Efficient Synchronization Between Chaotic Lorenz Systems in Unidirectional Coupling

Rosário Laureano, Diana A. Mendes and Manuel Alberto M. Ferreira

March 15, 2023

Efficient synchronization between chaotic Lorenz systems in unidirectional coupling

Rosário Laureano*, Diana A. Mendes# and Manuel A. Martins Ferreira×
*maria.laureano@iscte.pt #diana.mendes@iscte.pt ×manuel.ferreira@iscte.pt

Department of Quantitative Methods, IBS – ISCTE Business School Lisboa

Abstract

In order to obtain asymptotical synchronization, we combine active-passive decomposition for several driver signals, negative feedback control and dislocated negative feedback control with partial replacement on the nonlinear terms of the response system, a coupling version that was less explored. All these unidirectional coupling schemes are established between Lorenz systems with chaotic behavior/with control parameters that lead to chaotic behavior.

The sufficient conditions of global stable synchronization are obtained from a different approach of the Lyapynov direct method for the transversal system. In one coupling we apply a result based on classification of the symmetric matrix $\mathbf{A}^T + \mathbf{A}$ as negative definite, where $\mathbf{A}$ is the matrix characterizing the transversal system. In other couplings the sufficient conditions are based on derivative increase/accretion **(quero dizer majoração da derivada)** of an appropriate Lyapunov function. In fact, the effectiveness of a coupling between systems with equal dimension follows of the analysis of the synchronization error and, if the system variables can be bounded by positive constants, the derivative of an appropriate Lyapunov function can be increased.**(quero dizer majorada)** as required by the Lyapynov direct method.

In what follows we will always consider two chaotic dynamical systems, since they are sufficient to study the essential in the proposed coupling schemes. Our motivation for researching chaos synchronization methods is to explore their practical application in various scientific areas, such as physics, biology or economics.

1 Introduction

The ability of nonlinear oscillators to synchronize with each other is a basis for the explanation of many processes of nature. Therefore, chaos synchronization is thus a robust property expected to hold in mademan devices and plays a significant role in science. However, the possibility of two (or more) chaotic systems oscillate in a coherent and synchronized way is not an obvious phenomenon, since it is not possible to reproduce exactly the initial conditions andinfinitesimal perturbations to them/the initial conditions lead to divergence of nearby starting orbits. Contrary to expectation, when ensembles of chaotic oscillators are coupled, the attractive effect of a suitable coupling can counterbalance the trend of the trajectories to diverge. In many cases there are (coupling) parameters that control the strength of coupling between the systems, and the stability results of synchronous chaotic state depend on them.

Coupled dynamical systems are constructed from simple, low-dimensional dynamical systems and form new and more complex organizations. The chaotic dynamics introduces new degrees of freedom in ensembles of coupled systems. However, when two or more chaotic oscillators are coupled and synchronization is achieved, in general the number of dynamic degrees of freedom for the coupled system effectively decreases.

Asymptotical synchronization. Let X be a compact subset of $\mathbb{R}^m$ with $m \geq 3$ and consider (two) identical m-dimensional dynamical systems S_1 and S_2 defined on X by the nonlinear autonomous ordinary differential equations (ODE) $\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a})$ and $\dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a})$, respectively, where $\mathbf{a}$ is a vector of real control parameters.

Let $\mathbf{u}_1(0)$ and $\mathbf{u}_2(0)$ be (some) initial conditions for which, at certain value of $\mathbf{a}$, S_1 and S_2 evolve to an asymptotically stable chaotic attractor $\mathcal{A}$. The solutions $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ of the systems, starting at $\mathbf{u}_1(0) \neq \mathbf{u}_2(0)$ in the attraction basin $\mathcal{B}(\mathcal{A})$, are/represent independent trajectories in $\mathcal{A}$ after a period time of transient motion. This evolution is characterized by a positive Lyapunov exponent. Dynamical systems S_1 and S_2 are *asymptotically synchronized* if

$$\lim_{t \to +\infty} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| = 0. \tag{1}$$

The evolution of the difference $\mathbf{e}(t) = \mathbf{u}_2(t) - \mathbf{u}_1(t)$ between nearby starting orbits is described by

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{u}}_2(t) - \dot{\mathbf{u}}_1(t) = \mathbf{f}(\mathbf{u}_2(t); \mathbf{a}) - \mathbf{f}(\mathbf{u}_1(t); \mathbf{a}). \tag{2}$$

In case of asymptotical synchronization, this difference is the synchronization error and the system (2) is designated as transversal system (or error system). By (1), S_1 and S_2 achieve asymptotical synchronization if the transversal system (2) has an asymptotically stable equilibrium point at $\mathbf{e}(t) = \mathbf{0}$.

When asymptotical synchronization is achieved, the dynamics of $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ in $\mathcal{A}$, on the $2m$-dimensional phase space, are restricted to the m-dimensional smooth invariant manifold

$$\mathcal{M} \equiv \{(\mathbf{u}_1, \mathbf{u}_2) \in X \times X \mid \mathbf{u}_1 = \mathbf{u}_2\} \subset \mathbb{R}^{2m},$$

where occurs the synchronized dynamics defined by the symmetric synchronous chaotic state.

Transversal stability of the coupled system. The problem of synchronization can be understood as a problem of asymptotical stability of the chaotic attractor $\mathcal{A}$ (embedded in $\mathcal{M}$) in the $2m$-dimensional phase space of the coupled system (Fujisaka and Yamada [1], Pikovsky [2], Pecora and Carroll [3]).

It is necessary to distinguish between stability under tangent or transversal perturbations to the synchronization manifold $\mathcal{M}$. As stated by Pecora *et al.* [4], the limit (1) must be satisfied for all the initial conditions in a neighborhood of the equlibrium point $\mathbf{e}(t) = \mathbf{0}$. Since the system (2) characterizes the dynamics in the transversal direction to $\mathcal{M}$, it is necessary to analyze if small transversal perturbations to $\mathcal{M}$ are reduced or amplified by the evolution of S_1 and S_2. If they are reduced then $\mathcal{M}$ is transversely stable and the synchronous chaotic state $\mathbf{u}_1 = \mathbf{u}_2$ is stable. So, the synchronization stability is designated as *transversal stability*.

Usually the following criteria are applied:

(i) Criterion based on the eigenvalues of the Jacobian matrix corresponding to the flow over $\mathcal{M}$, suggested by Fujisaka and Yamada ([1],[5]); it requires that the largest eigenvalue is negative for the early stable synchronization;

(ii) Criterion based on the construction and study of an appropriate Lyapunov function $L(\mathbf{e}(t))$ (Lyapunov direct method) for the vector field of transversal perturbations to $\mathcal{M}$, developed by He and Vaidya [6]; it requires that L must be positive definite in a neighborhood of $\mathcal{M}$ ($L(\mathbf{e}(t)) \geq 0$), except in $\mathcal{M}$ where is null ($L(\mathbf{0}) = 0$), and its derivative is negative semidefinite ($\dot{L}(\mathbf{e}(t)) \leq 0$), and null in $\mathcal{M}$ ($\dot{L}(\mathbf{0}) = 0$);

(iii) Criterion based on the estimation of Lyapunov exponents, developed by Pecora and Carroll [3], which indicate if small transversal perturbations

$e_i(t)$, *for* $1 \le i \le m$, decrease or not; it requires that the largest transversal Lyapunov exponent is negative.

The criterion *(ii)* allows to prove the following/next proposition about global asymptotical stability of transversal system defined by (2).

Proposition 1 *Let* $\mathbf{A}$ *be the matrix characterizing the transversal system of a coupling between the identical systems* S_1 *and* S_2. *If there is a constant* $\delta < 0$ *such that the symmetric matrix* $\mathbf{A}^T + \mathbf{A}$ *is negative definite and* $\mathbf{A}^T + \mathbf{A} \le \delta \mathbf{I}$ *for any* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *in the phase space* X, *then the dynamics of the transversal system is globally stable and the systems* S_1 *and* S_2 *are in stable synchronization.*

Proof. Consider the Lyapunov function defined by $L(\mathbf{e}(t)) = [\mathbf{e}(t)]^T \cdot \mathbf{e}(t)$. Its derivative is given by

$$\frac{dL}{dt}(\mathbf{e}) = \frac{d\left(\mathbf{e}^T\right)}{dt} \cdot \mathbf{e} + \mathbf{e}^T \cdot \frac{d\mathbf{e}}{dt} = \mathbf{e}^T \cdot \mathbf{A}^T \cdot \mathbf{e} + \mathbf{e}^T \cdot \mathbf{A} \cdot \mathbf{e},$$

and verifies

$$\dot{L}(\mathbf{e}) = \mathbf{e}^T \left(\mathbf{A}^T + \mathbf{A}\right) \mathbf{e} \le \delta \left(\mathbf{e}^{\mathbf{T}} \cdot \mathbf{I} \cdot \mathbf{e}\right) = \delta \left(\mathbf{e}^T \cdot \mathbf{e}\right) < 0$$

for all $\mathbf{e} \neq \mathbf{0}$. The Lyapunov direct method guaranties the global asymptotical stability of transversal system ■

2 Unidirectional coupling schemes between continuous chaotic dynamical systems/chaotic dynamical systems defined by ODE

By partial replacement. Consider an (arbitrary) decomposition $\mathbf{u}_1 = (\mathbf{x}_1, \mathbf{y}_1)$ of the variable $\mathbf{u}_1$ into two subsystems

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}), \tag{3}$$

with variables $\mathbf{x}_1 = (u_1, \ldots, u_k)$ and $\mathbf{y}_1 = (u_{k+1}, \ldots, u_m)$, respectively, for $1 \le k \le m$. Since $\mathbf{f}(\mathbf{u}_1; \mathbf{a}) = (f_1(\mathbf{u}_1; \mathbf{a}), \ldots, f_m(\mathbf{u}_1; \mathbf{a}))$, the vector fields $\mathbf{g}$ and $\mathbf{h}$ are defined by the component functions of the vector field $\mathbf{f}$ as

$$\mathbf{g}(\mathbf{u}_1; \mathbf{a}) = (f_1(\mathbf{u}_1; \mathbf{a}), \ldots, f_k(\mathbf{u}_1; \mathbf{a}))$$

and

$$\mathbf{h}(\mathbf{u}_1; \mathbf{a}) = (f_{k+1}(\mathbf{u}_1; \mathbf{a}), \ldots, f_m(\mathbf{u}_1; \mathbf{a})).$$

They are respectively taken independent initial conditions $\mathbf{x}_1(0)$ and $\mathbf{y}_1(0)$ in the subsystems in (3). Let $\dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a})$ be a subsystem identical to $\dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a})$ with the variable $\mathbf{x}_1$ replaced by its corresponding $\mathbf{x}_2$,

$$\mathbf{x}_2 = \mathbf{x}_1 \qquad e \qquad \dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}).$$

So, the equations

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}), \tag{4}$$

with $\mathbf{y}_2(0) \neq \mathbf{y}_1(0)$, defined a dynamical system $\dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a})$ which shares some of the variables with the system $\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a})$. Pecora and Carroll [3] formalized this unidirectional coupling between the systems (3) and (4) through the variable $\mathbf{x}_1$, $\dot{\mathbf{u}}_2 = \mathbf{f}_{x_2 \to x_1}(\mathbf{u}_2; \mathbf{a}) = \mathbf{f}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a})$, where the coupled system

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}) \tag{5}$$

is obtained by complete replacement of the signal driver subsystem $\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a})$ in the response (or slave) system (4).

Instead of completely replacing one of the variables in the system response by its corresponding in drive (master or transport) system, a replacement can be partial as suggested by Guemez and Matthias [7]. In this case, a variable of response system gives rise to its corresponding in drive system only in some of its equations. In general, the stability results in partial replacement differ from those in complete replacement. In this paper it is studied the partial replacement in the nonlinear terms of response system.

By active-passive decomposition. Kocarev and Parlitz [8] proposed an unidirectional coupling more general than the complete replacement, in which the scalar signal transmitted from the drive subsystem to the response subsystem is a function of drive dynamical variables and, sometimes, a function of an information signal.

It is formally possible to rewrite the dynamical system S_1 defined by $\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a})$ as a non-autonomous system

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{s}(t); \mathbf{a}), \tag{6}$$

for some time vector function of time $\mathbf{s}(t)$, which possesses/with certain synchronization properties. If $\mathbf{s}$ is defined by $\mathbf{s}(t) = \mathbf{h}(\mathbf{x}_1(t))$ then $\mathbf{x}_1 = \mathbf{u}_1$. If $\mathbf{s}$ is given through an ordinary differential equation $\dot{\mathbf{s}}(t) = \mathbf{h}(\mathbf{x}_1(t), \mathbf{s}(t))$, the dimension of $\mathbf{x}_1$ may be lower than that of $\mathbf{u}_1$.

The vector functions $\mathbf{g}$ and $\mathbf{h}$ are a decomposition of the original vector field $\mathbf{f}$. The main feature of this decomposition is that, for appropriate choices of the function $\mathbf{h}$, any new system

$$\dot{\mathbf{x}}_2 = \mathbf{g}(\mathbf{x}_2, \mathbf{s}(t); \mathbf{a}) \tag{7}$$

synchronize with the initial system (6). The coupling between the systems is performed by the function $\mathbf{s}(t)$, designated by driver signal, which depends on the state vector $\mathbf{x}_1$ and is the same in both systems. The non-autonomous drive system (6) defined by $\mathbf{g}$ is a passive system while the component described by $\mathbf{h}$ is an active one. As such the decomposition given by $\mathbf{g}$ and $\mathbf{h}$ is said *active-passive decomposition* of drive system $\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a})$.

This coupling mechanism serves a large number of applications because, in many cases, the function $\mathbf{s}(t)$ is quite general. In particular, besides depending on $\mathbf{x}_1$, it may also depend on some information signal $i(t)$,

$$\mathbf{s}\,(t) = \mathbf{h}(\mathbf{x}_1\,(t)\,, i\,(t)) \quad \text{or} \quad \dot{\mathbf{s}}\,(t) = \mathbf{h}(\mathbf{x}_1\,(t)\,, \mathbf{s}\,(t)\,, i\,(t)).$$

In this case the active-passive decomposition can be used in communication schemes where $\mathbf{s}(t) = \mathbf{h}(\mathbf{x}_1\,(t)\,, i(t))$ is the transmitted and received signal. When synchronization identical occurs, the signal information $i\,(t)$ can be retrieved without error from the equation $\mathbf{s}(t) = \mathbf{h}(\mathbf{x}_1\,(t)\,, i(t)) = \mathbf{h}(\mathbf{x}_2\,(t)\,, i\,(t))$ whenever it has a unique solution for $i(t)$.

According Parlitz *et al.* [9], the coupling by active-passive decomposition is closely related to the Pyragas's approach [10] in chaos control. Instead of decomposing a given chaotic system, an appropriate nonlinear function $\mathbf{s} = \mathbf{h}(\mathbf{x}_1)$ can be added to a stable linear system $\dot{\mathbf{x}}_1 = \mathbf{A} \cdot \mathbf{x}_1$ such that $\dot{\mathbf{x}}_1 = \mathbf{A} \cdot \mathbf{x}_1 + \mathbf{s}$ is a chaotic system. In this case the synchronization error is given by the stable linear system $\dot{\mathbf{e}} = \mathbf{A} \cdot \mathbf{e}$, and occurs synchronization for all initial conditions and arbitrary signals $\mathbf{s}$.

By dislocated negative feedback control. Consider the coupling between S_1 and S_2 through the linear term $\boldsymbol{\rho}(\mathbf{u}_2 - \mathbf{u}_1)$,

$$\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a}) + \boldsymbol{\rho}(\mathbf{u}_2 - \mathbf{u}_1), \tag{8}$$

where $\boldsymbol{\rho} = (\rho_1, \rho_2, ..., \rho_m)$ is the vector coupling parameter, with $\rho_i > 0$ for all $i = 1, \ldots, m$. The unidirectional coupling in (8) is designated by *negative feedback control* through the damping term $\boldsymbol{\rho}(\mathbf{u}_2 - \mathbf{u}_1)$.

Let $\mathbf{u}_1 = (u_1, u_2, \ldots, u_m) \in X$ and $\mathbf{u}_2 = (u'_1, u'_2, \ldots, u'_m) \in X$ be the variables of S_1 and S_2, respectively. Suppose that the dynamical variable

$u_k(t)$, $1 \leq k \leq m$, and its corresponding $u'_k(t)$ can be measured. The addition of $\rho\left(u_k - u'_k\right)$, with $\rho > 0$, to the response system,

$$\left\{\begin{array}{l} \dot{u}'_1 = f_{\mathbf{a},1}\left(u'_1, u'_2, \ldots, u'_m\right), \\ \cdots \\ \dot{u}'_k = f_{\mathbf{a},k}\left(u'_1, u'_2, \ldots, u'_m\right) + \rho\left(u_k - u'_k\right) \\ \cdots \\ \dot{u}'_m = f_{\mathbf{a},m}\left(u'_1, u'_2, \ldots, u'_m\right) \end{array}\right., \tag{9}$$

leads to a particular case of (8) in which a single variable u_k makes the coupling. The term $\rho\left(u_k - u'_k\right)$ is used as a control signal (or perturbation signal) applied to the response system whereby negative feedback without changing its solution. The parameter ρ, designated by coupling strength, is experimentally adjustable and measures the perturbation intensity.

From initial conditions $\mathbf{u}_1(0)$ and $\mathbf{u}_2(0)$ such that $\mathbf{u}_1(0) \neq \mathbf{u}_2(0)$, the vector state of each systems S_1 and (9) are the same for certain value of ρ, after a certain time t_{sync}. When synchronization is achieved the control signal became 0 but the symmetric synchronous chaotic state $\mathbf{u}_1 = \mathbf{u}_2$ is established. In this paper it is studied the dislocated negative feedback control. After choosing the driver variable u_k, the control signal $\rho\left(u'_k - u_k\right)$ is applied to an equation j-th of response system S_2 with $j \neq k$. So, with $1 \leq j, k \leq m$, the response system is given by

$$\left\{\begin{array}{l} \dot{u}'_1 = f_{\mathbf{a},1}\left(u'_1, u'_2, \ldots, u'_m\right), \\ \cdots \\ \dot{u}'_j = f_{\mathbf{a},j}\left(u'_1, u'_2, \ldots, u'_m\right) + \rho\left(u_k - u'_k\right) \quad \text{para } j \neq k. \\ \cdots \\ \dot{u}'_m = f_{\mathbf{a},m}\left(u'_1, u'_2, \ldots, u'_m\right) \end{array}\right.$$

3 Case study: unidirectional couplings between nonlinear Lorenz systems

Consider the Lorenz system

$$\dot{x} = \sigma\left(y - x\right) \quad \wedge \quad \dot{y} = x\left(\alpha - z\right) - y \quad \wedge \quad \dot{z} = xy - \beta z$$

with parametric values σ, α and β that lead to chaotic behavior. In what follow it is considered unidirectional coupling between identical Lorenz systems.

By dislocated negative feedback control with partial replacement of x_2. (L2 da tese) Consider the driver variable x_1 by adding the control signal $\rho(x_1 - x_2)$, with $\rho > 0$, applied as dislocated negative feedback to the second equation of response system. Furthermore it is introduced the partial replacement of variable x_2 by the corresponding x_1 only in the nonlinear terms $x_2 z_2$ and $x_2 y_2$ of response system. Starting the coupled system

$$\left\{\begin{array}{l}\dot{x}_1 = \sigma(y_1 - x_1) \\ \dot{y}_1 = \alpha x_1 - x_1 z_1 - y_1 \\ \dot{z}_1 = x_1 y_1 - \beta z_1\end{array}\right. \wedge \left\{\begin{array}{l}\dot{x}_2 = \sigma(y_2 - x_2) \\ \dot{y}_2 = \alpha x_2 - \underline{x_1} z_2 - y_2 + \rho(x_1 - x_2) \\ \dot{z}_2 = \underline{x_1} y_2 - \beta z_2\end{array}\right. \qquad (10)$$

from (arbitrary) initial conditions such that $x_1(0) \neq x_2(0)$, $y_1(0) \neq y_2(0)$ and $z_1(0) \neq z_2(0)$, it is reached identical synchronization if the evolution of coupled system evolution (10) is continually confined to a hyperplane $\mathcal{M}$ in phase space. The coordinates $e_x = x_2 - x_1$, $e_y = y_2 - y_1$ and $e_z = z_2 - z_1$ of synchronization error $\mathbf{e}$ in the transversal subspace to $\mathcal{M}$ converge to 0 as $t \to +\infty$ if the point $(0,0,0)$ in the transversal subspace to $\mathcal{M}$ is an asymptotically stable equilibrium point (in this space). This leads to require that the dynamical system in $\mathbf{e} = (e_x, e_y, e_z)$ defining the transversal perturbations is asymptotically stable at the equilibrium point $(0,0,0)$.

Consider the function

$$\breve{\mathbf{f}} = (\sigma(y_2 - x_2), \alpha x_2 - x_1 z_2 - y_2 + \rho(x_1 - x_2), x_1 y_2 - \beta z_2)$$

obtained from the response in (10). For all values of ρ, the linearized equation which defines transversal perturbations to $\mathcal{M}$ is given by

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} \approx D_{(x_2,y_2,z_2)}\breve{\mathbf{f}} \cdot \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \alpha - \rho & -1 & -x_1 \\ 0 & x_1 & -\beta \end{bmatrix} \cdot \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}.$$

Studying the eigenvalues of Jacobian matrix $D_{(x_2,y_2,z_2)}\breve{\mathbf{f}}$, we conclude that locally stable synchronization is reached if $\rho_{sync} = \alpha - 1$.

Taking control parameters $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$ and strength coupling $\rho = 27.1$, we verify that $x_2 \to x_1$, $y_2 \to y_1$ and $z_2 \to z_1$ when systems evolve (Fig. 1a). After a certain time, the coordinates x, y and z of systems verify the equalities $x_2 = x_1$, $y_2 = y_1$ and $z_2 = z_1$ (Fig. 1b). So, the distances $|x_2 - x_1|$, $|y_2 - y_1|$ and $|z_2 - z_1|$ converge to 0 over time (Fig. 1c). Equations $x_2 = x_1$, $y_2 = y_1$ and $z_2 = z_1$ define a hyperplane $\mathcal{M}$ in the 6-dimensional phase space.

Applying criterion *(ii)*, it is obtained the threshold $\tilde{\rho}_{sync}$ of globally stable synchronization. It is greater than the threshold ρ_{sync} obtained for local stability, ,

$$\tilde{\rho}_{sync} = \alpha + \sigma > \alpha - 1 = \rho_{sync},$$

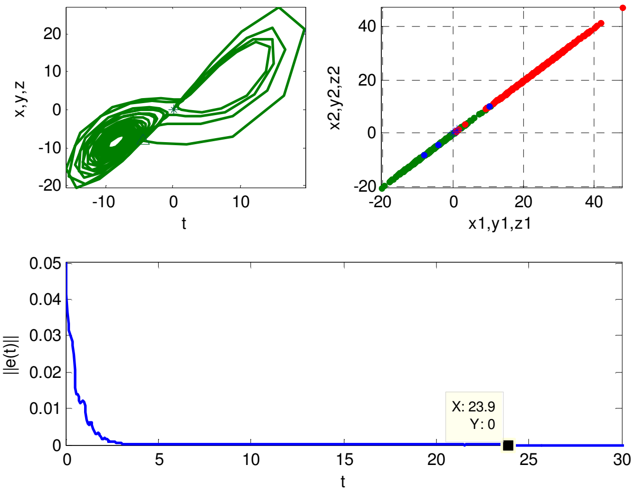

Figure 1: Parameter values $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$; coupling strength $\rho = 27.1$. (a) Coupled system attractor; (b) Synchronization manifold; (c) Evolution of synchronization error

leading to a range of values ρ more restrictive. In fact, consider the Lyapunov function $L(\mathbf{e}) = \left(e_x^2 + e_y^2 + e_z^2\right)/2$ which verifies $L(\mathbf{e}) > 0$ if $\mathbf{e} \neq \mathbf{0}$ and $L(\mathbf{0}) = 0$ for all $\rho > 0$. It is necessary to determine the strength coupling ρ such that the derivative of L satisfies $\dot{L}(\mathbf{e}) < 0$ if $\mathbf{e} \neq \mathbf{0}$ and $\dot{L}(\mathbf{0}) = 0$. Substituting the expression of $\dot{e}_x$, $\dot{e}_y$ and $\dot{e}_z$ in

$$\dot{L}(\mathbf{e}) = e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z$$

and simplifying, the derivative of L can be written as

$$\begin{aligned} \dot{L}(\mathbf{e}) &= -\sigma e_x^2 - e_y^2 - \beta e_z^2 + (\sigma + \alpha - \rho)\, e_x e_y \\ &\leq -\sigma e_x^2 - e_y^2 - \beta e_z^2 + (\sigma + \alpha - \rho)\, |e_x e_y|\,. \end{aligned}$$

Choosing a coupling strength satisfying $\tilde{\rho} > \alpha + \sigma$ the conditions required by Lyapunov direct method are guaranteed. So it is achieved globally stable synchronization in the coupled system with a coupling strength $\tilde{\rho} = \tilde{\rho}(\sigma, \alpha)$ which do not depend on the control parameter β. In Figure 2(a,b,c) are taken the same values for control parameters and the corresponding synchronization threshold $\tilde{\rho} = 38.1$. (As expected) The time synchronization t_{sync} for $\tilde{\rho} = 38.1$ is lower than the obtained for $\rho = 27.1 < 38.1$.

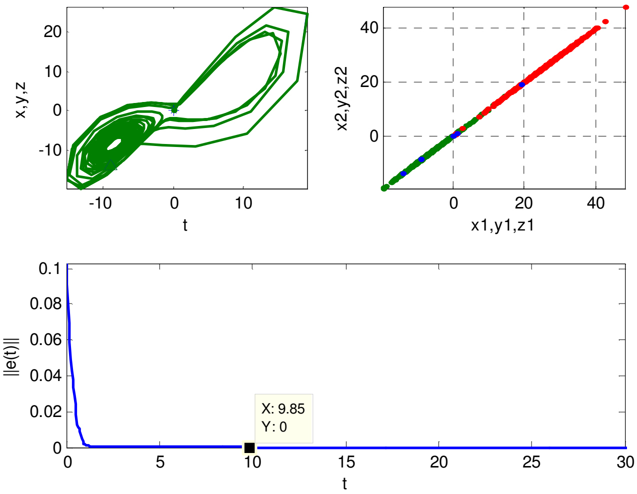

Figure 2: Parameter values $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$; coupling strength $\tilde{\rho} = 38.1$. (a) Coupled system attractor; (b) Synchronization manifold; (c) Evolution of synchronization error

Table 1 presents the sufficient conditions for globally stable synchronization obtained in the study of other similar cases. It is applied the dislocated control signal $\rho\,(x_1 - x_2)$ and, in some cases, also partial replacement of x_2 by its corresponding x_1 in some nonlinear terms of response. The constants ξ and K represent the expressions $\rho - \sigma - \alpha$ and $K_x + K'_x$, respectively.

Disloc.	**Replac.**	**Synchronization sufficient condition**
to 2^{th} eq.	——	$\beta\,(\xi + K_z)^2 < 4\sigma\beta - K_y^2$
to 2^{th} eq.	on 3^{th} eq.	$\beta\,(\xi + K_z)^2 < 4\sigma\beta - \sigma K^2$
to 3^{th} eq.	on 2^{th} eq.	$\xi^2 < 4\sigma \;\; \wedge \;\; \beta\xi^2 < 4\sigma\beta - KK_y\xi + K_y^2 + \sigma K^2$

Table 1: Unidirectional coupling by dislocated negative feedback control.

Unidirectional coupling by negative feedback control with partial replacement of x_2. (L4 da tese) Consider (two) identical chaotic Lorenz systems coupled by negative feedback control

$$\left\{\begin{array}{l}\dot{x}_1=\sigma\left(y_1-x_1\right)\\ \dot{y}_1=\alpha x_1-x_1z_1-y_1\\ \dot{z}_1=x_1y_1-\beta z_1\end{array}\right. \wedge \left\{\begin{array}{l}\dot{x}_2=\sigma\left(y_2-x_2\right)+\rho\left(x_1-x_2\right)\\ \dot{y}_2=\alpha x_2-\underline{x_1}z_2-y_2+\rho\left(y_1-y_2\right)\\ \dot{z}_2=\underline{x_1}y_2-\beta z_2+\rho\left(z_1-z_2\right)\end{array}\right.$$

where it is also made a partial replacement of variable x_2 by x_1 only in the nonlinear terms x_2z_2 and x_2y_2 of response system. Let $\breve{\mathbf{f}}$ be the function obtained from the response, which components are $\breve{\mathbf{f}}_1=\sigma\left(y_2-x_2\right)+\rho\left(x_1-x_2\right)$, $\breve{\mathbf{f}}_2=\alpha x_2-x_1z_2-y_2+\rho\left(y_1-y_2\right)$ and $\breve{\mathbf{f}}_3=x_1y_2-\beta z_2+\rho\left(z_1-z_2\right)$. Consider the components $e_x=x_2-x_1$, $e_y=y_2-y_1$ and $e_z=z_2-z_1$ of (synchronization error) $\mathbf{e}$. For all values of ρ, the linearized equation which defines transversal perturbations to (synchronization manifold) $\mathcal{M}$ is given by

$$\begin{bmatrix}\dot{e}_x\\ \dot{e}_y\\ \dot{e}_z\end{bmatrix}\approx D_{(x_2,y_2,z_2)}\breve{\mathbf{f}}\cdot\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix}=\begin{bmatrix}-\sigma-\rho & \sigma & 0\\ \alpha & -1-\rho & -x_1\\ 0 & x_1 & -\beta-\rho\end{bmatrix}\cdot\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix}.$$

It can take the matrix form $\dot{\mathbf{e}}=\mathbf{A}\left(x_1\right)\cdot\mathbf{e}$ with

$$\mathbf{A}=\begin{bmatrix}-\sigma-\rho & \sigma & 0\\ \alpha & -1-\rho & -x_1\\ 0 & x_1 & -\beta-\rho\end{bmatrix}.$$

The main determinants of the matrix

$$\mathbf{A}^T+\mathbf{A}=\begin{bmatrix}-2\left(\sigma+\rho\right) & \sigma+\alpha & 0\\ \sigma+\alpha & -2\left(1+\rho\right) & 0\\ 0 & 0 & -2\left(\beta+\rho\right)\end{bmatrix}$$

are $\Delta_1=-2\left(\sigma+\rho\right)$, $\Delta_2=4\left(\sigma+\rho\right)\left(1+\rho\right)-\left(\sigma+\alpha\right)^2$ and

$$\Delta_3=\left[2\left(\sigma+\alpha\right)^2-8\left(\sigma+\rho\right)\left(1+\rho\right)2\left(\sigma+\alpha\right)^2\right]\left(\beta+\rho\right).$$

We have $-\Delta_1>0$ and the condition $-\Delta_3>0$ is satisfied when/where $\Delta_2>0$ (since $\beta+\rho>0$). So, we conclude by Proposition 1 that occurs globally stable synchronization if the control and coupling parameters verify the inequality

$$4\left(\sigma+\rho\right)\left(1+\rho\right)>\left(\sigma+\alpha\right)^2.$$

Taking (the control parameters) $\sigma=10$, $\alpha=28$ and $\beta=2.(6)$, we present the Figure 3(a,b,c) obtained for (the coupling strength) $\rho=14.5$, which is the lowest value of ρ in a tenth step that verifies the previous inequality.

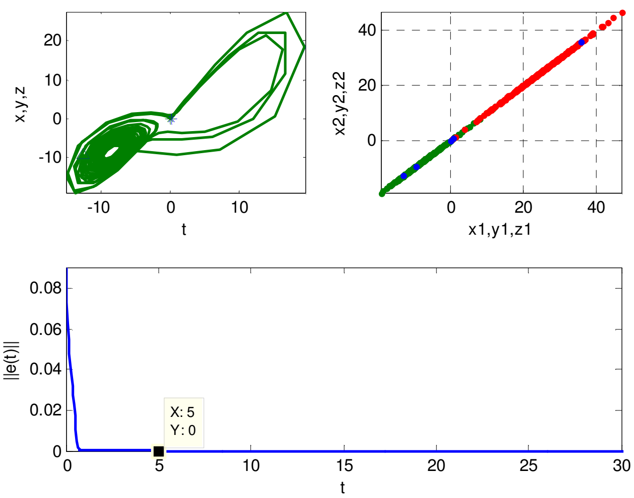

Figure 3: Parameter values $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$; coupling strength $\rho = 14.5$. (a) Coupled system attractor; (b) Synchronization manifold; (c) Evolution of the synchronization error

Such an/This approach is inconclusive when it is not done the partial replacement of the variable x_2 by x_1 in the nonlinear system response.

Unidirectional coupling by active-passive decomposition. (L3 da tese) Consider the active-passive decomposition between identical chaotic Lorenz systems

$$\left\{\begin{array}{l}\dot{x}_1 = -\sigma x_1 + \sigma y_1 \\ \dot{y}_1 = s(t) - y_1 \\ \dot{z}_1 = x_1 y_1 - \beta z_1\end{array}\right. \wedge \left\{\begin{array}{l}\dot{x}_2 = -\sigma x_2 + \sigma y_2 \\ \dot{y}_2 = s(t) - y_2 \\ \dot{z}_2 = x_2 y_2 - \beta z_2\end{array}\right.$$

using the driver signal $s = h(x_1, y_1, z_1) = x_1 (\alpha - z_1)$.

The linearized equation which defines transversal perturbations (to $\mathcal{M}$) is given by

$$\left[\begin{array}{c}\dot{e}_x \\ \dot{e}_y \\ \dot{e}_z\end{array}\right] \approx D_{(x_2,y_2,z_2)}\breve{\mathbf{f}} \cdot \left[\begin{array}{c}e_x \\ e_y \\ e_z\end{array}\right] = \left[\begin{array}{ccc}-\sigma & \sigma & 0 \\ 0 & -1 & 0 \\ y_2 & x_2 & -\beta\end{array}\right] \cdot \left[\begin{array}{c}e_x \\ e_y \\ e_z\end{array}\right],$$

where $\breve{\mathbf{f}} = (f_1, f_2, f_3) = (-\sigma x_2 + \sigma y_2, s(t) - y_2, x_2 y_2 - \beta z_2)$ and the differences e_x, e_y and e_z are considered sufficiently small. Since all eigenvalues

$\Lambda_1 = -\sigma$, $\Lambda_2 = -1$ and $\Lambda_3 = -\beta$of the Jacobian matrix $D_{(x_2,y_2,z_2)}\breve{\mathbf{f}}$ are negative, it is guaranteed by the criterion *(i)* the stable synchronization of the systems (see Fig. 4 a,b,c obtained with $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$).

In applying the criterion *(ii)*, note that $\dot{e}_y = -e_y$ so $e_y \to 0$ as $t \to +\infty$. Therefore, the 2-dimensional subsystem which describes the evolution of $e_x = x_1 - x_2$ and $e_z = z_1 - z_2$ can, when $t \to +\infty$, be written as just

$$\begin{cases} \dot{e}_x = -\sigma e_x \\ \dot{e}_z = y_2 e_x - \beta e_z \end{cases}.$$

Then consider $L(e_x, e_z) = \left(e_x^2 + e_z^2\right)/2$ which verifies $L(e_x, e_z) > 0$ whenever $(e_x, e_z) \neq (0,0)$ and $L(0,0) = 0$. Substituting the expressions $\dot{e}_x$ and $\dot{e}_z$ in the derivative $\dot{L}(e_x, e_z) = e_x\dot{e}_x + e_z\dot{e}_z$ is obtained

$$\dot{L}(e_x, e_z) = -\sigma e_x^2 - y_2 e_x e_z - \beta e_z^2 \leq -e_y^2 - \beta e_z^2 - y_2 \left|e_x e_z\right|.$$

Assuming that the function of real variable y_2 is bounded, let K_y be a positive constant such that $|y_2| \leq K_y$. As such is valid the inequality

$$\dot{L}(e_x, e_z) \leq -e_y^2 - \beta e_z^2 - K_y \left|e_x e_z\right| \leq 0,$$

therefore, by the Lyapunov direct method, the synchronization error tends to 0 as $t \to +\infty$ and synchronization is globally stable.

From the foregoing, it is even concluded that occurs globally stable synchronization for all signal $s(t)$ leading to the inequality $\dot{e}_y < 0$. Note that, for the transversal system to be asymptotically stable at origin, the constant symmetric matrix

$$\mathbf{P} = \begin{bmatrix} 1 & \frac{1}{2}K_y \\ \frac{1}{2}K_y & \beta \end{bmatrix}$$

associated with quadratic form $-\|\mathbf{e}\|^T \cdot \mathbf{P} \cdot \|\mathbf{e}\|$, with $\|\mathbf{e}\| = (|e_y|, |e_z|)$, must be positive definite. The main determinants Δ_i, $i = 1, 2$, of $\mathbf{P}$ are positive if $K_y^2 < 4\beta$. By the Lypaunov direct method, the synchronization error tends to 0 as $t \to +\infty$ whenever the positive constant K_y limiting the system variable satisfies this inequality, and the systems achieve globally stable synchronization.

Table 2 summarizes the study of couplings in the Lorenz attractor by active-passive decomposition with the signal conductors $s(t) = \sigma y_1(t)$, $s(t) =$

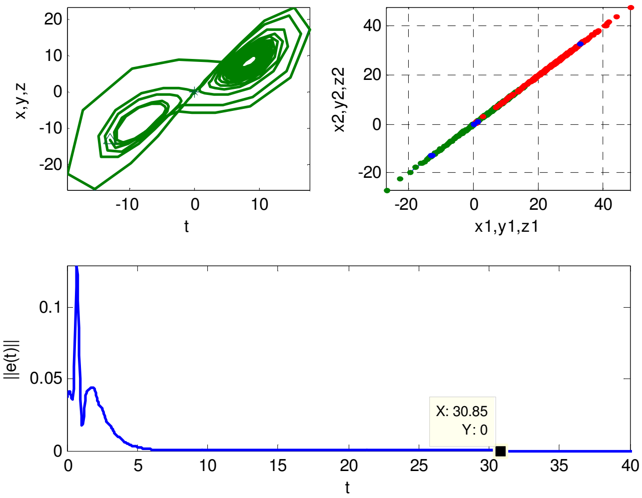

Figure 4: Parameter values $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$; coupling strength $\tilde{\rho} = 38.1$. (a) Coupled system attractor; (b) Synchronization manifold; (c) Evolution of synchronization error

$\alpha x_1(t)$ and $s(t) = -\beta z_1(t)$.

Driver signal	**Synchronization sufficient condition**
$s(t) = \sigma y_1(t)$	$4\sigma > (K_z - \alpha)^2 \quad \wedge \quad \beta \left[4\sigma - (K_z - \alpha)^2\right] > K_y^2.$
$s(t) = \alpha x_1(t)$	$(\sigma + K_z)^2 < 4\sigma \quad \wedge \quad \beta \left[4\sigma - (\sigma + K_z)^2\right] > K_y^2$
$s(t) = -\beta z_1(t)$	inconclusive

Table 2: Active-passive decomposition couplings.

Note that the active-passive decomposition using signal $s(t) = \sigma y_1(t)$ is equivalent to the substitution of y_2 by y_1, but only in the first equation of the response system. Similarly for the driver signals $s(t) = \alpha x_1(t)$ and $s(t) = -\beta z_1(t)$ corresponding, respectively, to the replacement of the variable x_2 by x_1 only in the second equation of the response system and the replacement of the variable z_2 by z_1 in the third equation. This shows that the Pecora-Carroll's criterion is included in the more general approach by active-passive decomposition.

4 Conclusions

Either using an usual negative feedback control or applying a control signal as dislocated negative feedback, the combination of each of these unidirectional couplings with replacement only shows advantages. Even if the replacement is partial, on the nonlinear terms of the response system, were obtained very simple sufficient conditions for globally stable synchronization between identical chaotic Lorenz systems. These conditions result from the classification of the symmetric matrix $\mathbf{A}^T + \mathbf{A}$ as negative definite (Proposition 1), where $\mathbf{A}$ is the matrix characterizing the transversal system of coupling, or are based on derivative increase/accretion of an appropriate Lyapunov function. Proposition 1 is not valid if the partial replacement is not applicable. The approach based on derivative increase/accretion of an appropriate Lyapunov function also leads to a sufficient condition for globally stable synchronization in a coupling by active-passive decomposition for several driver signals.

(referir algo muito breve sobre as mesmas ligações e abordagens com sistemas de Rössler?)

References

[1] T. Yamada and H. Fujisaka (1983), Stability theory of synchronized motion in coupled-oscillator systems, *Prog. Theoret. Phys.* **69** (1), 32-47.

[2] A.S. Pikovsky (1984), On the interaction of strange attractors, *Z. Physik B* **55** (2), 149-154.

[3] L.M. Pecora and T.L. Carroll (1990), Synchronization in chaotic systems, *Phys. Rev. Lett.* **64** (8), 821-824.

[4] L.M. Pecora, T.L. Carroll, G.A.. Johnson, D.J. Mar and J.F. Heagy (1997), Fundamentals of synchronization in chaotic systems, concepts, and applications, *Chaos* **7** (4), 520-543.

[5] T. Yamada and H. Fujisaka (1983), Stability theory of synchronized motion in coupled-oscillator systems. II, *Prog. Theoret. Phys.* **70** (5), 1240-1248.

[6] R. He and P.G. Vaidya (1992), Analysis and synthesis of synchronous periodic and chaotic systems, *Phys. Rev. A* **46** (12), 7387-7392.

[7] J. Guemez and M.A. Matias (1995), Modified method for synchronizing and cascading chaotic systems, *Phys. Rev. E* **52,** R2145-R2148.

[8] L. Kocarev and U. Parlitz (1995), General approach for chaotic synchronization with applications to communication, *Phys. Rev. Lett.* **74** (25), 5028-5031.

[9] U. Parlitz, L. Kocarev, T. Stojanovki and H. Preckel (1996), Encoding messages using chaotic synchronization, *Phys. Rev. E* **53,** 4351-4361.

[10] K. Pyragas (1993), Predictable chaos in slightly perturbed unpredictable chaotic systems, *Phys. Lett. A* **181,** 203-210.