The Cube: Its Relatives, Geodesics, Billiards, and Generalisations

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Abstract. Starting with a cube and its symmetry group one can get a set of related polyhedra via adding congruent pyramids to its faces. The height and the rotation angle of the added pyramids give rise to a two-parameter set of such polyhedra. Thereby occur Archimedian solids and their duals, as e.g. an "icosi-tetra deltahedron", but also starshaped solids. This approach can also be applied when taking a regular tetrahedron or a regular pentagon-dodecahedron as start figure. A hypercube in $\mathbb{R}^n$ (an "n-cube"), too, suits as start object and gives rise to interesting polytopes (c.f. [1], [2], [3]).

The cube's geodesics and (inner) billiards, especially the closed ones, are already well-known (see [4], [5]). Hereby, a ray's incoming angle equals its outcoming angle. There are many practical applications of reflections in a cube's corner, as e.g. the cat's eye and retroreflectors or reflectors guiding ships through bridges. Geodesics on a cube can be interpreted as billiards in the circumscribed rhombi-dodecahedron. This gives a hint, how to treat geodesics on arbitrary polyhedra.

Generalising reflections to refractions means that one has to apply Snellius' refraction law saying that the sine-ratio of incoming and outcoming angles is constant. Application of this law (or a convenient modification of it) to geodesics on a polyhedron will result in trace polygons, which might be called "quasi-geodesics". The concept "pseudo-geodesic", coined for curves c on smooth surfaces Φ, is defined by the property of c that its osculating planes enclose a constant angle with the normals n of Φ. Again, this concept can be modified for polyhedrons, too. We look for these three types of traces of rays in and on a 3-cube and a 4-cube.

Keywords: Polyhedron, Cube, Geodesic polygon, Billiard polygon, Snellius' Refraction law.

1 Introduction: The cube and related polyhedra

Well, it is not at all necessary to give a description of properties of a cube or "hexahedron", as it is called by the ancient Greeks. It is "the" space filling polyhedron in Euclidean space, and it also occurs, virtually or as a common set of vertices, at the other Platonic solids. Therefore, and roughly speaking, one can say that its symmetry group "dominates" that of the other Platonic solids and even those of Archimedian solids. Most of the latter are derived from the former by chamfering edges and vertices and even the snub cube and snub dodecahedron, too, can be related to the cube. The cube "generalizes" the square (which generalizes the segment) with respect to the dimension of the Euclidean spaces. The hypercube in 4-space resp. in an n-space are further generalizations in that sense. By the way, the projection of a 4-cube in direction of a diagonal plane of one of its face-cubes became the logo of the ISGG, see Fig.1.

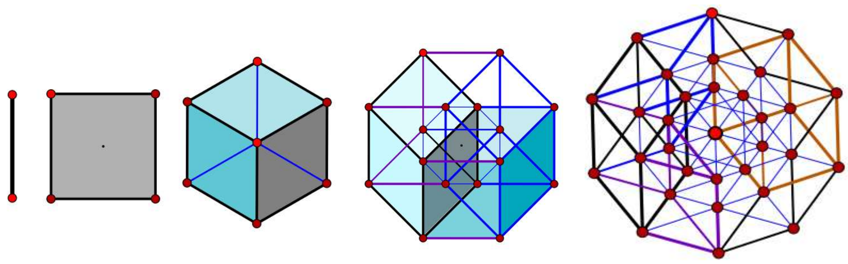

Fig. 1. Series of "cubes" from dimension 1 to 5. In the images of the 4-cube and the 5-cube some characteristic face-cubes are marked with edges or faces in different colors.

In contrast to cutting away corners and edges by a chamfering process one also can add congruent pyramids to the faces of a Platonic solid to receive polyhedra with congruent faces. In case of a cube we add quadratic right pyramids, the faces of which pass through vertices of the cube, see Fig.2.

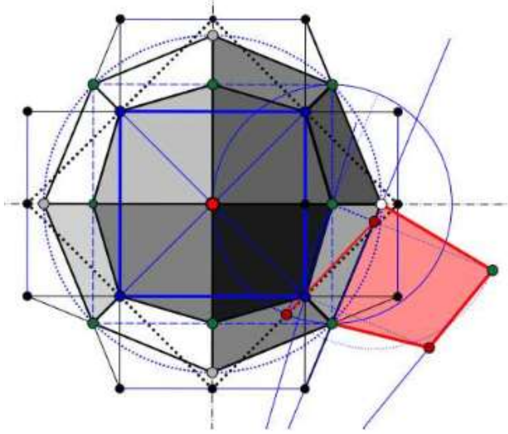

Fig. 2. Cube with added pyramids. The blue square is interpreted as front *and* top projection of the cube, the dotted square is the base of an attached pyramid in the plane of the cube's face.

Intersecting the 6 congruent pyramids results in a “deltoidal-icositetrahedron”. The rose deltoid shows the true size of its faces.

The rotation angle φ between the symmetry planes of the pyramid and those of the cube’s face is one parameter, the altitude h of the pyramid another one. The adding process delivers a two-parameter set of polyhedral with congruent faces. If $\varphi \neq \pi/2$, the faces are, in general, (irregular) quadrangles or pentagons; if $\varphi = \pi/2$, the faces are (in general) deltoids.

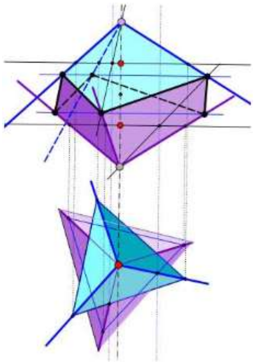

Fig. 3. Front and top projections of a three-sided double pyramid.
The twist angle between the two coaxial partial pyramids is φ=30°.

Fig. 4 shows a set of polyhedra with different ratio h: a, a the length of the cube’s edge. The polyhedra have, in general, 24 deltoids as faces. Starting with, $h > a$ we get a non-convex form of them. For the second object we put $\varphi = a$, and we receive the regular octahedron, where 3 deltoids become coplanar and form an equilateral triangle. For $0 < h < a/2$ the polyhedra are convex.

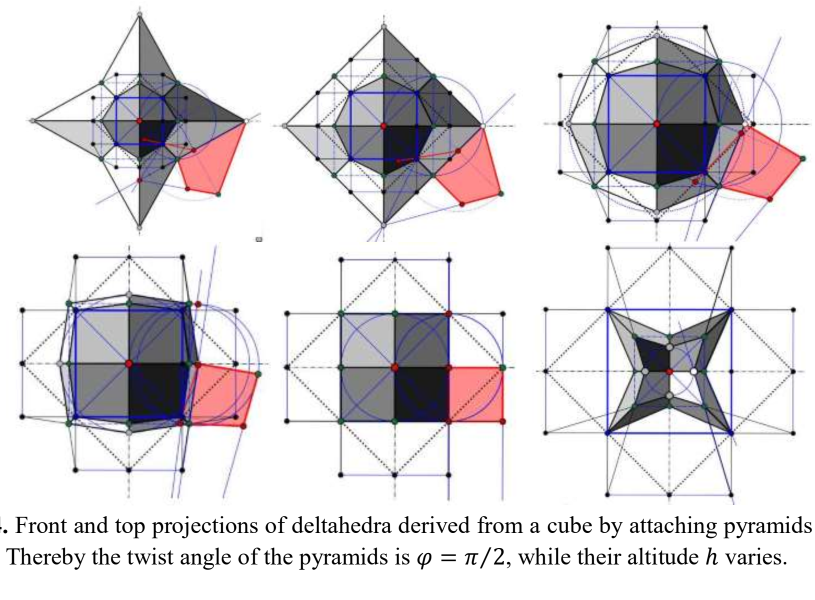

Fig. 4. Front and top projections of deltahedra derived from a cube by attaching pyramids to its faces. Thereby the twist angle of the pyramids is $\varphi = \pi/2$, while their altitude h varies.

The third object of the set in Fig. 4, is “the” *deltoidal-icositetrahedron*, meaning the Catalan solid dual to the dual to the *rhombicuboctahedron*. For this deltahedron we put $h = a/2$. For $h = 0$ we receive the cube itself, where 4 deltoids unite to a face of the cube, (fifth object in Fig. 4). For $0 > h > -a/2$ the polyhedra are again star shaped, (sixth object), and for $h = -a/2$ it becomes degenerate.

If we would have started with an octahedron and added three-sided right pyramids to its triangular faces we would get the same sequence of deltahedra, and, similar, if we started with a rhombicuboctahedron, Fig. 5.

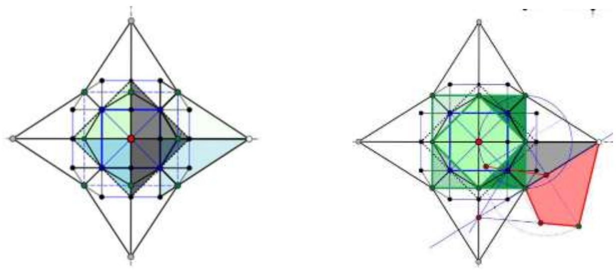

Fig. 5. 8 pyramids attached to faces of an octahedron or 6 to the square faces of a cuboctahedron result in the same set of deltahedra, as when attaching the pyramids to a cube’s faces.

Remark 1: By dualizing the rhombicuboctahedron one gets a single deltoidal-icositet-rahedron, others are not mentioned in references. The dualizing process replaces each polyhedron face by its center to get the vertices of the dual polyhedron. As long the original polyhedron has regular polygons as faces, this process is well-defined. But if the polyhedron has irregular faces, deltoids or with no symmetry pentagons with no symmetry, the choice of a suitable face center has some arbitrariness and it is not at all trivial to find the “right” center, such that the dualizing process becomes involutoric. Fig. 5 shows the (incomplete) dualizing of a rhombicuboctahedron (above left) and of Miller’s pseudo-rhombicuboctahedron (above right), see e.g. [7]. The latter belongs to the family of the Johnson polyhedral and has the Johnson number J_{37}, c.f. [7] and [8].

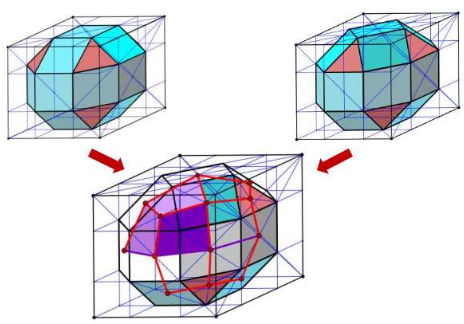

Fig. 5. (Incomplete) dualizing of the cuboctahedron and the pseudo-cuboctahedron. The resulting two polyhedra are named “deltoidal-icositerahedron” and “pseudo-icosi- tetrahedron”. Both have 24 congruent deltoids as faces.

Remark 2: If we take the twist angle $\varphi = 0$, the attached pyramids pass through the edges of the start polyhedron. For a cube as start polyhedron and arbitrary height h we receive, in general, a polyhedron with 24 face triangles. For pyramids with height $h = a/2$ the resulting polyhedron is the *rhombic-dodecahedron*, where two face triangles become coplanar and form a rhombus. This polyhedron is dual to the cuboctahedron and therefore a Catalan solid. Its face planes are the outer symmetry planes of two adjacent cube faces, a property we later will take into consideration.

2 Reflections and shortest paths

The reflection law states that "the ray's incoming angle equals its outcoming angle". We imagine that, at the point P, where the ray meets the reflecting (hyper-) surface Φ, we replace Φ by a planar mirror tangent to Φ and its normal n. A first and fundamental consequence of this consideration is that P is a regular point of Φ.
If Φ is the boundary of two media with different refractivity one applies *Snell's law of refraction* stating that "the sine ratio of the angles of incoming and outcoming rays is constant", (see e.g. [9]).
A light ray r starting from point A passing through point B and meeting Φ in between, must trace the fastest path from A over P to B. If the refractivity on both sides Φ is the same or r is reflected at Φ , this fastest path is also the shortest (with respect to Euclidean geometry).
The shortest path problem connects the topic to the concept of *geodesics*, a concept of differential geometry. For a curve c on a (regular) surface Φ to be the shortest connection between two points $A, B \in \Phi$, the osculating planes of c contain the normals n of Φ along c. The heuristic imagination that, locally, c is reflected at the tangent plane of Φ at each point $P \in c$, suits very well to that orthogonality property of geodesics.

We aim at dealing with polyhedra and construct (closed) geodesics on them. E.g. a rubber band between two points of adjacent faces will cross the common edge such that incoming angle equals the outcoming angle. This means that the trace of the rubber band can be interpreted as a reflection path at the (outer) symmetry plane of the two faces. Fig. 6 shows such a closed geodesic on a cube passing each face of the cube only once. It is also an inner (closed) reflection path, a so-called *billiard trace*, in the subscribed rhombi-dodecahedron.

The closed geodesic in Fig. 6 is also a closed billiard trace in the subscribed rhombi-dodecahedron for a starting point on a short diagonal of is rhombic faces. It meets only 6 of the 12 faces of the rhombi-dodecahedron

The 2-dimensional case, namely the construction of a closed billiard path in a square (Fig. 7) gives a hint, how to proceed in higher dimensions.

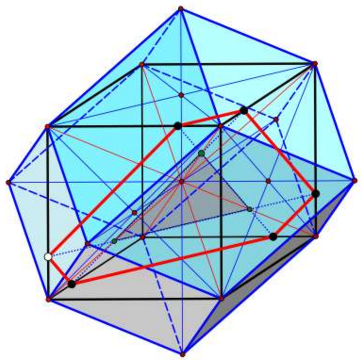

Fig. 6. A closed geodesic (red) of a cube is a planar hexagon with sides parallel to face diagonals of the cube.

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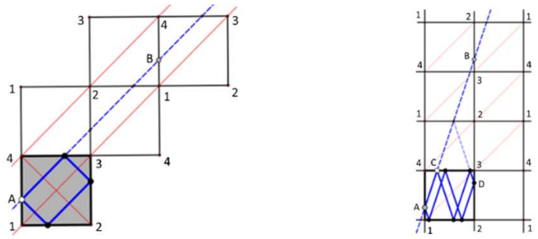

Fig. 7. Closed billiards in a square. Suitable reflections of the square give a rectification of the billiard. It occurs a closed billiard automatically, when choosing start- and endpoint on corresponding edges at the same position.

By suitable reflections of the square we get a rectification of the path. By choosing start- and endpoint of this rectification on corresponding edges at the same position forces the path to become closed. A path meeting all sides of the square and being shortest must have sides parallel to the diagonals. In Fig. 7 the exceptional billiard paths through vertices of the square are marked red. A ray meeting a vertex of a face or a polyhedron will be excluded from consideration, even though such cases make sense as limits.

In the planar case, for the billiard we start from a (regular) point A of a side of a polygon. In the three-dimensional case we start from an inner point of a face of a polyhedron, and, for a hypercube, the starting point can be arbitrarily chosen as inner point of its hyperface. The method to receive a closed billiard path is the same as for the square, using reflections of the (hyper-) cube at its faces. Fig. 8 shows the shortest closed billiard trace in a cube.

If the trace meets all six face squares of the cube, the partial segments must be parallel to three diagonals of the cube, c.f. [4]. Because of three diagonals of the cube are not coplanar, the resulting trace hexagon cannot be planar. But it is centric symmetric with respect to the cube's center

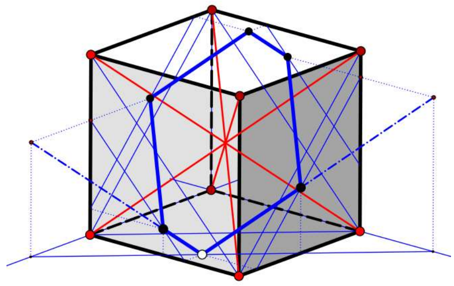

Fig. 8. Closed billiard trace in a cube meeting each face exactly once. The sides of the trace hexagon are parallel to 3 diagonals of the cube.

3 Geodesics, pseudo- and quasi-geodesics

In chapter 2 we started with the reflection law and Snell's refraction law and mentioned that, for a (regular) surface in Euclidean 3-space, its geodesics have at each point P an osculating plane containing the surface normal n at P. A generalization of the concept "geodesic" reads as follows, see [10] and Fig. 9:

Definition 1: A curve on a surface in Euclidean 3-space are called a *pseudo-geodesic*, if, at each of its points P, its osculating plane includes a fixed angle δ with the surface normal n at P. For $\delta = 0$ this the curve is an ordinary geodesic, for $\delta = \pi/2$ it is an asymptotic curve.

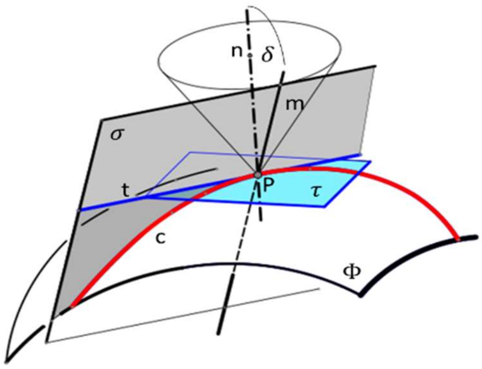

Fig. 9. Sketch to definition 1 of a pseudo-geodesic c on a surface Φ. σ…osculating plane at $P \in c$, τ…tangent plane of Φ at P, t…tangent of c,n…normal of Φ at P, m…main normal of c, $\delta = \sphericalangle nm$.

A natural extension of Definition 1 considers straight lines, too, as pseudo-geodesic curves and we will meet them in faces of a polyhedron. There seem to be several possibilities, how to proceed at a point P of a common edge of two faces and we start with describing one following Definition 1, see Fig. 10:

Let a convex polyhedron be given. We consider two adjacent faces and a point P on their common edge e. As replacement for the tangent plane in the regular case we use the outer symmetry plane such that also the replacement of the normal n through P becomes well-defined. A ray g_1 in the first face ε_1 with endpoint P shall proceed in face ε_2 as g_2 such that the plane $\gamma := g_1 \vee g_2$ includes a given angle δ with the normal n. Fig. 10 shows the situation in front- and top-projection, where e is a projecting line. All possible planes γ envelop a cone of revolution. Supposing that $\delta < \sphericalangle\varepsilon_1\varepsilon_2$ there are two solutions g_2 to a given ray g_1. If we orient the half lines g_1, g_2, n emanating from P, the two solutions can be distinguished by the sign of the determinant $\det(g_1 g_2 n)$.

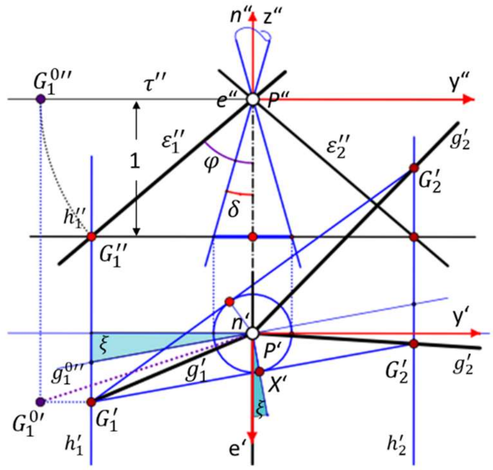

Fig. 10. Front and top projection of two face planes $\varepsilon_1, \varepsilon_2$ with the two solutions $g_2 \subset \varepsilon_2$ to an incoming ray $g_1 \subset \varepsilon_1$.

Intersecting the object depicted in Fig. 10 with a plane parallel to the outer symmetry plane τ of $\varepsilon_1, \varepsilon_2$ allows to find the traces of the planes $\gamma = g_1 \vee g_2$ as tangents to the trace circle of a cone with half apex angle δ. Using the touching point X of the trace of γ resp. the angle $\xi := \sphericalangle X'P'x'$ as parameter, the dependence of $\alpha_{1(\varphi,\delta;\xi)} := \sphericalangle eg_1$ and $\alpha_{2(\varphi,\delta;\xi)} := \sphericalangle eg_2$ can be described with $p := \tan\varphi$, $q := \tan\delta$ as follows:

$$G_i = \left(\frac{q}{\cos\xi} \pm p.\tan\xi, -p, -1\right),\ i = 1,2 \tag{1}$$

$$\cos\alpha_i(\xi) = \left(\frac{q}{\cos\xi} \pm p.\tan\xi\right) \Big/ \sqrt{\left(\frac{q}{\cos\xi} \pm p.\tan\xi\right)^2 + p^2 + 1}\ . \tag{2}$$

For a given α_1 we can exploit equations (2) only numerically. Looking for a better practicable modification we remind us to Snell's refraction law and the fact that the construction of a geodesic uses the net of a polyhedron.

Definition 2: A oriented polygon with vertices on edges of a polyhedron Φ and sides in faces of Φ is called a *quasi-geodesic*, if, at each vertex P, which is an inner point of an edge e of Φ, the polygon's sides g_1, g_2 fulfil a "Snell's refraction condition" with respect to the edge e. If the α_1, α_2 denote the angles $\sphericalangle e g_i$, the following Snell's conditions SRC_i are convenient:

(SRC_1) $\sin \alpha_1 : \sin \alpha_2 = s_1 = const.$ (Snell's law),
(SRC_2) $\alpha_1 : \alpha_2 = s_2 = const.$,
(SRC_3) $\tan \alpha_1 : \tan \alpha_2 = s_3 = const.$

Obviously, also other simple functions $\alpha_2 = f(\alpha_1)$ might be used. We will focus on (SRC_3), as it is easiest to handle. Fig. 11 gives a sketch of the local situation in general in the net of a polyhedron, Fig. 12 shows the front and top projection of two face planes of a polyhedron. In case of (SRC_3), for any ratio $\tan \alpha_1 : \tan \alpha_2 = s_3$, the enveloped cone of the planes $\gamma = g_1 \vee g_2$ degenerates into a line a normal to edge e, which can be considered as a proper replacement for the normal n.

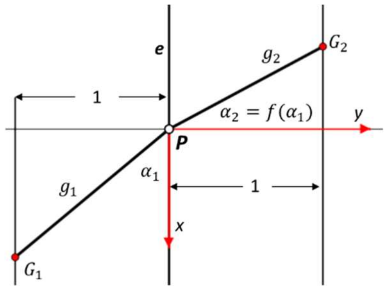

Fig. 11. Sketch to definition 2 of a quasi-geodesic on a polyhedron.

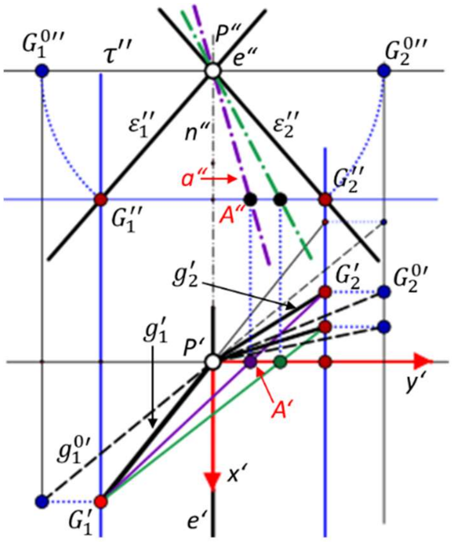

Fig. 12. Front and top projection of two face planes $\varepsilon_1, \varepsilon_2$ with projecting common edge e. In case of (SRC_3) the planes $\gamma = g_1 \vee g_2$ of incoming and outcoming rays pass through a line a.

For a cube's net we find the trace of a quasi-geodesic polygon based on the condition (SRC_3) $\tan\alpha_1 : \tan\alpha_2 = 1/2$, see Fig. 13.

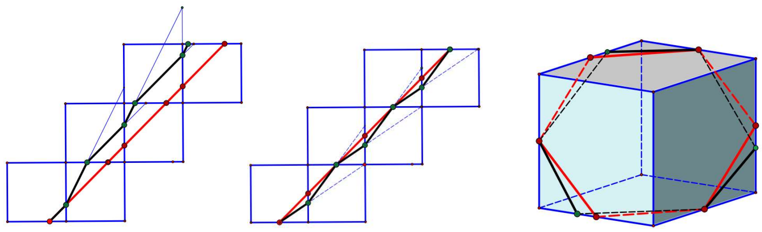

Fig. 13. Net of a cube with developed geodesic (red) and quasi-geodesic (black). The axonometric image of the cube shows a closed geodesic and a closed quasi-geodesic.

4 Square, Cube, and Hypercube: Closed geodesics and billiards

In Chapter 2 we discuss the connection between billiards and geodesics. For example, in Fig. 7, the (closed) billiard trace in the square is constructed in the same way as the closed geodesic on a cube, using the net of the cube. The shortest billiard path in the square has sides parallel to the square's diagonals, and so does the shortest geodesic on the cube. The billiard trace in the cube has segments parallel to 3 diagonals and we can expect that a closed geodesic on the hypercube in the Euclidean 4-space, which meets all 8 face cubes, has similar directed segments. To construct such a geodesic polygon we use a "net" of the hypercube in the 3-space of one of its face cubes. Such a net was depicted by S. Dali in his famous painting "corpus hypercubicus". We proceed analogue to the cube case Fig. 13 and label the vertices in the net according to their position on the hypercube, see Fig. 14.

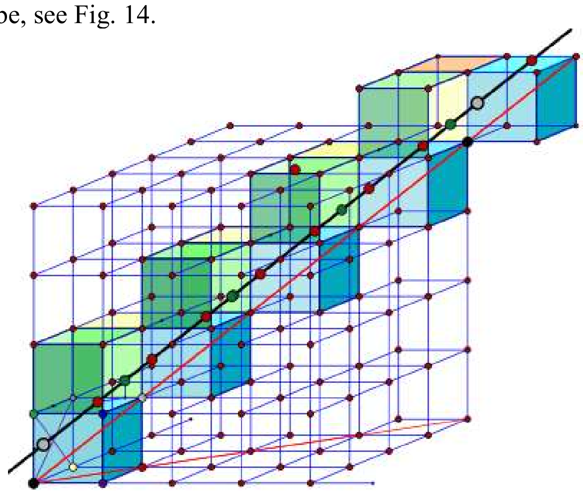

Fig. 14. Net of a hypercube with developed shortest closed geodesic (black) together with the critical position of a geodesic through vertices of the hypercube (red).

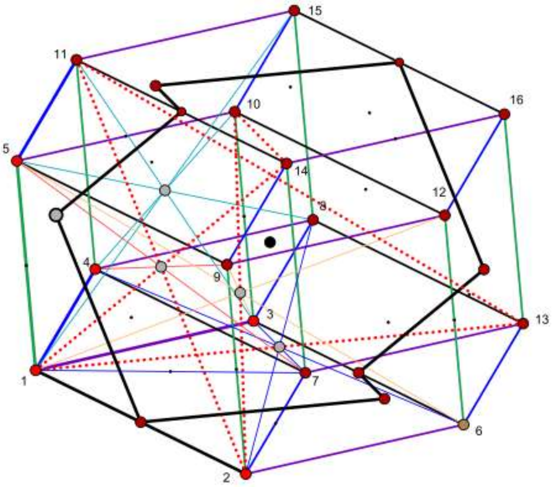

Fig. 15. A special geodesic closed polygon on a hypercube. The sides of the polygon are parallel to diagonals of the face cubes of the hypercube.

In Fig. 14 the diagonals (red) of the cubes symbolize the rectified "critical" geodesic passing through vertices of the hypercube, while the black one represents the rectification of one of the general shortest closed geodesics. Fig. 15 represents an axonometric view of the hypercube with a special case of such a closed geodesic polygon. It meets four edges of the hypercube and has therefore only 8 sides instead of the twelve sides in the general case. The sides are parallel to diagonals of face cubes, (marked as dotted lined star shaped hexagon in Fig. 15).

5 Conclusion

The main aim of this article is to give a hint, how to modify the differential geometric concept "pseudo-geodesic", such that it becomes applicable for polyhedra. Thereby a new concept "quasi-geodesic" is coined, which is based on generalizations of Snell's refraction law. The idea of interpreting geodesics as billiards in a lower dimensional case can be combined with the concept of quasi-geodesics. Here the reader should quicken his appetite to further and deeper going research.

A second aim is a didactical one: Even though there seems nothing new to be said about the cube itself, it is a surprisingly good basic object for generalizations and research, too, and one can teach many mathematical concepts based on them. There is a rich fundus of materials in references, see e.g. Wikipedia, to satisfy one's curiosity.

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