Near-Square Primes Conjecture

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ABSTRACT. In 1912, Edmund Landau listed four basic problems about prime numbers in the International Congress of Mathematicians. These problems are now known as Landau's problems. Landau's fourth problem asked whether there are infinitely many primes which are of the form $n^2 + 1$ for some integer n. This problem remains open and it is known as the Near-square primes conjecture. We prove this conjecture is indeed true.

1. INTRODUCTION

As usual $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are the infinite set of natural, integer and real numbers respectively [1]. Given a function $f : \mathbb{N} \to \mathbb{R}$, we say $\mathsf{Whole}(f)$ holds provided when

$$\exists n_0 \in \mathbb{N} \textit{ such that } \forall n \geq n_0 : f(n) \in \mathbb{Z}.$$

It is easy to deduce the following lemma:

Lemma 1.1. *Given two functions $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$, $\mathsf{Whole}(g)$ does not hold when $\mathsf{Whole}(f)$ holds, $g(n) = \alpha \times f(n)$ and α is an irrational number.*

Certainly, there is not any integer number k with any irrational number α, such that $\alpha \times k$ could have the chance of being an integer no matter how large we choose the value of k [1].

In number theory, Wilson's theorem states that a natural number $n > 4$ is a composite number if and only if the product of all the positive integers less than n is multiple of n [1]. That is the factorial $(n-1)! = 1 \times 2 \times 3 \times \cdots \times (n-1)$ satisfies

$$(n-1)! \equiv 0 \pmod{n}$$

exactly when n is a composite number [1]. In this way, if the Near-square primes conjecture is false, then we would have that $n^2 + 1$ must be a composite number when n tends to infinity. In this way, we prove our main theorem:

Theorem 1.2. $\mathsf{Whole}(\frac{n^2!}{n^2+1})$ *does not hold and therefore, the Near-square primes conjecture is true.*

2. PROOF OF MAIN THEOREM

Proof. If we assume the Near-square primes conjecture is false, then we would have that $n^2 + 1$ must be a composite number when n tends to infinity. Consequently, we obtain that $\mathsf{Whole}(\frac{n^2!}{n^2+1})$ holds. We know

$$\prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)} = 1$$

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where p_j is the j^{th} prime number. We also know that $\mathsf{Whole}(\frac{n^2!}{n^2+1} \times 1)$ holds and thus

$$\mathsf{Whole}(\frac{n^2!}{n^2+1} \times \prod_{j=1}^{\infty} \frac{(p_j^2-1)}{(p_j^2-1)})$$

holds as well. This is equivalent to

$$\mathsf{Whole}(\prod_{j=1}^{\infty}(p_j^2) \times g(n) \times \prod_{j=1}^{\infty} \frac{(p_j^2-1)}{(p_j^2-1)})$$

where

$$\mathsf{Whole}(g(n))$$

since no matter how large could be the value of n, we can always be able to divide the fraction $\frac{n^2!}{n^2+1}$ without eliminating any square of a prime number p_j^2 from the numerator $n^2!$. Certainly, the possible composite number (n^2+1) could be represented in the form of $x \times y$ such that $x, y < n^2$ and the numbers x and y cannot be any square of some prime number. We remove the numbers x and y from the numerator $n^2!$ in the fraction $\frac{n^2!}{n^2+1}$. In addition, we can transform this into

$$\mathsf{Whole}(\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2-1} \times h(n))$$

where we know that

$$\mathsf{Whole}(h(n)) = \mathsf{Whole}(g(n) \times \prod_{j=1}^{\infty}(p_j^2-1)) = \mathsf{Whole}(g(n) \times \prod_{p_j < n^2+1}(p_j^2-1))$$

since every prime number p_j would be lesser than n^2+1 when n tends to infinity. However,

$$\mathsf{Whole}(\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2-1} \times h(n))$$

would be the same as

$$\mathsf{Whole}(\frac{\pi^2}{6} \times h(n))$$

since we have

$$\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2-1} = \prod_{j=1}^{\infty} \frac{1}{1-p_j^{-2}} = \frac{\pi^2}{6}$$

because of the result in the Basel problem [1]. Hence, we obtain a contradiction since

$$\mathsf{Whole}(\frac{\pi^2}{6} \times h(n))$$

does not hold according to the Lemma 1.1. Hence, our assumption that the Near-square primes conjecture would be false is incorrect and therefore, we obtain the conjecture should be necessarily true. □

References

[1] David G. Wells. *Prime Numbers, The Most Mysterious Figures in Math.* John Wiley & Sons, Inc., 2005.

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