$-classical, in which case it is simple, in view of Example 3.2 and Lemma 3.4. Finally, as $\{n \times \{d\} \mid d \in F\} \subseteq D$ and $n \neq 0$, $e \triangleq \{\langle d, n \times \{d\}\rangle \mid d \in F\}$ is an embedding of $\mathcal{F}$ into $\mathcal{D}$, in which case, $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{F},\mathcal{E})$, and so Corollary 2.14 completes the argument. □

Theorem 5.14. *C is $\wr$-subclassical iff $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case the logic of $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is the only $\wr$-classical extension of C.*

Proof. Let $\mathcal{B}$ be a $\wr$-classical model of C, in which case it is two-valued, and so finite (in particular, finitely generated), consistent and simple (cf. Example 3.2 and Lemma 3.4) but not $\sim$-paraconsistent.

First, by Lemma 5.13(i), $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

Conversely, assume $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case, by (2.5), $\mathcal{D} \triangleq (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})$ is a $\wr$-classical model of C, and so, by (2.5), Corollary 3.11 and Lemma 5.13(ii), we eventually get $\mathcal{D} = \mathcal{B}$, as required. □

In view of Theorem 5.14, the unique $\wr$-classical extension of C (if any) is referred to as *characteristic for C* and is denoted by C^{PC}.

Theorem 5.15. *Let C' be a consistent extension of C. Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is a model of C'.*

Proof. Then, $x_0 \notin C'(\varnothing)$, while, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, C'(\varnothing)\rangle$ is a model of C' (in particular, of C), and so is its consistent finitely generated submatrix $\langle \mathfrak{Fm}^1_\Sigma, \mathrm{Fm}^1_\Sigma \cap C'(\varnothing)\rangle$, in view of (2.5). In this way, (2.5) and Lemma 5.13(ii) complete the argument. □

5.1.3. *Disjunctive three-valued paraconsistent logics with subclassical negation.* Fix (in addition to $\wr$) a (possibly, secondary) binary connective $\veebar$ of Σ and a $\wr$-superclassical Σ-matrix $\mathcal{A}$. Let C be the logic of $\mathcal{A}$. Then, by Corollary 3.17, we first have:

Corollary 5.16. *C is[weakly] $\veebar$-disjunctive iff $\mathcal{A}$ is so.*

Corollary 5.17. *Any $\wr$-classical extension of C is[weakly] $\veebar$-disjunctive, whenever C is so.*

Theorem 5.18. *Let $\mathcal{B}$ be a $\wr$-superclassical Σ-matrix. Suppose $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$) and C is $\veebar$-disjunctive. Then, $\mathcal{B} = \mathcal{A}$.*

Proof. In that case, by Corollary 3.17, Lemma 3.4 and Remark 5.3, $\mathcal{B}$ is a $\veebar$-disjunctive simple $\wr$-paraconsistent finite (in particular, finitely-generated) model of C. Hence, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}\}$, there are some finite set I, some I-tuple $\overline{\mathcal{C}}$ of consistent submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{B})$. Then, by Remark 3.14 and (2.5), $\mathcal{D}$ is $\veebar$-disjunctive and $\wr$-paraconsistent, in which case it is consistent, and so, by Corollary 3.15, there is some $i \in I$ such that $h \triangleq (\pi_i{\restriction}D) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{C}_i)$. Moreover, as $\mathcal{C}_i$ is consistent, we have $\mathsf{f} \in C_i$, and so $\mathsf{t} = \wr^{\mathfrak{A}}\mathsf{f} \in C_i$. And what is more, since $\mathcal{D}$ is $\wr$-paraconsistent, there is some $a \in D^{\mathcal{D}}$ such that $\wr^{\mathfrak{D}} a \in D^{\mathcal{D}}$, in which case, by Lemma 4.1, $C_i \ni \pi_i(a) = \mathsf{b}$, and so $\mathcal{C}_i = \mathcal{A}$. On the other hand, by Lemma 3.4 and Remark 5.3, $\mathcal{A}$ is simple. Therefore, by Proposition 2.16, we have $(\ker h) = \partial(\mathcal{D}) = (\ker g)$. In this way, by Proposition 2.15, we eventually conclude that $g \circ h^{-1}$ is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, in which case Lemma 5.7 completes the argument. □

5.1.3.1. Subclassical three-valued paraconsistent disjunctive logics. Note that $\mathbf{S}_*(\mathcal{A}) \setminus \{\mathcal{A}\}$ is either the singleton $\{\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}\}$, if $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, or empty, otherwise. In this way, the fact that $\wr$-[super]classical matrices are not [resp., are] $\wr$-paraconsistent, by Corollary 5.16, Lemma 4.26 and Theorem 3.21, we then get:

Theorem 5.19. *Suppose C is $\veebar$-disjunctive and $\{\mathsf{f},\mathsf{t}\}$ does not form [resp., forms] a subalgebra of $\mathfrak{A}$. Then, there is no [resp., a unique] proper consistent $\veebar$-disjunctive extension of C[, in which case it is defined by $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ and relatively axiomatized by (4.10)].*

Recall that (4.10) is nothing but the *Resolution* rule. Since any $\wr$-classical Σ-logic is consistent but not $\wr$-paraconsistent, as opposed to C, by (2.5), Corollary 5.17 and Theorem 5.19, we eventually get the following "disjunctive" analogue of Theorem 5.14:

Corollary 5.20. *[Suppose C is $\veebar$-disjunctive. Then,]C is $\wr$-subclassical if[f] $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case the logic of $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is a[unique] $\wr$-classical extension of C.*

Remark 5.21. Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is weakly $\veebar$-disjunctive. Then, $\wr(\wr x_1 \veebar \wr x_2)$ is clearly a ternary b-relative conjunction for $\mathfrak{A}$. □

Combining Corollaries 5.16, 5.20, Remarks 3.14, 5.21, Proposition 5.1 and Theorem 5.6, we then get:

Theorem 5.22. *Any $\veebar$-disjunctive $\wr$-subclassical three-valued $\wr$-paraconsistent Σ-logic is maximally $\wr$-paraconsistent.*

5.2. **Minimally n-valued maximally paraconsistent subclassical logics.** Fix any $n \in (\omega \setminus 3)$.

Let $\Sigma_{[+]} \triangleq ([\Sigma^+ \cup]\{\supset, \sim\} \cup \{\nabla_i \mid i \in ((n-1)\setminus 1)\})$, where $\supset$ is binary, while other connectives [beyond Σ^+] are unary, $\mathcal{A}_{[+]}$ the $\Sigma_{[+]}$-matrix such that $A_{[+]} \triangleq n$, $D^{\mathcal{A}_{[+]}} \triangleq (n\setminus 1)$, $\sim^{\mathfrak{A}_{[+]}} \triangleq \sim^{\mathfrak{K}_n}[$, $(\mathfrak{A}_{[+]}{\restriction}\Sigma^+) \triangleq \mathfrak{D}_n]$,

$$\nabla_i^{\mathfrak{A}_{[+]}}(a) \triangleq \begin{cases} a & \text{if } a \in \{0, n-1\}, \\ i & \text{otherwise,} \end{cases}$$

for all $i \in ((n-1)\setminus 1)$ and all $a \in n$, and

$$(a \supset^{\mathfrak{A}_{[+]}} b) \triangleq \begin{cases} n-1 & \text{if } a \leqslant b, \\ 0 & \text{otherwise,} \end{cases}$$

for all $a, b \in n$, and $C_{[+]}$ the logic of $\mathcal{A}_{[+]}$, in which case it is $\sim$-paraconsistent[and both $\wedge$-conjunctive and $\vee$-disjunctive], for $\mathcal{A}_{[+]}$ is so[, in view of Corollary 3.16]. Note that the injection $e' \triangleq \{\langle 0, \mathsf{f}\rangle, \langle n-1, \mathsf{t}\rangle\}$ is an isomorphism from $\mathcal{A}_{[+]}{\restriction}\{0, n-1\}$ onto the $\sim$-classical matrix with underlying algebra $e'[\mathfrak{A}_{[+]}{\restriction}\{0, n-1\}]$, in which case, by (2.5), $C_{[+]}$ is $\sim$-subclassical, so, in particular, $\sim$ is a subclassical negation for $C_{[+]}$.

The following key result "kills two birds (both minimal n-valuedness and maximal paraconsistency of $C_{[+]}$) with one stone":

Lemma 5.23 (Many-Valued Key Lemma)**.** *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of $C_{[+]}$. Then, there is a submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}_{[+]}$ is embeddable into $\mathcal{D}/\eth(\mathcal{D})$.*

Proof. In that case, there are some $a \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} a \in D^{\mathcal{B}}$ and some $b \in (B \setminus D^{\mathcal{B}})$. Let $\mathfrak{D}$ be the subalgebra of $\mathfrak{B}$ generated by $\{a, b\}$. Then, in view of (2.5), the submatrix $\mathcal{D} \triangleq (\mathcal{B}{\restriction}D)$ of $\mathcal{B}$ is a finitely-generated $\sim$-paraconsistent model of $C_{[+]}$. Therefore, by Lemma 2.20 with $\mathsf{M} = \{\mathcal{A}_{[+]}\}$, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}_{[+]}$, some subdirect product $\mathcal{E}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{D}/\eth(\mathcal{D}))$, in which case, by (2.5), $\mathcal{E}$ is $\sim$-paraconsistent (in particular, consistent), and so $I \neq \varnothing$. Take any $c \in D^{\mathcal{E}}$ such that $\sim^{\mathfrak{E}} c \in D^{\mathcal{E}}$. Then, by Lemma 4.1, $c \in ((n-1)\setminus 1)^I$. Hence, for every $j \in ((n-1)\setminus 1)$, we have $E \ni \nabla_j^{\mathfrak{E}} c = (I \times \{j\})$. Moreover, $E \ni (c \supset^{\mathfrak{E}} c) = (I \times \{n-1\})$ and $E \ni \sim^{\mathfrak{E}}(c \supset^{\mathfrak{E}} c) = (I \times \{0\})$. Thus, $\{I \times \{k\} \mid k \in n\} \subseteq E$, in which case, as $I \neq \varnothing$, $e \triangleq \{\langle k, I \times \{k\}\rangle \mid k \in n\}$ is an embedding of $\mathcal{A}_{[+]}$ into $\mathcal{E}$, and so $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}_{[+]}, \mathcal{D}/\eth(\mathcal{D}))$. Moreover, $\{x_0 \supset x_1, x_1 \supset x_0\}$ is clearly a binary equality determinant for $\mathcal{A}_{[+]}$. In this way, Corollary 2.14 and Lemma 3.4 complete the argument. □

Theorem 5.24. *$C_{[+]}$ is maximally $\sim$-paraconsistent.*

Proof. Consider any $\sim$-paraconsistent extension C' of $C_{[+]}$, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\})$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a $\sim$-paraconsistent model of C', and so of $C_{[+]}$. Then, by Lemma 5.23 and (2.5), $\mathcal{A}_{[+]}$ is a model of C', as required. □

Theorem 5.25. *Let M be a class of $\Sigma_{[+]}$-matrices. Suppose $C_{[+]}$ is defined by M. Then, there is some $\mathcal{B} \in \mathsf{M}$ such that $n \leqslant |B|$. In particular, $C_{[+]}$ is minimally n-valued.*

Proof. As $C_{[+]}$ is $\sim$-paraconsistent, there must be some $\sim$-paraconsistent $\mathcal{B} \in \mathsf{M}$, in which case it is a model of $C_{[+]}$, and so, by Lemma 5.23, there is some submatrix $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{A}_{[+]}$ is embeddable into $\mathcal{D}/\eth(\mathcal{D})$. Thus, $n = |A_{[+]}| \leqslant |D/\eth(\mathcal{D})| \leqslant |D| \leqslant |B|$, as required. □

On the other hand, we have:

Proposition 5.26. *Let $\Sigma'_{[+]} \triangleq (\Sigma_{[+]} \setminus \{\supset\})$. Then, the $\Sigma'_{[+]}$-fragment of $C_{[+]}$ is defined by a[both $\wedge$-conjunctive and $\vee$-disjunctive] $\sim$-superclassical $\Sigma'_{[+]}$-matrix[, being a definitional expansion of $\mathfrak{DM}_4{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, and so the fragment is a definitional expansion of LP]. In particular, it is not minimally n-valued, unless $n = 3$.*

Proof. Let $\mathcal{S}_{[+]}$ be the[both $\wedge$-conjunctive and $\vee$-disjunctive] $\sim$-superclassical $\Sigma'_{[+]}$-matrix given by $\sim^{\mathfrak{S}_{[+]}}\mathsf{b} \triangleq \mathsf{b}[$, $(\mathfrak{S}_{[+]}{\restriction}\Sigma^+) \triangleq (\mathfrak{D}^2_2{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\})]$ and $\nabla_i^{\mathfrak{S}_{[+]}}(a) \triangleq a$, for all $a \in \{\mathsf{f},\mathsf{b},\mathsf{t}\}$ and all $i \in ((n-1)\setminus 1)[$, in which case $\mathcal{S}_+$ is an expansion of $\mathfrak{DM}_4{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ by diagonal operations, and so a definitional one]. Then, $(\{\langle n-1, \mathsf{t}\rangle, \langle 0, \mathsf{f}\rangle\} \cup (((n-1)\setminus 1) \times \{\mathsf{b}\})) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}_{[+]}{\restriction}\Sigma'_{[+]}, \mathcal{S}_{[+]})$. In this way, (2.5) completes the argument. □

This highlights the special role of involving the implication connective $\supset$ and shows that the implication-less fragment of C_+ yields nothing else that the logic of paradox had done in this connection (cf. Theorem 2.1 of [14] and Subsubsection 6.2.1).

6. Applications and Examples

6.1. **Four-valued expansions of Belnap's logic.** Here, we consider applications of Theorems 4.17, 4.25, 4.20, 4.15, 4.53(i)⇔(x), Lemmas 4.13, 4.14 and Corollaries 4.56 and 4.57 normally not mentioning them explicitly and implicitly following the conventions adopted in Section 4.

6.1.1. *Fragments of the classical expansion.* Here, we deal with the basic signature $\Sigma \triangleq (\Sigma_{01} \cup \{\neg\})$, where $\neg$ (classical negation) is unary, and its subsignature $\Sigma' \supseteq \Sigma_0$. Put $\neg^{\mathfrak{A}}\vec{a} \triangleq \langle 1 - a_i \rangle_{i \in 2}$, for all $\vec{a} \in 2^2$. Then, $\mu \in \hom(\mathfrak{A}, \mathfrak{A})$. Moreover, $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}{\restriction}\Sigma'$ iff $\neg \notin \Sigma'$. Likewise, $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}{\restriction}\Sigma'$ iff $\Sigma' = \Sigma_0$. In this way, we have:

Corollary 6.1. *Let $\Sigma_0 \subseteq \Sigma' \subseteq \Sigma$. Then, the logic of $\mathcal{A}{\restriction}\Sigma'$:*

(i) *is self-extensional, and so $\sim$-subclassical;*
(ii) *is maximally $\sim$-paraconsistent iff $\neg \in \Sigma'$;*
(iii) *is purely inferential iff it has no consistent formula iff it holds Relevance Principle iff $\Sigma' = \Sigma_0$.*

In this way, the classical expansion of C_{B} becomes a first instance of a *minimally four*-valued *maximally* paraconsistent *subclassical* logic (further but non-subclassical ones are provided by the next subsubsection). In this connection, we should like to highlight that, as opposed to the generic examples provided by Subsection 5.2, the four-valued ones provided by this and the next subsubsections are not definable by false-singular matrices (cf. Corollary 4.6).

6.1.2. *Bilattice expansions.* Here, it is supposed that $\{\sqcap, \sqcup\} \subseteq \Sigma$, where $\sqcap$ and $\sqcup$ are binary (*knowledge* conjunction and disjunction, respectively), while $(\langle a, b\rangle \sqcap^{\mathfrak{A}} \langle c, d\rangle) = \langle \min(a, c), \max(b, d)\rangle$, for all $a, b, c, d \in 2$, in which case $(\mathsf{f} \sqcap^{\mathfrak{A}} \mathsf{t}) = \mathsf{n}$, whereas $(\langle a, b\rangle \sqcup^{\mathfrak{A}} \langle c, d\rangle) = \langle \max(a, c), \min(b, d)\rangle$, for all $a, b, c, d \in 2$, in which case $(\mathsf{f} \sqcup^{\mathfrak{A}} \mathsf{t}) = \mathsf{b}$. In that case, neither $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ nor $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. And what is more, $\{\mathsf{b}\}$ and $\{\mathsf{n}\}$ are exactly all proper subalgebras of $\mathfrak{A}$ in the *purely-bilattice* case $\Sigma = (\Sigma_0 \cup \{\sqcap, \sqcup\})$, $\mathcal{A}{\restriction}\{\mathsf{n}\}$ being the only proper consistent submatrix of $\mathcal{A}$, in that case. Hence, we immediately obtain the following universal negative and positive results, respectively:

Corollary 6.2. *Any bilattice expansion of C_{B} is not $\sim$-subclassical, and so not self-extensional.*

Corollary 6.3. *Any [purely-]bilattice expansion of C_{B} [holds Relevance Principle and]is inferentially maximal, and so maximally $\sim$-paraconsistent.*

And what is more, in case $\Sigma_{01} \subseteq \Sigma$, $\mathcal{A}$ has no proper submatrix at all. Thus, by Theorem 4.17 and Lemma 4.13, we also get:

Corollary 6.4. *C is maximal iff it is not purely inferential if $\Sigma_{01} \subseteq \Sigma$.*

6.1.3. *Implicative expansions.* Here, it is supposed that Σ contains a binary $\supset$ (implication) such that

$$(a \supset^{\mathfrak{A}} b) = \begin{cases} b & \text{if } \pi_0(a) = 1, \\ \mathsf{t} & \text{otherwise}, \end{cases}$$

for all $a, b \in 2^2$ (cf. [16]). Then, $(\mathsf{n} \supset^{\mathfrak{A}} \mathsf{n}) = \mathsf{t} \neq \mathsf{b} = (\mathsf{b} \supset^{\mathfrak{A}} \mathsf{b})$, so $\mu \notin \hom(\mathfrak{A}, \mathfrak{A})$, in which case we immediately get:

Corollary 6.5. *The logic of $\mathcal{A}$ is neither self-extensional nor purely-inferential, ad so does not hold Relevance Principle.*

It is remarkable that, as opposed to bilattice expansions, implicative ones are not, generally speaking, covered by Corollary 4.56 because $\{\mathsf{f}(, \mathsf{b}/\mathsf{n}), \mathsf{t}\}$ does form a subalgebra of $\mathfrak{B}_{[01]} \triangleq (\mathfrak{A}{\restriction}(\Sigma_{0[1]} \cup \{\supset\}))$, in which case, by Theorem(s) 4.20(and 4.25), C is $\sim$-subclassical(and is not maximally $\sim$-paraconsistent), whenever $\Sigma \subsetneq (\Sigma_{01} \cup \{\supset\})$. It is also remarkable that $\{\mathsf{b}\}$ does [not]form a subalgebra of $\mathfrak{B}_{[01](,\not{\mathsf{n}})}$, while $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{B}_{[01]}$. On the other hand, $\supset^{\mathfrak{B}_{01(,\not{\mathsf{n}})}}$, being the only non-regular operation of $\mathfrak{B}_{01(,\not{\mathsf{n}})}$, for $\mathfrak{DM}_{4,01}$ is regular, and so is $\mathfrak{DM}_{4,01,\not{\mathsf{n}}}$, while $(\mathsf{f} \supset^{\mathfrak{A}} \mathsf{f}) = \mathsf{t} \not\sqsubseteq \mathsf{f} = (\mathsf{b} \supset^{\mathfrak{A}} \mathsf{f})$, whereas $\mathsf{f} \sqsubseteq \mathsf{b}$, is both binary and b-idempotent. This is why Theorem 4.44 has proved equally applicable to both bounded and unbounded *purely*-implicational cases that have been due to [23] (collectively with both [15] and [18]) *ad hoc.*

6.1.4. *Disjunctive extensions of expansions of Belnap's logic.* In view of Theorem 4.1 of [13], $\vee$-disjunctive extensions of C_{B} are exactly *De Morgan logics* in the sense of the reference [Pyn 95a] of [14]. In this way, the present subsection incorporates the material announced therein. We use Theorems 3.21 and 3.24 tacitly.

From now on, unless otherwise specified, C is supposed to be both self-extensional and not maximally $\sim$-paraconsistent. Then, by Theorem 4.25 and Corollaries 4.56 and 4.57, under identification of submatrices of expansions of $\mathcal{DM}_4$ with underlying algebras of their carriers, we have $\mathbf{S}^*_*(\mathcal{A}) = \mathsf{S}_{01} \triangleq \mathbf{S}(\mathcal{DM}_{4,01}) = \{\{\mathsf{f}, \mathsf{t}, \mathsf{b}, \mathsf{n}\}, \{\mathsf{f}, \mathsf{t}, \mathsf{n}\}, \{\mathsf{f}, \mathsf{t}, \mathsf{b}\}, \{\mathsf{f}, \mathsf{t}\}\}$ and $\mathsf{S}_{01} \subseteq \mathbf{S}_*(\mathcal{A}) \subseteq \mathsf{S} \triangleq \mathbf{S}_*(\mathcal{DM}_4) = (\mathsf{S}_{01} \cup \{\{\mathsf{n}\}\})$, in which case there are at most nine and at least [resp., exactly] six lower cones of $\mathsf{S}_{[01]}$ (actually given by their generating anti-chains):

$$\mathsf{C}_4 \triangleq \{\{\mathsf{f}, \mathsf{t}, \mathsf{b}, \mathsf{n}\}\}^{\triangledown}, \qquad \mathsf{C}^{\mathsf{b}}_3 \triangleq \{\{\mathsf{f}, \mathsf{t}, \mathsf{b}\}\}^{\triangledown}, \qquad \mathsf{C}^{\mathsf{n}}_3 \triangleq \{\{\mathsf{f}, \mathsf{t}, \mathsf{n}\}\}^{\triangledown},$$

$$\mathsf{C}_3 \triangleq (\mathsf{C}^{\mathsf{b}}_3 \cup \mathsf{C}^{\mathsf{n}}_3), \qquad \mathsf{C}_2 \triangleq \{\{\mathsf{f}, \mathsf{t}\}\}, \qquad \mathsf{C}_1 \triangleq \{\{\mathsf{n}\}\},$$

$$\mathsf{C}_0 \triangleq \varnothing, \qquad \mathsf{C}^{\mathsf{b}}_{3\uplus 1} \triangleq (\mathsf{C}^{\mathsf{b}}_3 \cup \mathsf{C}_1), \qquad \mathsf{C}_{2\uplus 1} \triangleq (\mathsf{C}_2 \cup \mathsf{C}_1).$$

Those at most eight and at least [resp. exactly] five ones, which are proper (viz. distinct from $\mathsf{S}_{[01]} = \mathsf{C}_4$) are relatively axiomatized by the following calculi, respectively:

(6.1) $$x_0 \vee {\sim} x_0,$$

(6.2) $$(4.17),$$

(6.3) $$(4.1),$$

(6.4) $$\{(6.1), (6.2)\},$$

(6.5) $$x_0 \vdash x_1,$$

(6.6) $$x_0,$$

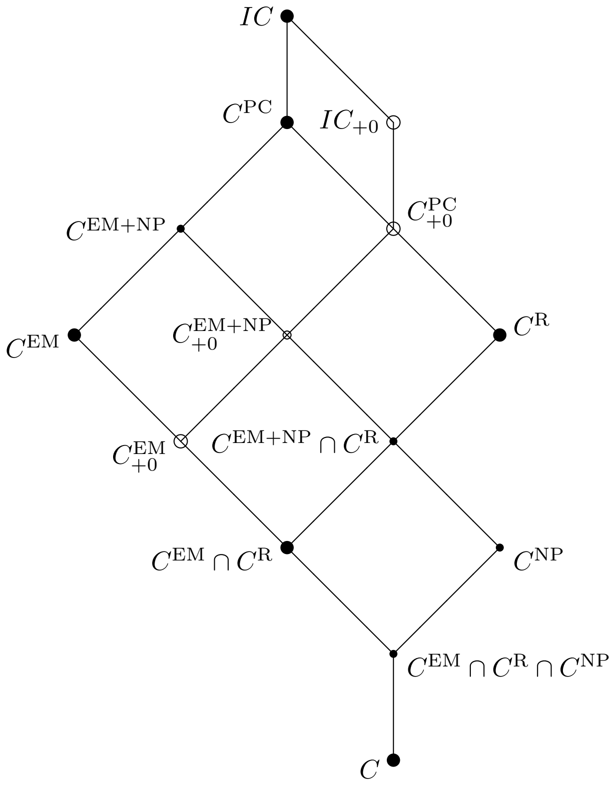

FIGURE 1. The lattices of(all $\vee$-disjunctive) and all Kleene extensions of C(with solely big circles).

$$x_0 \vdash (x_1 \vee {\sim}x_1), \tag{6.7}$$

$$\{(6.2), (6.7)\}. \tag{6.8}$$

The logic of C_4 is then C itself, while that of $\mathsf{C}_3^{\mathfrak{b}}$ is $C^{\not\mathfrak{n}'} = C^{\mathrm{EM}}$ (cf. Theorem 4.25), whereas the one of $\mathsf{C}_3^{\mathfrak{n}}$ is C^{R} (cf. Corollary 4.31), in which case the one of C_3 is their self-extensional meet $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ (cf. Lemma 4.60), while the logic of C_2 being the unique $\sim$-classical extension C^{PC} of C (cf. Theorem 4.20 and Corollary 4.56), whereas the one of C_0 being the inconsistent logic IC defined by $\varnothing$. And what is more, $\sigma_{+1}(6.5) \vee x_0$ is equivalent to (6.5) under (3.3) and (3.5). Likewise, $\sigma_{+1}(6.7) \vee x_0$ is equivalent to (6.7) under (3.3), (3.4) and (3.5). In this way, taking Remarks 2.8, 2.10, Proposition 2.19 and Lemmas 4.13 and 4.26 into account, we eventually get:

Theorem 6.6. *$\vee$-Disjunctive[merely non-pseudo-axiomatic, if C is purely inferential, and arbitrary, otherwise,] extensions of C form the non-chain distributive lattice depicted at Figure 1[with solely solid circles]. Moreover, those of them, which are proper, are axiomatized relatively to C by the following calculi, respectively (in the above order):*

$$(6.1), \tag{6.9}$$

$$(4.8), \tag{6.10}$$

$$(4.19), \tag{6.11}$$

$$\{(6.1), (4.8)\}, \tag{6.12}$$

$$(6.5), \tag{6.13}$$

$$(6.6), \tag{6.14}$$

$$(6.7), \tag{6.15}$$

$$\{(6.7), (4.8)\}. \tag{6.16}$$

Theorem 6.6 is quite easily extended to any given four-valued expansion of C_{B} *mutatis mutandis*, for possible lower cones and their relative axiomatizations (clearly inherited by respective expansions) have been found, in which case the lattice of $\vee$-disjunctive[merely non-pseudo-axiomatic, if the expansion is purely inferential, and arbitrary, otherwise,] extensions of the expansion becomes a sublattice of that of $C_{[\mathrm{B}]\mathrm{B}}$ depicted at Figure 1. For instance, when dealing with the classical expansion CB_4 (cf. Subsubsection 6.1.1), $\mathbf{S}_*(\mathcal{A})$ becomes equal to $\{A, \{\mathsf{f}, \mathsf{b}\}\}$, in which case $\vee$-disjunctive (viz., self-extensional; cf. Corollary 4.62) extensions of CB_4 form the three-element chain $CB_4 \subsetneq CB_4^{\mathrm{PC}} = CB_4^{\mathrm{EM}} = CB_4^{\mathrm{R}} \subsetneq IC$ (cf. Theorems 4.20, 4.25 and 4.36). Likewise, given any bilattice expansion BL_4 (cf. Subsubsection 6.1.2), $\mathbf{S}_*(\mathcal{A})$ becomes equal to $\{A[, \{\mathsf{n}\}]\}$, in which case $\vee$-disjunctive extensions of BL_4 form the two-[three-]element chain $BL_4[\subsetneq IC_{+0}] \subsetneq IC$, exhausting *all* extensions of BL_4, in view of its [inferential]maximality. Implicative expansions (cf. Subsubsection 6.1.3) are analyzed in a very similar way. In case of the *purely*-implicational expansion $B^{\supset}_{4[,01]}$ of $C_{[\mathrm{B}]\mathrm{B}}$ (viz., by $\supset$ alone), $\mathbf{S}_*(\mathcal{A}) = \mathsf{S}_{01}$, so the lattice of $\vee$-disjunctive extensions of $B^{\supset}_{4[,01]}$ is exactly the six-element one, each lower cone C of S_{01} clearly being relatively axiomatized by the axiomatic calculus $\mathcal{R}^{\supset} \triangleq ((\mathcal{R} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cup \{\phi \supset \varphi \mid (\{\phi\} \vdash \varphi) \in \mathcal{R}\} \cup \{(\phi \wedge \psi) \supset \varphi \mid (\{\phi, \psi\} \vdash \varphi) \in \mathcal{R}, \phi \neq \psi\}$, where $\mathcal{R}$ is the relative axiomatization of C found above, in which case the corresponding $\vee$-disjunctive extension of $B^{\supset}_{4[,01]}$ is the axiomatic one relatively axiomatized by $\mathcal{R}^{\supset}$. In this connection, recall that Subsection 5.3 of [23] collectively with both [18] and [15] have provided the nineteen[thirty]-element non-chain distributive lattice of *all* extensions of $B^{\supset}_{4[,01]}$ as well as their defining sets of matrices, their relative axiomatizations, though having been already-known by that time, eventually appearing beyond the scopes of the mentioned study.

It is remarkable that, in view of Theorem 5.2 of [13] providing an axiomatization of C_{B} given by Definition 5.1 therein,[5] Theorem 6.6 yields axiomatizations of all $\vee$-disjunctive extensions of C_{B} (in particular, of K_3 relatively axiomatized by the Resolution rule (4.8)).

On the other hand, to find *all* extensions of C is a much more complicated problem, a first idea of which having been due to Theorems 4.64, 4.35, 4.44 and Corollaries 4.4 and 4.63. A partial solution of it is presented below.

6.1.4.1. Kleene extensions.

Corollary 6.7. *Let I be a finite set, $\overline{\mathcal{C}} \in \{\mathcal{A}_{\not\mathsf{b}}, \mathcal{A}_{\not\mathsf{n}}\}^I$, and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent submatrix of $\prod_{i\in I} \mathcal{C}_i$. Then,* $\mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not\mathsf{b}}) \neq \varnothing$.

Proof. In that case, by Lemma 4.39, there is some $h \in \mathrm{hom}(\mathcal{B}, \overrightarrow{\mathcal{A}}) \neq \varnothing$, in which case $\mathfrak{D} \triangleq (\mathfrak{A}{\restriction}(\mathrm{img}\, h))$ satisfies (3.13), for $\mathfrak{B}$ does so, while $h \in \mathrm{hom}(\mathfrak{B}, \mathfrak{D})$ is surjective. Hence, $\{\mathsf{n}, \mathsf{b}\} \nsubseteq B$, for otherwise, (3.13) would not be true in $\mathfrak{D}$ under $[x_0/\mathsf{n}, x_1/\mathsf{b}]$. Thus, $\mathcal{D} \triangleq (\overrightarrow{\mathcal{A}}{\restriction}D)$ is a submatrix of $\overrightarrow{\mathcal{A}}{\restriction}A_{\not a}$, for some $a \in \{\mathsf{n}, \mathsf{b}\}$, in which case $h \in \mathrm{hom}(\mathcal{B}, \overrightarrow{\mathcal{A}}{\restriction}A_{\not a})$, and so the fact that $\mu{\restriction}A_{\not\mathsf{n}}$ is an isomorphism from $\overrightarrow{\mathcal{A}}{\restriction}A_{\not\mathsf{n}}$ onto $(\overrightarrow{\mathcal{A}}{\restriction}A_{\not\mathsf{b}}) = \mathcal{A}_{\not\mathsf{b}}$ completes the argument. □

Lemma 6.8. *Suppose $\mathfrak{A}_{\not\mathsf{b}}$ is regular. Then,* $(\mathcal{A}_{\not\mathsf{n}} \times \mathcal{A}_{\not\mathsf{b}}) \in \mathrm{Mod}(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})$.

Proof. Since, by Corollaries 4.31, 4.57 and Theorems 4.25 and 4.41, $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$ is defined by $\{\mathcal{A}_{\not\mathsf{b}}, \mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})\}$, $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}))$, being isomorphic to $\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}))$, is a model of $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$, in view of (2.5). Moreover, by Lemma 4.33, $(\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})){\restriction}K^{\mathsf{n}}_4$ is a submatrix of $\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})$, in which case $\mathcal{A}_{\not\mathsf{n}} \times ((\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})){\restriction}K^{\mathsf{n}}_4)$ is a submatrix of $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}))$, and so it is a model of $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$, in view of (2.5). And what is more, $h \triangleq (\pi_0{\restriction}K^{\mathsf{n}}_4) \in \mathrm{hom}_{\mathrm{S}}((\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})){\restriction}K^{\mathsf{n}}_4, \mathcal{A}_{\not\mathsf{b}})$ is surjective, and so is $(\Delta_{A_{\not\mathsf{n}}} \times h) \in \mathrm{hom}_{\mathrm{S}}(\mathcal{A}_{\not\mathsf{n}} \times ((\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})){\restriction}K^{\mathsf{n}}_4), \mathcal{A}_{\not\mathsf{n}} \times \mathcal{A}_{\not\mathsf{b}})$. In this way, (2.5) completes the argument. □

Corollary 6.9. *Suppose $\mathfrak{A}_{\not\mathsf{b}}$ is regular. Then, $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$ is the extension of $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ relatively axiomatized by* (4.17).

Proof. By Corollaries 4.31, 4.57 and Theorem[s] 4.25[and 4.41], $C^{\mathrm{EM[+NP]}} \cap C^{\mathrm{R}}$ is defined by $\{\mathcal{A}_{\not\mathsf{n}}[\times(\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})], \mathcal{A}_{\not\mathsf{b}}\}$[, both matrices being non-$\sim$-paraconsistent, and so being the logic involved]. Conversely, consider any model $\mathcal{B} \in \mathbf{S}(\mathbf{P}_{\omega}(\{\mathcal{A}_{\not\mathsf{b}}, \mathcal{A}_{\not\mathsf{n}}\}))$ of (4.17), in which case there is some finite set I, some $\overline{\mathcal{C}} \in \{\mathcal{A}_{\not\mathsf{b}}, \mathcal{A}_{\not\mathsf{n}}\}^I$ such that $\mathcal{B}$ a submatrix of $\prod_{i\in I} \mathcal{C}_i$. Put $J \triangleq \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not\mathsf{n}} \times \mathcal{A}_{\not\mathsf{b}})$ and $K \triangleq \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not\mathsf{b}})$. Consider any $a \in (B \setminus D^{\mathcal{B}})$, in which case $\mathcal{B}$ is consistent and there is some $i \in I$ such that $\pi_i(a) \notin D^{\mathcal{C}_i}$. Consider the following exhaustive cases:

(1) $\mathfrak{C}_i = \mathcal{A}_{\not\mathsf{n}}$.
Then, by Corollary 6.7, there is some $h \in \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not\mathsf{b}}) \neq \varnothing$, in which case $g \triangleq ((\pi_i{\restriction}B) \times h) \in J$ and $g(a) \notin D^{\mathcal{A}_{\not\mathsf{n}} \times \mathcal{A}_{\not\mathsf{b}}}$.

(2) $\mathfrak{C}_i = \mathcal{A}_{\not\mathsf{b}}$.
Then, $(\pi_i{\restriction}B) \in K$.

In this way, $f \triangleq ((\prod \Delta_J) \times (\prod \Delta_K)) \in \mathrm{hom}_{\mathrm{S}}(\mathcal{B}, (\mathcal{A}_{\not\mathsf{n}} \times \mathcal{A}_{\not\mathsf{b}})^J \times \mathcal{A}^K_{\not\mathsf{b}})$, and so (2.5), Corollary 2.21, Lemma 6.8 and the finiteness of A complete the argument. □

Theorem 6.10. *Suppose $\mathfrak{A}$ is regular (in particular, $\Sigma = \Sigma_{0(1)}$). Then,[merely non-pseudo-axiomatic, if C is purely inferential, and arbitrary, otherwise,] extensions of $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ form the non-chain distributive lattice depicted at Figure 1[with solely solid circles]. Moreover, those of them, which are not $\vee$-disjunctive, are relatively axiomatized as follows:*

$$\begin{aligned} C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}} & \quad \textit{by} \quad (4.17), \\ C^{\mathrm{EM+NP}}_{+0} & \quad \textit{by} \quad \{(4.17), (6.7)\}, \\ C^{\mathrm{EM+NP}} & \quad \textit{by} \quad \{(4.17), (6.1)\}. \end{aligned}$$

Proof. We use Theorems 4.25, 4.41, 6.6, Corollary 6.9, Remarks 2.8, 4.42 and Proposition 2.19 tacitly. First, as C^{EM} is $\sim$-paraconsistent, $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})/C^{\mathrm{EM+NP}}_{+0}/C^{\mathrm{EM+NP}}$ is distinct from $(C^{\mathrm{EM}} \cap C^{\mathrm{R}})/C^{\mathrm{EM}}_{+0}/C^{\mathrm{EM}}$, respectively. Likewise, since (4.18) is not true in $\mathcal{A}_{\not\mathsf{n}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})$ under $[x_0/\langle\mathsf{b},\mathsf{t}\rangle, x_1/\langle\mathsf{f},\mathsf{t}\rangle]$, $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})/C^{\mathrm{EM+NP}}_{+0}/C^{\mathrm{EM+NP}}$ is distinct from $C^{\mathrm{R}}/C^{\mathrm{PC}}_{+0}/C^{\mathrm{PC}}$, respectively. Finally, consider any extension C' of $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ and the following exhaustive cases [but (3) and (4)]:

(1) $IC \subseteq C'$.
Then, $C' = IC$.

(2) $C^{\mathrm{PC}} \subseteq C'$ but $IC \nsubseteq C'$.
Then, C' is consistent, so, by Corollary 3.10, $C' = C^{\mathrm{PC}}$.

(3) $IC_{+0} \subseteq C'$ but $C^{\mathrm{PC}} \nsubseteq C'$.
Then, $IC \nsubseteq C'$, so, by the following claim, C' is purely-inferential:

Claim 6.11. *Let C' and C'' be Σ-logics. Suppose $C' \nsubseteq C''$ is non-pseudo-axiomatic and $C'_{+0} \subseteq C''$. Then, C'' is purely-inferential.*

Proof. By contradiction. For suppose C'' is not purely inferential, in which case $\varnothing \notin (\mathrm{img}\, C'')$, and so $C''_{-0} = C''$. In this way, by Remark 2.8, we get $C' = (C'_{+0})_{-0} \subseteq C''_{-0} = C''$. This contradiction completes the argument. □

[5]In this connection, we should also like to take the opportunity to notice that Footnote 3 on p. 443 of [13] has proved absolutely irrelevant and is to be disregarded.

In this way, as $C'_{-0} \subseteq IC$, we have $C' = (C'_{-0})_{+0} \subseteq IC_{+0}$, and so we get $C' = IC_{+0}$.

(4) $C^{\mathrm{PC}}_{+0} \subseteq C'$ but both $C^{\mathrm{PC}} \nsubseteq C'$ and $IC_{+0} \nsubseteq C'$.
Then, by Claim 6.11, C' is purely inferential. Moreover, (6.7), being satisfied in C^{PC}_{+0}, is so in C', in which case, by the structurality of C', $(x_0 \vee \sim x_0) \in (\bigcap((\mathrm{img}\, C') \setminus \{\varnothing\})) = C'_{-0}(\varnothing)$, and so $C^{\mathrm{PC}} \subseteq C'_{-0}$. On the other hand, $IC = (IC_{+0})_{-0} \nsubseteq C'_{-0}$, so C'_{-0} is consistent. Hence, by Corollary 3.10, $C'_{-0} = PC$. In this way, $C' = (C'_{-0})_{+0} = C^{\mathrm{PC}}_{+0}$.

(5) $(C^{\mathrm{PC}}_{+0}[\cup C^{\mathrm{PC}}]) \nsubseteq C'$ but $C^{\mathrm{R}} \subseteq C'$.
Then,[(6.1), and so, in view of the non-pseudo-axiomaticity of C',] (6.7) is not satisfied in C', in which case, by Theorem 4.35, $C' = C^{\mathrm{R}}$.

(6) $C^{\mathrm{R}} \nsubseteq C'$.
Then, (4.18) is not satisfied in C', in which case, by Lemma 4.43, $C' \subseteq C^{\mathrm{EM+NP}}$, and so we have the following exhaustive subcases[but (c) and (d)]:
   (a) $C^{\mathrm{EM+NP}} \subseteq C'$.
   Then, $C' = C^{\mathrm{EM+NP}}$.
   (b) $C^{\mathrm{EM+NP}} \nsubseteq C'$ but $C^{\mathrm{EM}} \subseteq C'$.
   Then, C' is $\sim$-paraconsistent, so, by Corollary 4.24, $C' = C^{\mathrm{EM}}$.
   (c) $C^{\mathrm{EM+NP}}_{+0} \subseteq C'$ but $C^{\mathrm{EM}} \nsubseteq C'$.
   Then, $C^{\mathrm{EM+NP}} \nsubseteq C'$, so, by Claim 6.11, C' is purely-inferential. Therefore, $C^{\mathrm{EM+NP}} = (C^{\mathrm{EM+NP}}_{+0})_{-0} \subseteq C'_{-0}$, $(C^{\mathrm{EM}} \cap C^{\mathrm{R}}) = (C^{\mathrm{EM}} \cap C^{\mathrm{R}})_{-0} \subseteq C'_{-0}$ and $C^{\mathrm{R}} \nsubseteq C'_{-0}$, for, otherwise, we would have $C^{\mathrm{R}} = (C^{\mathrm{R}})_{+0} \subseteq (C'_{-0})_{+0} = C'$. Hence, by Lemma 4.43, we have $C'_{-0} \subseteq C^{\mathrm{EM+NP}}$, in which case we get $C' = (C'_{-0})_{+0} \subseteq C^{\mathrm{EM+NP}}_{+0}$, and so $C' = C^{\mathrm{EM+NP}}_{+0}$.
   (d) $C^{\mathrm{EM}}_{+0} \subseteq C'$ but both $C^{\mathrm{EM}} \nsubseteq C$ and $C^{\mathrm{EM+NP}}_{+0} \nsubseteq C'$.
   Then, by Claim 6.11, C' is purely inferential. Moreover, (6.7), being satisfied in C^{EM}_{+0}, is so in C', in which case, by the structurality of C', $(x_0 \vee \sim x_0) \in (\bigcap((\mathrm{img}\, C') \setminus \{\varnothing\})) = C'_{-0}(\varnothing)$, and so $C^{\mathrm{EM}} \subseteq C'_{-0}$, while $(C^{\mathrm{EM}} \cap C^{\mathrm{R}}) = (C^{\mathrm{EM}} \cap C^{\mathrm{EM}})_{-0} \subseteq C'_{-0}$. On the other hand, $C^{\mathrm{EM+NP}} = (C^{\mathrm{EM+NP}}_{+0})_{-0} \nsubseteq C'_{-0}$, so C'_{-0} is $\sim$-paraconsistent. Hence, by Corollary 4.24, $C'_{-0} = C^{\mathrm{EM}}$. In this way, $C' = (C'_{-0})_{+0} = C^{\mathrm{EM}}_{+0}$.
   (e) $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}) \subseteq C'$ but $(C^{\mathrm{EM+NP}}_{+0}[\cup C^{\mathrm{EM+NP}}]) \nsubseteq C'$.
   Then,[(6.1), and so, in view of the non-pseudo-axiomaticity of C',] (6.7) is not satisfied in C', in which case, by Theorem 4.35, $C' = (C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})$.
   (f) $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}) \nsubseteq C'$ and $(C^{\mathrm{EM}}_{+0}[\cup C^{\mathrm{EM}}]) \nsubseteq C'$.
   Then, C' is both $\sim$-paraconsistent and inferentially paracomplete[, in view of the non-pseudo-axiomaticity of C'], and so, by Corollary 4.57(ii), $C' = (C^{\mathrm{EM}} \cap C^{\mathrm{R}})$.

This completes the argument. □

As an immediate consequence of Theorems 6.6 and 6.10, we have:

Corollary 6.12. *Suppose $\mathfrak{A}$ is regular (in particular, $\Sigma = \Sigma_{0[1]}$). Then, extensions of C^{R} are all $\vee$-disjunctive.*

On the other hand, the $\vee$-disjunctivity of extensions holds for neither $C^{\mathrm{EM}}(\cap C^{\mathrm{R}})$ nor C, even if $\mathfrak{A}$ is regular, as it follows from Theorems 6.6 and 6.10.

Concluding this discussion, we should like to highlight that the technique elaborated here has proved well-applicable to finding all extensions of LP that has been done in [14] with using an advanced algebraic method based upon finding the lattice of all subprevarieties of KL going back to finding that of ones of DML being due to [17]. However, the mentioned method is not applicable to K_3 (as well as to both $LP_{[01]} \cap K_{3[,01]}$ and $C_{[\mathrm{B}]\mathrm{B}}$) at all. And what is more, it is hardly applicable to LP_{01} (as well as to $K_{3,01}$), because the structure of subprevarieties of BKL is much more complicated than the five-element chain one of those of KL (cf. [17] and [18]) — more specifically, as it is well-known, there are continuously many quasivarieties of Kleene algebras, the lattice of which has never been analyzed in detail for this reason. This highlights the special value of the technique elaborated here.

6.1.4.1.1. Some proper non-Kleene extensions. Finally, we explore some of proper non-Kleene (and so non-$\vee$-disjunctive, in view of Theorem 6.6) extensions of C. First of all, notice that (4.19) is not true in $\overrightarrow{\mathcal{A}}$ under $[x_0/\mathsf{n}, x_1/\mathsf{b}, x_2/\mathsf{n}]$. Therefore, by Theorems 4.44 and 4.63, C^{MP} and C^{NP} become first distinct examples of such a kind. (In particular, this shows that Remark 4.42 is not inherited by non-Kleene extensions of C). Moreover, by Theorem 4.44, we get two more distinct proper non-Kleene extensions $C^{\mathrm{EM[+NP]}} \cap C^{\mathrm{MP}}$, for $C^{\mathrm{EM}} \cap C^{\mathrm{MP}}$ is $\sim$-paraconsistent (cf. Theorem 4.25), while $C^{\mathrm{EM+NP}} \cap C^{\mathrm{MP}}$ is an extension of C^{NP}. Then, a one more example of such a kind is as follows:

Theorem 6.13. *$C^{\mathrm{EM}} \cap C^{\mathrm{R}} \cap C^{\mathrm{NP}}$ is the proper extension of C relatively axiomatized by the rule* (4.1).

Proof. Let C' be the extension of C relatively axiomatized by the rule (4.1). Since (4.1) is a logical consequence of (4.17) and is true in C_3, $C^{\mathrm{EM}} \cap C^{\mathrm{R}} \cap C^{\mathrm{NP}}$ is an extension of C'. Conversely, consider any $\mathcal{B} \in (\mathrm{Mod}(C') \cap \mathsf{K})$, where $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}(\mathbf{S}_*(\mathcal{A}))$. Assume, (4.17) is not true in $\mathcal{B}$, in which case there is some $a \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{A}} a \in D^{\mathcal{B}}$, and so, by (4.1) and (3.3), the conclusion of (4.19) is true in $\mathcal{A}$, and so is the rule (4.19) itself. Thus, $(\mathrm{Mod}(C') \cap \mathsf{K}) \subseteq ((\mathrm{Mod}(C^{\mathrm{NP}}) \cap \mathsf{K}) \cup (\mathrm{Mod}(C^{\mathrm{EM}} \cap C^{\mathrm{R}}) \cap \mathsf{K}))$. Hence, by Corollary 2.21, we eventually conclude that $C' = (C^{\mathrm{EM}} \cap C^{\mathrm{R}} \cap C^{NP})$. Finally, recall that (4.1) is not true in $\mathcal{A}$ under $[x_0/\mathsf{b}, x_1/\mathsf{n}]$, as required. □

And what is more, we also have:

Theorem 6.14. *The extension of $C^{\mathrm{EM}} \cap C^{\mathrm{MP}}$ relatively axiomatized by* (4.17)*, i.e., the join of $C^{\mathrm{EM}} \cap C^{\mathrm{MP}}$ and C^{NP} is defined by $\{\overrightarrow{\mathcal{A}}, \mathcal{A}_{\not\mathfrak{n}} \times \overrightarrow{\mathcal{A}}\}$.*

Proof. By Theorem[s 4.25 and] 4.63, $[C^{\mathrm{EM}}\cap]C^{\mathrm{MP}}$ is defined by $\{\overrightarrow{\mathcal{A}}[,\mathcal{A}_{\not{n}}]\}$. In particular, $\overrightarrow{\mathcal{A}}$ is a model of (4.17). Moreover, by (2.5) and Theorem 4.64, $\mathcal{A}_{\not{n}}\times\overrightarrow{\mathcal{A}}$, being a submatrix of $\mathcal{A}\times\overrightarrow{\mathcal{A}}$, is a model of (4.17) too. Conversely, consider any finite set I, any $\overline{\mathcal{C}}\in\mathbf{S}_*(\{\overrightarrow{\mathcal{A}},\mathcal{A}_{\not{n}}\})^I$ and any subdirect product $\mathcal{D}$ of it being a model of (4.17). Put $J\triangleq\hom(\mathcal{D},\mathcal{A}_{\not{n}}\times\overrightarrow{\mathcal{A}})$ and $K\triangleq\hom(\mathcal{D},\overrightarrow{\mathcal{A}})$. Consider any $a\in(D\setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent and there is some $i\in I$, in which case $h\triangleq(\pi_i\restriction D)\in\hom(\mathcal{D},\mathcal{C}_i)$, such that $h(a)\notin D^{\mathcal{C}_i}$. Consider the following exhaustive cases:

(1) $\mathcal{C}_i=\mathcal{A}_{\not{n}}$.
Then, by Lemma 4.39, there is some $g\in\hom(\mathcal{D},\overrightarrow{\mathcal{A}})\neq\varnothing$, in which case $f\triangleq(h\times g)\in J$ and $f(a)\notin D^{\mathcal{A}_{\not{n}}\times\overrightarrow{\mathcal{A}}}$.

(2) $\mathcal{C}_i=\overrightarrow{\mathcal{A}}$.
Then, $h\in K$.

In this way, $((\prod\Delta_J)\times(\prod\Delta_K))\in\hom_{\mathrm{S}}(\mathcal{D},(\mathcal{A}_{\not{n}}\times\overrightarrow{\mathcal{A}})^J\times\overrightarrow{\mathcal{A}}^K)$. Hence, by (2.5) and Corollary 2.21, the extension involved is finitely-defined by $\{\overrightarrow{\mathcal{A}},\mathcal{A}_{\not{n}}\times\overrightarrow{\mathcal{A}}\}$. Then, the finiteness of A completes the argument. □

Finally, note that the rule:

$$\{x_0,\sim x_0\vee x_2\}\vdash((\sim x_1\vee x_1)\vee x_2), \tag{6.17}$$

being satisfied in $C^{\mathrm{EM}}\cap C^{\mathrm{MP}}$, in view of (3.3) and (3.4), is not true in $\mathcal{A}\times\overrightarrow{\mathcal{A}}$ under $[x_0/\langle\mathsf{b},\mathsf{t}\rangle,x_1/\langle\mathsf{n},\mathsf{t}\rangle,x_2/\langle\mathsf{f},\mathsf{t}\rangle]$. Therefore, by Theorem 4.64, we get:

Corollary 6.15. $C^{\mathrm{EM}[+\mathrm{NP}]}\cap C^{\mathrm{R/MP}}$ *is a proper extension of* $(C^{\mathrm{EM}}\cap C^{\mathrm{R}}\cap C^{\mathrm{NP}})[\cup C^{\mathrm{NP}}]$.

In this way, $C^{\mathrm{EM}}\cap C^{\mathrm{R}}\cap C^{\mathrm{NP}}$ and C^{NP} are properly depicted at Figure 1. And what is more, we have found here at least five proper non-Kleene extensions of C. The main non-trivial problems remaining still open are then:

- Is the join of $C^{\mathrm{EM}}\cap C^{\mathrm{MP}}$ and C^{NP} equal to $C^{\mathrm{EM+NP}}\cap C^{\mathrm{MP}}$ (at least, under regularity of $\mathfrak{A}$)?
- What is a relative axiomatization of $C^{\mathrm{EM}}\cap C^{\mathrm{MP}}$? Is this (6.17) (at least, under regularity of $\mathfrak{A}$)?
- What are *all* proper non-Kleene extensions of C? Are these exactly the five/six ones found above (at least, under regularity of $\mathfrak{A}$)?

6.2. **Three-valued paraconsistent logics.** Here, we follow Subsection 5.1 supposing that $\wr\triangleq\sim\in\Sigma$. It is remarkable that all *particular* examples considered below are $\sim$-subclassical, because $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of the underlying algebras of the $\wr$-superclassical matrices under consideration.

6.2.1. *The logic of paradox and its expansions.* Here, it is supposed that $\Sigma_0\subseteq\Sigma$.

Given any $n\in(\omega\setminus 2)$, put $\mathcal{K}_n\triangleq\langle\mathfrak{K}_n,n\setminus 1\rangle$. Then, the bijection $e_{3,1}:3\to\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ is an isomorphism from $\mathcal{K}_3$ onto the $\sim$-superclassical $\wedge$-conjunctive $\vee$-disjunctive Σ_0-matrix with underlying algebra $e_{3,1}[\mathfrak{K}_3]$. Let $\mathfrak{A}$ be a Σ-algebra such that $(\mathfrak{A}\restriction\Sigma_0)=e_{3,1}[\mathfrak{K}_3]$. As usual, it is supposed that $\bot^{\mathfrak{A}}=\mathsf{f}$ and $\top^{\mathfrak{A}}=\mathsf{t}$, whenever $\Sigma_{01}\subseteq\Sigma$. Likewise, in case $\supset\in\Sigma$, it is supposed that $\supset^{\mathfrak{A}}$ is the restriction of the operation specified in Subsubsection 6.1.3. Finally, let $\mathcal{A}$ be the $\sim$-superclassical Σ-matrix with underlying algebra $\mathfrak{A}$. Since the *logic of paradox LP* [11] is defined by $\mathcal{K}_3$ (cf. [14]), in view of (2.5), the logic of $\mathcal{A}$ is an expansion of LP. And what is more, in case $\Sigma=(\Sigma_{0[1]}\cup\{\supset\})$, the logic of $\mathcal{A}$ is exactly the *logic of antinomies LA* [1] [resp., a definitional copy of the notorious $J3$]. The maximal paraconsistency of LP/both LA and $J3$ has been due to [14]/both [18] and [23], respectively. In this way, Corollary 5.10 provides a new *and uniform* insight into those particular results proved *ad hoc* therein.

Concluding this subsubsection, remark that LP, being defined by the three-valued matrix $\mathcal{K}_3$, in view of (2.5), is equally defined by the n-valued matrix $\mathcal{K}_n$, where $n\in(\omega\setminus 4)$, for $\hbar_n\in\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}_n,\mathcal{K}_3)$. Thus, for every $n\in(\omega\setminus 4)$, LP is an n-valued maximally $\sim$-paraconsistent logic but is not minimally n-valued, as opposed to the classical and bilattice expansions of C_{B} as well as the examples proposed in Subsection 5.2 (actually arisen by proper expanding $\mathcal{K}_n$ with providing both minimal n-valuedness and maximal $\sim$-paraconsistency) that highlights their particular meaning.

6.2.2. *Sette's logic P1.* Let $\Sigma\triangleq\{\supset,\sim\}$ and $\mathcal{S}_3$ the $\sim$-superclassical Σ-matrix such that $\sim^{\mathfrak{S}_3}\mathsf{b}\triangleq\mathsf{t}$, in which case $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{S}_3$, and

$$(a\supset^{\mathfrak{S}_3}b)\triangleq\begin{cases}\mathsf{t} & \text{if } (a\neq\mathsf{f})\Rightarrow(b\neq\mathsf{f}),\\ \mathsf{f} & \text{otherwise,}\end{cases}$$

for all $a,b\in\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ (cf. [24]). In this way, Theorem 5.6 yields:

Corollary 6.16. *The logic* $P1$ *of* $\mathcal{S}_3$ *is maximally* $\sim$*-paraconsistent.*

This strengthens the maximality result of [24], according to which $P1$ has no proper $\sim$-paraconsistent *axiomatic* extension, properly strengthened in [12] by proving the fact that the $\sim$-classical logic of $\mathcal{S}_2\triangleq(\mathcal{S}_3\restriction\{\mathsf{f},\mathsf{t}\})$ is the only proper axiomatic extension of $P1$, equally ensuing from Proposition 2.12 and the fact $\mathcal{S}_2$ is the only proper submatrix of $\mathcal{S}_3$ and is a model of the axiom $x_0\supset\sim\sim x_0$, not being true in $\mathcal{S}_3$ under $[x_0/\mathsf{b}]$, in which case the classical logic involved is axiomatized by the axiom involved relatively to $P1$.

Concluding this subsubsection, note that $P1$ is both $\underline{\vee}$-disjunctive and $\bar{\wedge}$-conjunctive, for $\mathcal{S}_3$ is so, where:

$$\begin{aligned}(x_0\,\underline{\vee}\,x_1) &\triangleq ((x_0\supset x_1)\supset x_1),\\ (x_0\,\bar{\wedge}\,x_1) &\triangleq \sim(x_0\supset(x_1\supset\sim(x_0\supset x_0))).\end{aligned}$$

6.2.3. *Hałkowska-Zajac' logic HZ.* Let $\Sigma \triangleq \Sigma_0$ and $\mathcal{HZ}$ the $\sim$-superclassical Σ-matrix such that $\sim^{\mathfrak{H}3}\mathsf{b} \triangleq \mathsf{b}$, while $\wedge^{\mathfrak{H}3}$ and $\vee^{\mathfrak{H}3}$ are defined as min and max, respectively, but with respect to rather the chain partial ordering $\leqq$ given by $\mathsf{b} \leqq \mathsf{f} \leqq \mathsf{t}$ (cf. [5]) than the point-wise natural partial ordering $\leqslant$ on 2^2, as in the case of the logic of paradox. Then, $(x_1 \wedge x_2)$ is a ternary b-relative conjunction for $\mathcal{HZ}$, so, by Theorem 5.6, the logic HZ of $\mathcal{HZ}$ is maximally $\sim$-paraconsistent, as it has been proved *ad hoc* in [19].

Concluding this subsubsection, note that HZ is both $\underline{\vee}$-disjunctive and $\overline{\wedge}$-conjunctive, for $\mathcal{HZ}$ is so, where:

$$(x_0 \underline{\vee} x_1) \triangleq \sim(\sim x_0 \wedge \sim x_1),$$
$$(x_0 \overline{\wedge} x_1) \triangleq \sim(\sim x_0 \vee \sim x_1),$$

but is neither $\wedge$-conjunctive nor $\vee$-disjunctive, because $(\mathsf{f} \wedge^{\mathfrak{H}3} \mathsf{b}) = \mathsf{b}$ and $(\mathsf{f} \vee^{\mathfrak{H}3} \mathsf{b}) = \mathsf{f}$.

7. Conclusions

Aside from the quite non-trivial general results and their numerous illustrative applications, the present paper demonstrates the special value of the conception of congruence/equality determinant, initially suggested in [20] just for the sake of construction of *two-side* sequent calculi for *many*-valued logics.

Perhaps, the main problems remaining still open (within this study) are completing finding the lattice of *all* extensions of $C_{[\mathrm{B}]}$ as well as finding that of CB_4. Among other things, solving the latter is based upon an algebraic technique going essentially beyond the scopes of the present study and, for this reason, is going to appear elsewhere.

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