A Look on Mathematical Fundamentals for Minimax Theorem and Nash Equilibrium Existence

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A LOOK ON MATHEMATICAL FUNDAMENTALS FOR MINIMAX THEOREM AND NASH EQUILIBRIUM EXISTENCE

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Abstract. We present two results of Game Theory, very important in Economics: minimax theorem and Nash equilibrium existence, together with their mathematical fundamentals, obtained in the scope of Real Analysis. For minimax theorem, **t**he mathematical structure considered is real Hilbert Spaces. Moreover, the convex sets strict separation play here an important role. For Nash equilibrium existence, Kakutani's theorem is the key result to consider.

Key words. Minimax theorem, Nash equilibrium, convex sets strict separation, Kakutani's theorem.

Mathematics Subject Classification: 91A40.

1 Introduction

We will see how the convex sets strict separation result, allows obtaining a fundamental result in Game Theory: minimax theorem. The mathematical structure considered is real Hilbert Spaces; see [7]. Then we do the same for Nash equilibrium existence result using mainly Kakutani's theorem, see [7, 12 and 15].Finally, we present and briefly discuss some results, trying to find contact points between minimax theorem and Nash equilibrium existence result.

2 Minimax Theorem

Consider Games with two players and null sum:

- Be $\Phi(\boldsymbol{x},\boldsymbol{y})$ a two variables real function, $\boldsymbol{x},\boldsymbol{y} \in H$, being H a real Hilbert Space.

- Be A and B two convex sets in H.

- One of the players chooses strategies (points) in A in order to maximize $\Phi(\boldsymbol{x},\boldsymbol{y})$ (or minimize $-\Phi(\boldsymbol{x},\boldsymbol{y})$): it is a *maximizing* player.

- The other player chooses strategies (points) in B in order to minimize $\Phi(\boldsymbol{x},\boldsymbol{y})$ (or maximize $-\Phi(\boldsymbol{x},\boldsymbol{y})$): it is the *minimizing* player.

The function $\Phi(\boldsymbol{x},\boldsymbol{y})$ is the *payoff function.* The value $\Phi(\boldsymbol{x}_0,\boldsymbol{y}_0)$ represents, simultaneously, the maximizing player gain and the minimizing player loss in a move where they choose, respectively, the strategies $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$. Therefore, the gain of one of the players is identical to the loss of the other. Therefore, the game is a null sum game.

In these conditions the game has value c if

$$\sup_{\boldsymbol{x}\in A}\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) = c = \inf_{\boldsymbol{y}\in B}\sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y}). \qquad (2.1)$$

If, for any $(\boldsymbol{x}_0,\boldsymbol{y}_0)$, $\Phi(\boldsymbol{x}_0,\boldsymbol{y}_0) = c$, $(\boldsymbol{x}_0,\boldsymbol{y}_0)$ is a pair of *optimal strategies*. It is also a saddle point if it verifies in addition

$$\Phi(\boldsymbol{x},\boldsymbol{y}_0) \le \Phi(\boldsymbol{x}_0,\boldsymbol{y}_0) \le \Phi(\boldsymbol{x}_0,\boldsymbol{y}), \boldsymbol{x}\in A, \boldsymbol{y}\in B. \qquad (2.2)$$

It is conceptually easy to generalize this situation to an n players null sum game, although algebraically fastidious.

The fundamental result in this section is:

Theorem 2.1 (minimax theorem)
The sets A and B in H are both closed and convex, being A also bounded; $\Phi(\boldsymbol{x},\boldsymbol{y})$ is a real function defined for x in A and $\boldsymbol{y}$ in B such that:

- $\Phi(\boldsymbol{x},(1-\theta)\boldsymbol{y}_1+\theta\boldsymbol{y}_2) \le (1-\theta)\Phi(\boldsymbol{x},\boldsymbol{y}_1)+\theta\Phi(\boldsymbol{x},\boldsymbol{y}_2)$ for $\boldsymbol{x}$ in A and $\boldsymbol{y}_1,\boldsymbol{y}_2$ in B, $0\le\theta\le 1$ (that is: $\Phi(\boldsymbol{x},\boldsymbol{y})$ is convex in $\boldsymbol{y}$ for each $\boldsymbol{x}$),

- $\Phi\big((1-\theta)\boldsymbol{x}_1+\theta\boldsymbol{x}_2,\boldsymbol{y}\big) \ge (1-\theta)\Phi(\boldsymbol{x}_1,\boldsymbol{y})+\theta\Phi(\boldsymbol{x}_2,\boldsymbol{y})$ for $\boldsymbol{y}$ in B and $\boldsymbol{x}_1,\boldsymbol{x}_2$ in A, $0\le\theta\le 1$ (that is: $\Phi(\boldsymbol{x},\boldsymbol{y})$ is concave in $\boldsymbol{x}$ for each $\boldsymbol{y}$),

- $\Phi(\boldsymbol{x},\boldsymbol{y})$ is continuous in $\boldsymbol{x}$ for each $\boldsymbol{y}$.

So (2.1) holds, that is the game has a value.
Dem.:
Beginning by the most trivial part of the demonstration:

$$\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \le \Phi(\boldsymbol{x},\boldsymbol{y}) \le \sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y})$$

and so

$$\sup_{\boldsymbol{x}\in A}\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \le \inf_{\boldsymbol{y}\in B}\sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y}).$$

Then, as $\Phi(\boldsymbol{x},\boldsymbol{y})$ is concave and continuous in $\boldsymbol{x}\in A$, A convex, closed and bounded, it follows that $\sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y}) < \infty$.

Be $c = \inf_{\boldsymbol{y}\in B}\sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y})$. Suppose now that there is $\boldsymbol{x}_0\in A$ such that $\Phi(\boldsymbol{x}_0,\boldsymbol{y}) \ge c$, for any $\boldsymbol{y}$ in B. In this case, $\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x}_0,\boldsymbol{y}) \ge c$ or $\sup_{\boldsymbol{x}\in A}\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \ge c$ as it is appropriate.

Then the existence of such a $\boldsymbol{x}_0$ will be demonstrated.

For any $\boldsymbol{y}$ in B, be $A_{\boldsymbol{y}} = \{\boldsymbol{x}\in A: \Phi(\boldsymbol{x},\boldsymbol{y}) \ge c\}$. The set $A_{\boldsymbol{y}}$ is closed, bounded and convex. Suppose that, for a finite set $(\boldsymbol{y}_1,\boldsymbol{y}_2,\dots,\boldsymbol{y}_n)$, $\bigcap_{i=1}^{n} A_{\boldsymbol{y}_i} = \emptyset$. Consider the transformation from A to E_n defined by

$$f(\boldsymbol{x}) = (\Phi(\boldsymbol{x},\boldsymbol{y}_1) - c, \Phi(\boldsymbol{x},\boldsymbol{y}_2) - c, \dots, \Phi(\boldsymbol{x},\boldsymbol{y}_n) - c).$$

Call G the $f(A)$ convex hull closure. Be P the E_n closed positive cone. Now we show $P \cap G = \emptyset$: indeed, being $\Phi(\boldsymbol{x},\boldsymbol{y})$ concave in $\boldsymbol{x}$, for any $\boldsymbol{x}_k$ in A, $k = 1, 2, \dots, n, 0 \leq \theta_k \leq 1, \sum_{k=1}^{n} \theta_k = 1$,

$$\sum_{k=1}^{n} \theta_k (\Phi(\boldsymbol{x}_k,\boldsymbol{y}) - c) \leq \Phi\left(\sum_{k=1}^{n} \theta_k \boldsymbol{x}_k, \boldsymbol{y}\right) - c$$

therefore, the convex extension of $f(A)$ does not intersect P.
Consider now a sequence $\boldsymbol{x}_n$ of points in A, such that $f(\boldsymbol{x}_n)$ converges to $\boldsymbol{v}, \boldsymbol{v} \in E_n$. As A is closed, bounded and convex, it is possible to define a subsequence, designated $\boldsymbol{x}_m$ such that $\boldsymbol{x}_m$ converges weakly for an element of A (call it $\boldsymbol{x}_0$). In addition, for any $\boldsymbol{y}_i$ as $\Phi(\boldsymbol{x},\boldsymbol{y}_i)$ is concave in $\boldsymbol{x}$,

$$\overline{lim}\,\Phi(\boldsymbol{x}_m,\boldsymbol{y}_i) \leq \Phi(\boldsymbol{x}_0,\boldsymbol{y}_i), \text{or } f(\boldsymbol{x}_0) \geq \overline{lim}\, f(\boldsymbol{x}_m = \boldsymbol{v}).$$

So $P \cap G = \emptyset$. Then, G and P may be strictly separated, and it is possible to find a vector in E_n with coordinates a_k, such that

$$\sup_{x \in A} \sum_{i=1}^{n} a_i (\Phi(\boldsymbol{x},\boldsymbol{y}_i) - c) < \sum_{i=1}^{n} a_i \boldsymbol{e}_i,$$

with the whole a_i greater or equal than zero.

Obviously, the a_i cannot be simultaneously null. So, calculating the ratio over $\sum_{i=1}^{n} a_i$ and having in mind the convexity of $\Phi(\boldsymbol{x},\boldsymbol{y})$ in $\boldsymbol{y}$

$$\sup_{x \in A} \Phi(\boldsymbol{x},\overline{\boldsymbol{y}}) - c < 0, \text{where } \overline{\boldsymbol{y}} = \frac{\sum_{k=1}^{n} a_k y_k}{\sum_{k=1}^{n} a_k}.$$

Moreover, evidently, either $\overline{\boldsymbol{y}} \in B$ or $\inf_{y \in B} \sup_{x \in A} \Phi(\boldsymbol{x},\boldsymbol{y}) < c$. This contradicts the definition of c. So,

$$\bigcap_{i=1}^{n} A_{y_i} \neq \emptyset.$$

Indeed,

$$\bigcap_{y \in B} A_{\boldsymbol{y}} \neq \emptyset,$$

as we will demonstrate in the sequence using that result and proceeding by absurd. Note that $A_{\boldsymbol{y}}$ is a closed and convex set and so it is also weakly closed. Moreover, as it is a bounded set, it is compact in the weak topology[1], such as A. Calling $G_{\boldsymbol{y}}$ the complement of $A_{\boldsymbol{y}}$ it results that $G_{\boldsymbol{y}}$ is an open set in the weak topology. So, if $\bigcap_{y \in B} A_{\boldsymbol{y}}$ is empty:

$$\bigcap_{y \in B} G_{\boldsymbol{y}} \supset H \supset A.$$

However, being A compact, a finite number of G_{y_i} is enough to cover A:

[1] See, for instance, [13].

$$\bigcup_{i=1}^{n} G_{y_i} \supset A.$$

That is: $\bigcap_{i=1}^{n} A_i$ is in the complement of A and so it must be $\bigcap_{i=1}^{n} A_{y_i} = \emptyset$, leading to a contradiction.
Suppose then that $\boldsymbol{x}_0 \in \bigcap_{y \in B} A_{\boldsymbol{y}}$. So, actually $\boldsymbol{x}_0$ satisfies $\Phi(\boldsymbol{x}_0, \boldsymbol{y}) \geq c$, as requested. □

Then it follows a Corollary of Theorem 2.1, obtained strengthening its hypothesis.

Corollary 2.1
Suppose that the functional $\Phi(\boldsymbol{x}, \boldsymbol{y})$ defined in Theorem 2.1 is continuous in both variables, separately, and that *B* is a bounded set. Therefore, there is an optimal pair of strategies, with the property of being a saddle point.
Dem.:
It was already seen that exists $\boldsymbol{x}_0$ such that

$$\Phi(\boldsymbol{x}_0, \boldsymbol{y}) \geq c \qquad (2.3)$$

for each $\boldsymbol{y}$. As $\Phi(\boldsymbol{x}_0, \boldsymbol{y})$ is continuous in $\boldsymbol{y}$ and *B* is a bounded set

$$\inf_{\boldsymbol{y} \in B} \Phi(\boldsymbol{x}_0, \boldsymbol{y}) = \Phi(\boldsymbol{x}_0, \boldsymbol{y}_0) \geq c \qquad (2.4)$$

for any $\boldsymbol{y}_0$ in *B*[2]. But $\inf_{\boldsymbol{y} \in B} \Phi(\boldsymbol{x}_0, \boldsymbol{y}) \leq \sup_{x \in A} \inf_{y \in B} \Phi(\boldsymbol{x}, \boldsymbol{y}) = c$ and, so

$$\Phi(\boldsymbol{x}_0, \boldsymbol{y}_0) = c. \qquad (2.5)$$

The saddle point property follows immediately from (2.3), (2.4) and (2.5). □

3 Nash Equilibrium Existence

The formulation and resolution of a game is very important in Game Theory. There are several game solution concepts. However, some of these concepts are restrict to a certain kind of games. John Nash, see [20], defined the most important solution concept. We will see that Nash equilibrium exists for a large class of games.

Call E_n the finite set of available strategies for a player. E denotes the Cartesian product of these sets. A typical element of this set is $\boldsymbol{e} = (e_1, e_2, \ldots, e_N)$, called a pure strategy profile, where each e_n is a pure strategy for player *n*.

Definition 3.1
A mixed strategy of a player *n* is a lottery over the pure strategies of player *n*.

Obs.:

- Denote σ_n a player *n* particular mixed strategy, and Σ_n the player *n* set of all its mixed strategies.

[2] A continuous convex functional in a Hilbert space has minimum in any bounded closed convex set.

- Thus $\boldsymbol{\sigma}_n = \left(\sigma_n(e_n^1), \sigma_n(e_n^2), \dots, \sigma_n\left(e_n^{k_n}\right)\right)$ where k_n is the number of pure strategies of player *n* and $\sigma_n\left(e_n^i\right) \geq 0, i = 1, 2, \dots, k_n$ and $\sum_{i=1}^{k_n} \sigma_n\left(e_n^i\right) = 1$.

- The cartesian product $\Sigma = \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_N$ is the set of all mixed strategy profiles.

- Therefore, the mixed strategy set for each player is the probability distribution set over its pure strategy set.

Definition 3.2
A *n*-dimensional simplex defined by the $n+1$ points $\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ in $\mathbb{R}^p, p \geq n$, is denoted $\langle \boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n \rangle$ and is defined by the set

$$\left\{\mathbb{R}^p : \boldsymbol{x} = \sum_{j=0}^{n} \theta_j \boldsymbol{x}_j, \sum_{j=0}^{n} \theta_j = 1, \theta_j \geq 0\right\}.$$

Obs.:

- The simplex is non-degenerate if the *n* vectors $\boldsymbol{x}_1 - \boldsymbol{x}_0, \dots, \boldsymbol{x}_n - \boldsymbol{x}_0$ are linearly independent.

- If $\boldsymbol{x} = \sum_{j=0}^{n} \theta_j \boldsymbol{x}_j$, the numbers $\theta_0, \theta_1, \dots, \theta_n$ are called the barycenter coordinates of $\boldsymbol{x}$.

- The barycentre of the simplex $\langle \boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n \rangle$ is the point having the whole coordinates equal to $(n+1)^{-1}$.

Definition 3.3
Call $u_n(\sigma)$ the expected payoff function of player *n* associated to the mixed strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$.

Definition 3.4
A Nash equilibrium of a game is a profile of mixed strategies $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$ such that for each $n = 1, 2, \dots, N$ for each e_n and e_n' in E_n, if $\sigma_n(e_n) > 0$ then

$$u_n(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, e_n, \sigma_{n+1}, \dots, \sigma_N) \geq u_n(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, e_n', \sigma_{n+1}, \dots, \sigma_N).$$

Obs.:

- Therefore, an equilibrium is a profile of mixed strategies such that a player knows what strategies the other players will go to choose, and no player has incentive to deviate from the equilibrium since that it cannot improve its payoff through a unilateral change of its strategy.

- A Nash equilibrium induces a necessary condition of strategic stability.

For the remaining, it is necessary the following result:

Theorem 3.1 (Kakutani's theorem)
Let $M \subset \mathbb{R}^n$ be a compact convex set. Be $F: M \to M$ an upper hemi-continuous convex valued correspondence. Then the correspondence F has a fixed point.

Theorem 3.2 (Nash)
The mixed extension of every finite game has, at least, one strategic equilibrium.

Dem.:
Consider the set-valued mapping that maps each strategy profile, $\boldsymbol{x}$, to all strategy profiles in which each player's component strategy is a best response to $\boldsymbol{x}$. That is, maximizes the player's payoff given that the others are adopting their components of $\boldsymbol{x}$**.** If a strategy profile is contained in the set to which it we mapped it (is a fixed point) then it is an equilibrium. This is so because we defined a strategic equilibrium, in effect, as a profile that is a best response itself.

Thus, the proof of existence of equilibrium amounts to a demonstration that the best response correspondence has a fixed point. The fixed – point theorem of Kakutani asserts the existence of a fixed point for every correspondence from a convex and compact subset of Euclidean Space into itself. This happens if two conditions hold: 1) The image of every set must be convex; 2) The graph of the correspondence (the set of pairs $(\boldsymbol{x}, \boldsymbol{y})$ where $\boldsymbol{y}$ is the image of $\boldsymbol{x}$) must be closed. Now, in the mixed extension of a finite game, the strategies set of each player consists of all vectors (with as many components as there are pure strategies) of non-negative numbers that sum *1*; that is, it is a simplex. Thus, the set of all strategy profiles is a product of simplexes. In particular, it is a convex and compact subset of Euclidean Space. Given a particular choice of strategies by the other players, a player's best responses consist of all (mixed) strategies that put positive weight on those pure strategies that highest expected payoff among all the pure strategies. Thus, the set of best responses is a sub simplex. In particular, it is convex.

Finally, note that the conditions that needed a given strategy to be a best response to a given profile are all weak polynomial inequalities, so the graph of the best response correspondence is closed.

Thus, all the conditions of Kakutani's theorem hold, and this completes the proof of Theorem 3.2. □

4 Minimax Theorem versus Nash Equilibrium

Begin to refer that in [25] it is established that Sion's minimax theorem, see [26], is equivalent to the existence of Nash equilibrium in a symmetric multi-person zero - sum game. If a zero-sum game is asymetric, players maximin strategies and minimax strategies do not correspond to Nash equilibrium strategies. However, if it is symmetric, the maximin strategy and the minimax strategy constitute a Nash equilibrium.

In [11], Hattori, Satoh, and Tanaka consider a symmetric multi-players zero-sum game with two strategic variables. There are n players,$n \geq 3$. Each player is denoted by i. Two strategic variables are t_i and s_i, $i \in \{1, \dots, n\}$. They are related by invertible functions. Using the Sion's Minimax Theorem, see again Sion [26], they show that Nash equilibria in the following states are equivalent: 1) All players choose t_i, $i \in \{1, \dots, n\}$, as their strategic variables, 2) Some players choose $t_i's$ and the other players choose $s_i's$, and 3) All players choose s_i, $i \in \{1, \dots, n\}$.

In short, Hattori, Satoh, and Tanaka have shown that in a symmetric multi-player zero-sum game with two strategic variables, the choice of strategic variables is irrelevant to the Nash equilibrium. Indeed, in an asymmetric situation Nash equilibrium depends on the choice of strategic variables by players other than two-player case, see [24].

5 Conclusions

Minimax theorem, see [21], and Nash equilibrium, see [20], were two main achievements that give raise to a great spread of the Game Theory applications namely in the Economic domain. The minimax theorem is more important in domains like Operations Research than in Economics. The opposite happens with Nash equilibrium. In particular in the famous Cournot-Nash Model, among others.

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